The Einstein relation for random walks on Galton–Watson trees

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Joint with G. Ben Arous, Y. Hu, S. Olla

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One dimensional Brownian motion:
\( W_t; \ E W_t^2 = t. \)

Add drift \( \alpha \) locally: \( W_t^\alpha = W_t + \alpha t; \ v_\alpha = \lim_{t \to \infty} \frac{W_t^\alpha}{t} \)

Of course, \( v_\alpha = \alpha \), hence
\[
\lim_{t \to \infty} \frac{E W_t^2}{t} = \frac{v_\alpha}{\alpha}
\]

In general, can re-parametrize \( \alpha \), i.e., have drift \( d = d(\alpha) \) with \( d'(\alpha)|_{\alpha=0} = 1 \). Einstein relation is then the statement
\[
\lim_{t \to \infty} \frac{E W_t^2}{t} = \lim_{\alpha \to 0} \frac{v_\alpha}{\alpha} = \lim_{\alpha \to 0} \lim_{t \to \infty} \frac{W_t^\alpha}{t}.
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Random walk setup (on $\mathbb{Z}$)

Ct’s time random walk $X_t$, rate of jumps $e^{\alpha}$ to right, $e^{-\alpha}$ to left. At $\alpha = 0$, $EX_t^2/t \to 2$.

When $\alpha \neq 0$, we get

$$\lim_{\alpha \to 0} \lim_{t \to \infty} \frac{|X_t|}{t} = 2,$$

verifying ER.

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What can be said in disordered systems?
In a rather general setup, a tagged particle $X_t$ moves in a random environment, satisfying the invariance principle, and $EX_t^2 / t \to \sigma^2$.

This is usually proved by considering the environment viewed from the point of view of particle, and applying the Kipnis-Varadhan theory; works well in reversible situations.

Apply external force $\alpha f$ and obtain process $X_t^\alpha$.

**Theorem (Lebowitz-Rost)**

*Under quite general conditions, for any $c > 0$,*

$$\lim_{\alpha \to 0} \frac{X_t^\alpha c / \alpha^2}{\alpha c / \alpha^2} = \frac{f \sigma^2}{2}.$$ 

Verified for tagged particle in environment of interacting particles, for random walk on random conductance network and for Ornstein-Uhlenbeck process in random medium.

Argument uses a Girsanov transformation that eliminates the drift, and an estimate on the resulting Radon-Nykodim derivative.
Lebowitz-Rost (1994)

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Loulakis ’02 Tagged particle in symmetric exclusion process, \( d \geq 3 \).
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Gantert-Mathieu-Piatnitskii ’10 Diffusion in random potential/random conductance model.
Approach of [GMP] uses regeneration times: [LR] tell us ER holds by time \( c/\alpha^2 \). Control on regeneration times says that by that time, relaxation to equilibrium in perturbed system has occured. Control on regeneration times is uniform in environment because traps are of bounded size.
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Galton-Watson trees

$T$: random tree, Galton Watson, offspring distribution $\{p_k\}$, $p_0 = 0$, $p_1 < 1$, $m = \sum kp_k$ mean offspring.

Start random walk on $T$: if $u$ is an offspring of $v$ then jump rate is 1. Jump rate to parent is $\lambda$.

Theorem (Lyons, Pemantle, Peres ’95)

- $\lambda > m$ (drift toward root): $\{X_n\}$ positive recurrent, $|X_n|/n \to 0$.
- $\lambda < m$ (drift away from root): $\{X_n\}$ transient, $|X_n|/n \to \nu > 0$ (ballistic). There is a sequence of regeneration times $\tau_i$, such that $(\tau_{i+1} - \tau_i, |X|_{\tau_{i+1}} - |X|_{\tau_i})$ are i.i.d. (annealed).
- $\lambda = 1 < m$: an explicit invariant measure for the environment viewed from the point of view of particle is known. Speed $\nu = \sum p_k(k - 1)/(k + 1)$.
- $\lambda = m$: critical case. Walk is null recurrent (Lyons).
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Galton-Watson trees - CLT

Assume \( \{p_k\}_{k \geq 1} \) has exponential moments.

Theorem (Peres-Z ’08)

\( \lambda = m \) There is a deterministic \( \sigma^2 > 0 \) such that, for almost every \( T \),

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\left\{ \frac{X_{[nt]}}{\sqrt{\sigma^2 n}} \right\}_{t \geq 0} \rightarrow \{ |B_t| \}_{t \geq 0}.
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For \( \lambda < m \), walk has positive speed, and regeneration times can be used.

For \( \lambda = m \), crucial role played by an explicit invariant measure of environment viewed from particle, and \( \sigma^2 \) becomes explicit.
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Galton-Watson trees - invariant measure

Invariant measure for $\lambda = m$

For $\lambda < m$: not explicit (and possibly not absolutely continuous with respect to IGW)
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Set $\lambda = me^{-\alpha}$, $v_\alpha = \lim_{t \to \infty} |X_t^\alpha|/t$. Recall that $|X_{nt}|/\sqrt{n} \to \sigma^2 |B_t|$. 

Theorem (Ben Arous, Hu, Olla, Z ’11)

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There is also a statement when $\alpha < 0$, walk on extended tree, using (explicit) expression for invariant measure.
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- No explicit expression for invariant measure when $\alpha > 0$.
- Due to existence of arbitrary large traps, no uniform control on slow-down.
- Lack of uniform control translates to bad control of moments of regeneration times (as function of $\alpha$).

But... Tree structure allows for recursions, which can be used to compute hitting times.
Galton-Watson trees - Proof of Einstein Relation

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A relevant quantity is $\beta(x) = P_{\lambda}(\{X_n\}_{n \geq 1} \cap x = \emptyset)$. Set $B(x) = \lambda^{-1} \sum_{x_i \text{child of } x} \beta(x_i)$.

$$B(x) = \lambda^{-1} \sum_i \frac{B(x_i)}{1 + B(x_i)}.$$  

This implies

$$EB = e^\alpha E(B/(1 + B)) \leq e^\alpha EB/(1 + EB) \Rightarrow EB \leq (e^\alpha - 1).$$

Computation: for some $C$ independent of $\alpha$,

$$EB \geq C(e^\alpha - 1), \quad EB^2 \leq C(E(B))^2$$

Hence $B/EB$ is tight, i.e. $B/\alpha$ is tight, and converges (as $\alpha \to 0$) to a random variable $Y$. 
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Some elements of proof of ER - basic recursion

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Computation: for some \( C \) independent of \( \alpha \),

\[
EB \geq C(e^\alpha - 1), \quad EB^2 \leq C(E(B))^2
\]

Hence \( B/EB \) is tight, i.e. \( B/\alpha \) is tight, and converges (as \( \alpha \to 0 \)) to a random variable \( Y \).
$B/\alpha \to Y$, $Y$ satisfies

$$Y \stackrel{d}{=} \frac{1}{m} \sum_i Y_i,$$

This allows to identify law of $Y$, but also that

$$\lim_{\alpha \to 0} \frac{EB}{\alpha} = \frac{\sigma^2}{2m}.$$

Missing element: with $T_n = \min \{ t : |X_t| = n \}$, evaluate $ET_n$. Uses recursions similar to $B$, but also a representation of expectations in terms of a spine random walk, and a renewal argument.
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