Mode-locked laser pulse fluctuations

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Acknowledgement: Rafi Weill, Oded Basis, Alex Bekker, Vladimir Smulakovsky
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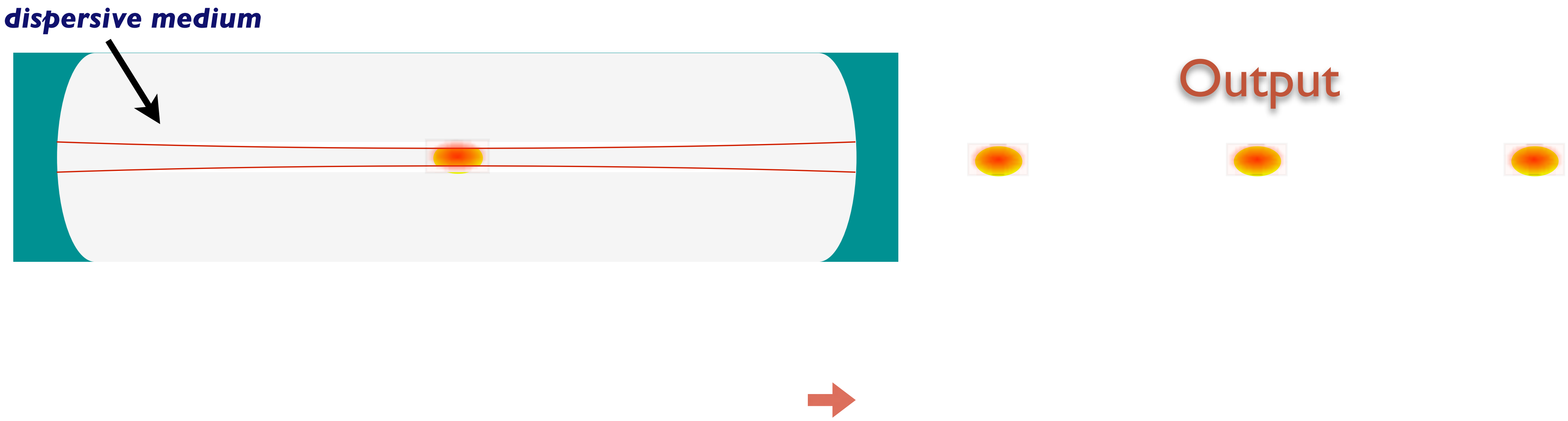
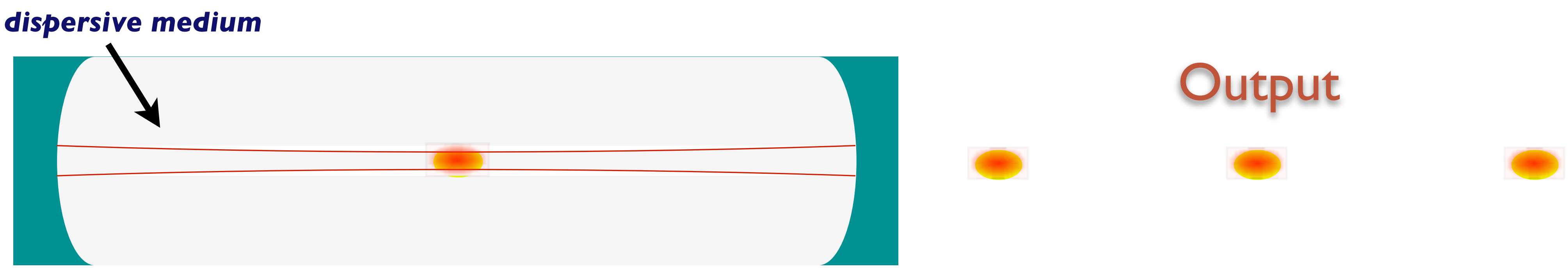
Synopsis

- Mode-locked soliton lasers and noise
- The statistical steady state
- Fluctuations in the steady state
    1. Pulse-continuum interactions
    2. Gain fluctuations
    3. Slow modes of pulse dynamics
    4. Pulse parameter equations of motion
- Autocorrelation and diffusion of pulse parameters

