The Einstein relation for random walks on Galton–Watson trees

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One dimensional Brownian motion:
W_t; $EW_t^2 = t$.
Add drift α locally: $W_t^\alpha = W_t + \alpha t$; $v_\alpha = \lim_{t\to\infty} \frac{W_t^\alpha}{t}$
Of course, $v_\alpha = \alpha$, hence

$$\lim_{t\to\infty} \frac{EW_t^2}{t} = \frac{v_\alpha}{\alpha}$$

In general, can re-parametrize α, ie have drift $d = d(\alpha)$ with $d'(\alpha)|_{\alpha=0} = 1$. Einstein relation is then the statement

$$\lim_{t\to\infty} \frac{EW_t^2}{t} = \lim_{\alpha\to 0} \frac{v_\alpha}{\alpha} = \lim_{\alpha\to 0} \lim_{t\to\infty} \frac{W_t^\alpha}{t}.$$

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Random walk setup (on $\mathbb{Z}$)

Ct's time random walk X_t, rate of jumps e^{α} to right, $e^{-\alpha}$ to left. At $\alpha = 0$, $EX_t^2/t \to 2$.

When $\alpha \neq 0$, we get

$$\lim_{\alpha\to 0}\lim_{t\to\infty}\frac{|X_t|}{t} = 2\,,$$

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Lebowitz-Rost (1994)

In a rather general setup, a tagged particle X_t moves in a random environment, satisfying the invariance principle, and $EX_t^2/t \to \sigma^2$.

This is usually proved by considering the environment viewed from the point of view of particle, and applying the Kipnis-Varadhan theory; works well in reversible situations.

Apply external force αf and obtain process X_t^α.

Theorem (Lebowitz-Rost)

Under quite general conditions, for any $c > 0$,

$$\lim_{\alpha \to 0} \frac{X^\alpha_{c/\alpha^2}}{\alpha c/\alpha^2} = \frac{f\sigma^2}{2}.$$

Verified for tagged particle in environment of interacting particles, for random walk on random conductance network and for Orenstein-Uhlenbeck process in random medium.

Argument uses a Girsanov transformation that eliminates the drift, and an estimate on the resulting Radon-Nykodim derivative.

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Verification using Kipnis-Varadhan theory and control on relaxation time of dynamics:

Loulakis '02 Tagged particle in symmetric exclusion process, $d \geq 3$.

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Approach of [GMP] uses regeneration times: [LR] tell us ER holds by time c/α^2. Control on regeneration times says that by that time, relaxation to equilibrium in perturbed system has occured.

Control on regeneration times is uniform in environment because traps are of bounded size.

What about systems with arbitrarily large traps?

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$\mathcal{T}$: random tree, Galton Watson, offspring distribution $\{p_k\}$, $p_0 = 0$, $p_1 < 1$, $m = \sum kp_k$ mean offspring.
Start random walk on $\mathcal{T}$: if u is an offspring of v then jump rate is 1. Jump rate to parent is λ.

Theorem (Lyons, Pemantle, Peres '95)

• $\lambda > m$ (drift toward root): $\{X_n\}$ positive recurrent, $|X_n|/n \to 0$.
• $\lambda < m$ (drift away from root): $\{X_n\}$ transient, $|X_n|/n \to v > 0$ (ballistic). There is a sequence of regeneration times τ_i, such that $(\tau_{i+1} - \tau_i, |X|_{\tau_{i+1}} - |X|_{\tau_i})$ are i.i.d. (annealed).
•: $\lambda = 1 < m$: an explicit invariant measure for the environment viewed from the point of view of particle is known. Speed $v = \sum p_k(k-1)/(k+1)$.
• $\lambda = m$: critical case. Walk is null recurrent (Lyons).

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Galton-Watson trees - CLT

Assume $\{p_k\}_{k\geq 1}$ has exponential moments.

Theorem (Peres-Z '08)

($\lambda = m$) There is a deterministic $\sigma^2 > 0$ such that, for almost every $\mathcal{T}$,

$$\left\{\frac{|X_{[nt]}|}{\sqrt{\sigma^2 n}}\right\}_{t\geq 0} \to \{|B_t|\}_{t\geq 0}\,.$$

($\lambda < m$) (easier):

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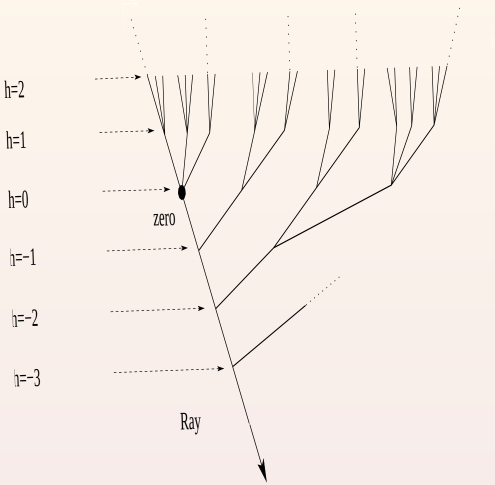

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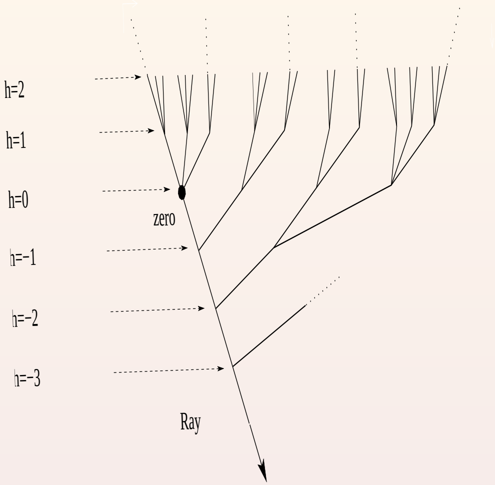
For $\lambda < m$: not explicit (and possibly not absolutely continuous with respect to IGW)

Galton-Watson trees - invariant measure



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There is also a statement when $\alpha < 0$, walk on extended tree, using (explicit) expression for invariant measure.

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Difficulties

• No explicit expression for invariant measure when $\alpha > 0$.

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• Lack of uniform control translates to bad control of moments of regeneration times (as function of α).

But. . . Tree structure allows for recursions, which can be used to compute hitting times.

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A relevant quantity is $\beta(x) = P_T^x(\{X_n\}_{n\geq 1} \cap x = \emptyset)$. Set $B(x) = \lambda^{-1} \sum_{x_i \text{child of } x} \beta(x_i)$.

$$B(x) = \lambda^{-1} \sum_i \frac{B(x_i)}{1 + B(x_i)} .$$

This implies

$$EB = e^{\alpha} E(B/(1+B)) \leq e^{\alpha} EB/(1+EB) \Rightarrow EB \leq (e^{\alpha} - 1) .$$

Computation: for some C independent of α,

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Some elements of proof of ER - basic recursion II

$B/\alpha \to Y$, Y satisfies

$$Y \stackrel{d}{=} \frac{1}{m} \sum_i Y_i \,,$$

This allows to identify law of Y, but also that

$$\lim_{\alpha \to 0} \frac{EB}{\alpha} = \frac{\sigma^2}{2m} \,.$$

Missing element: with $T_n = \min\{t : |X_t| = n\}$, evaluate ET_n.
Uses recursions similar to B, but also a representation of expectations in terms of a spine random walk, and a renewal argument.

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