Mode-locked soliton lasers

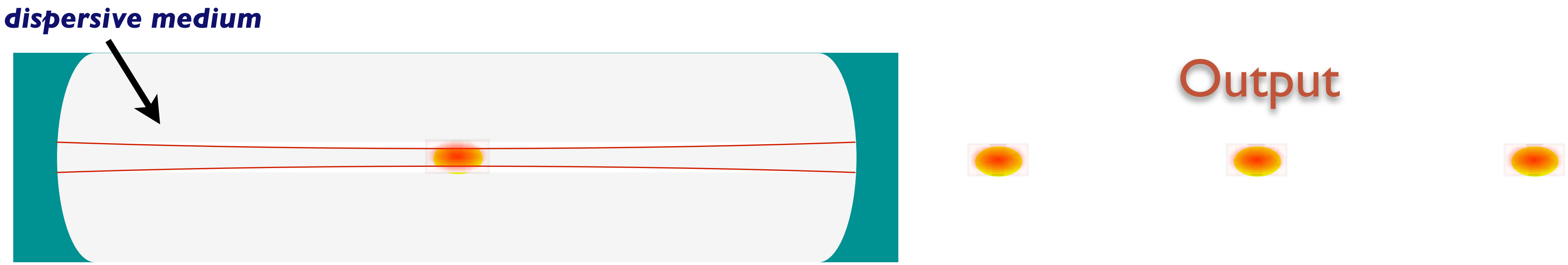


Mode-locked soliton lasers



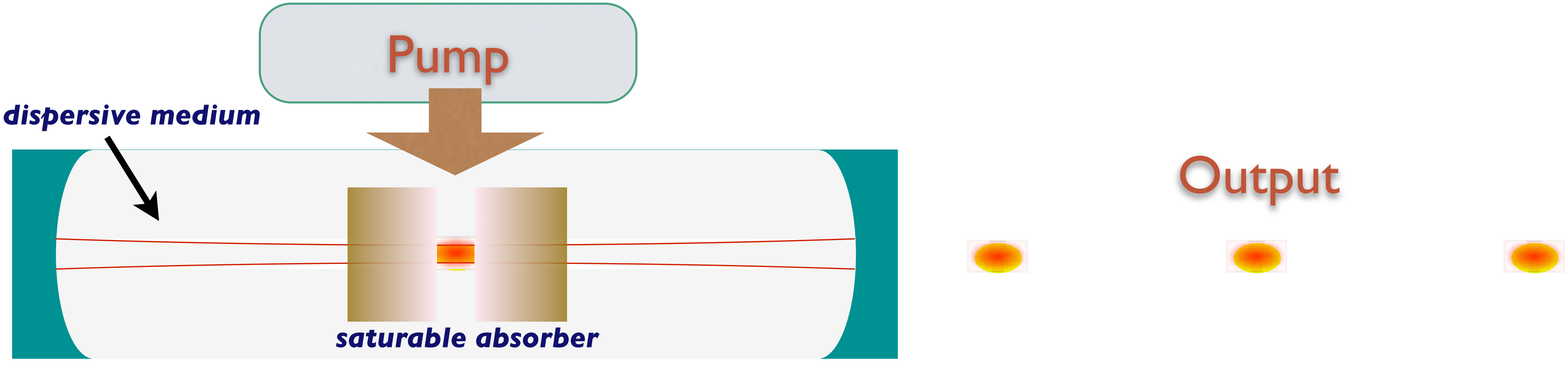
- Ultrashort light pulses <1ps ➡ high intensity, broad bandwidth

Mode-locked soliton lasers



- Ultrashort light pulses <1ps ➜ high intensity, broad bandwidth
- Dominant dispersive effects:
  - Chromatic dispersion ("anomalous") — *linear*
  - Kerr effect — *nonlinear*

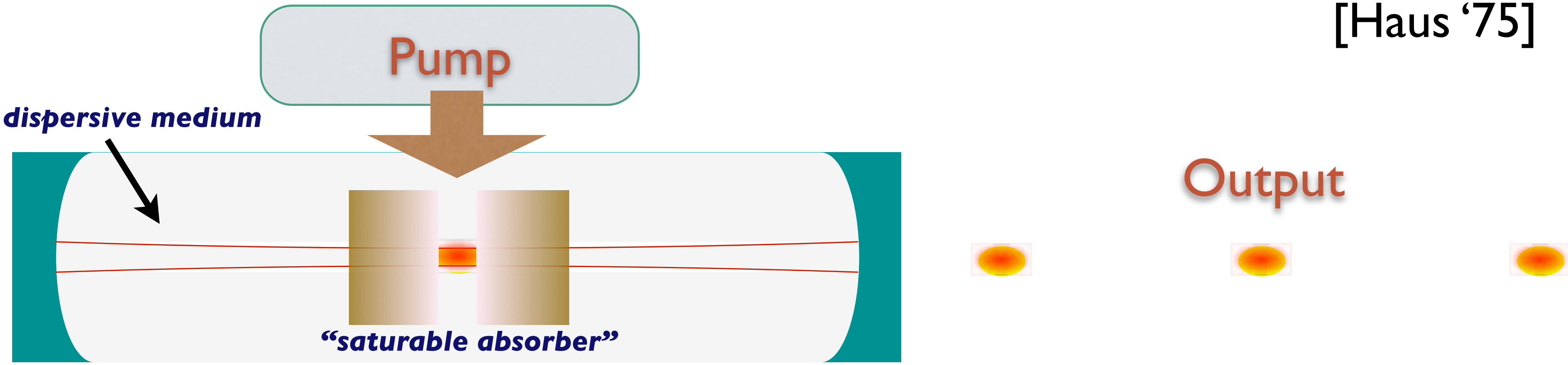
Mode-locked soliton lasers



- Pulse shaping effects:
  - Overall gain & gain filtering — *linear*
  - "Saturable absorption": Intensity-bleached absorbing element — *nonlinear*
- Relatively weak: proportional to $\mu \ll 1$

Mode-locked soliton lasers

[Haus '75]



- "Master" equation of motion for the field envelope ψ

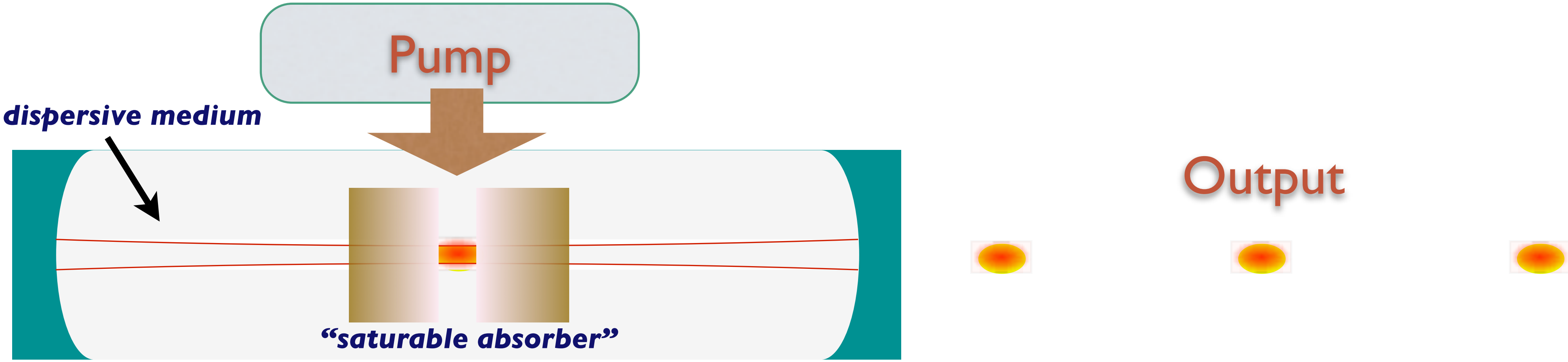
$$\boxed{\partial_z \psi = (i+\mu)\left(\frac{1}{2}\partial_t^2 \psi + |\psi|^2 \psi\right) + g\psi}$$

dispersive effects (*non-dimensionalized*)

pulse shaping

overall net gain (<0)

Mode-locked soliton lasers



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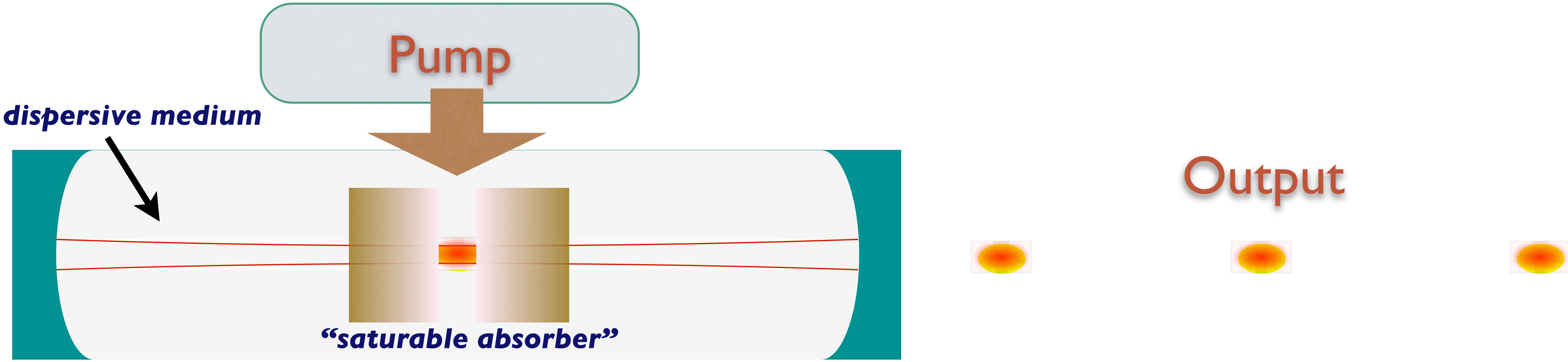
- Soliton-like pulse: $\boxed{\psi_s(t,z) = a\,\mathrm{sech}(\frac{t-C(z)}{\tau})e^{i\Phi(t,z)}}$

- Parameters: $a = \frac{1}{2}P \qquad C = c - \int V dz$

$$\tau = \frac{1}{a} \qquad \Phi = \phi + \frac{1}{4}Vt + \frac{1}{8}\int(P^2+V^2)dz$$

Mode-locked soliton lasers



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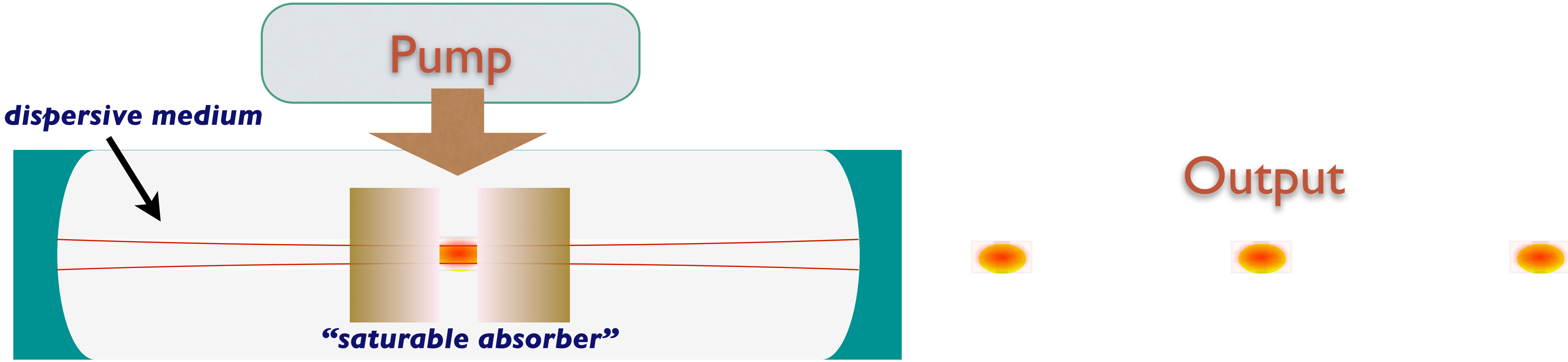
- Timing
- Phase
- Power
- Frequency

Mode-locked soliton lasers



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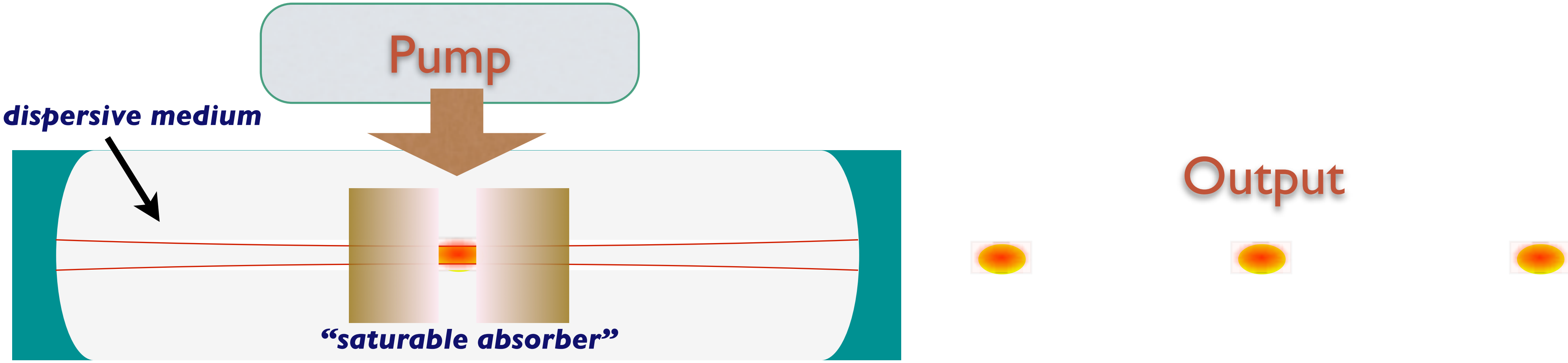

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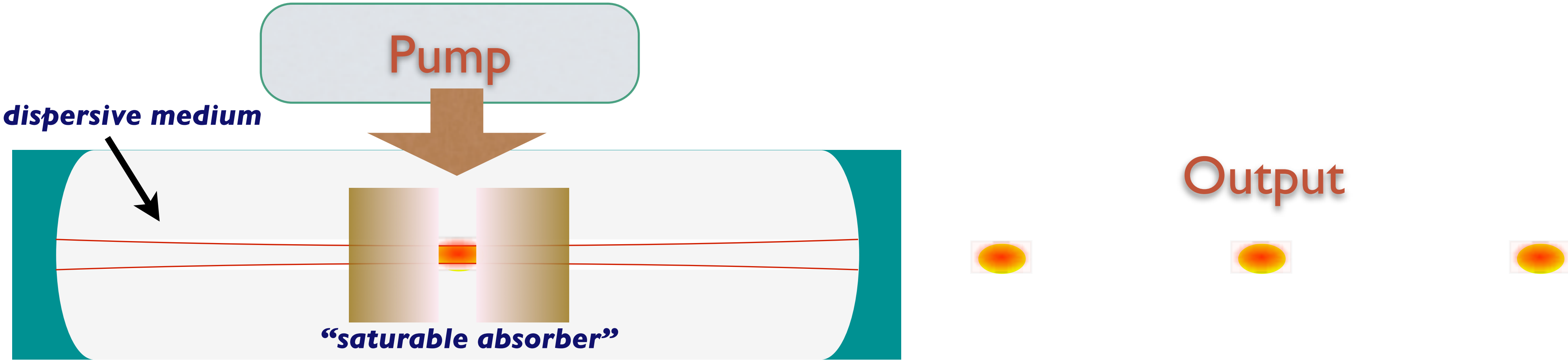

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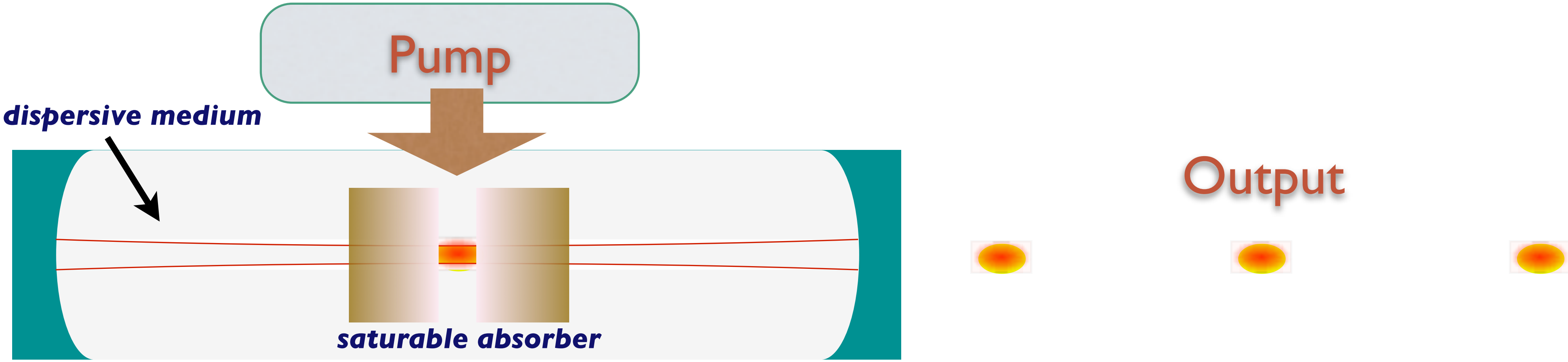

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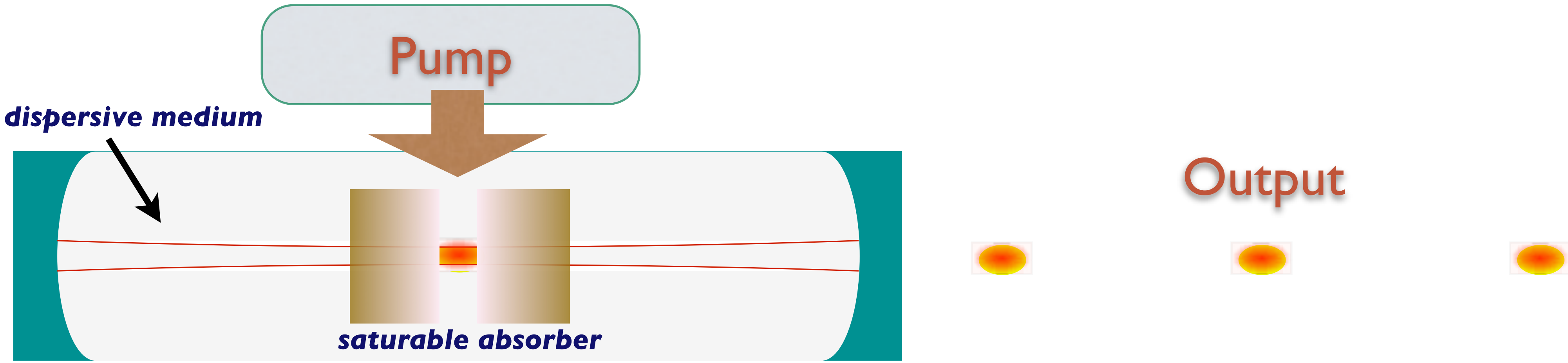
- Pulse parameters:
  - Power $\boxed{P = \sqrt{\frac{8|g|}{\mu}}}$ & frequency $\boxed{V = 0}$ fixed

Pulse shaping: Singular perturbation

  - Timing c and phase ϕ free — exact symmetries

Mode-locked soliton lasers



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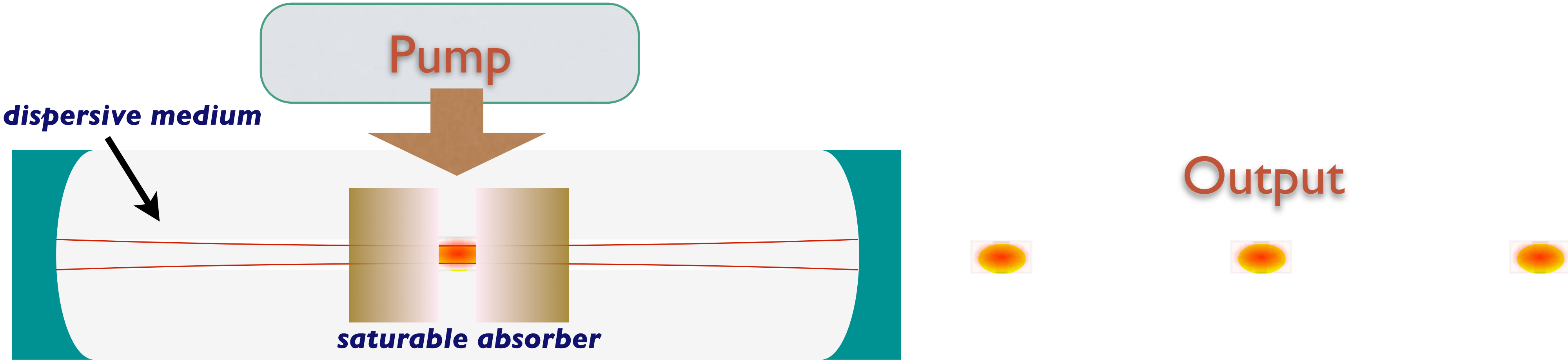

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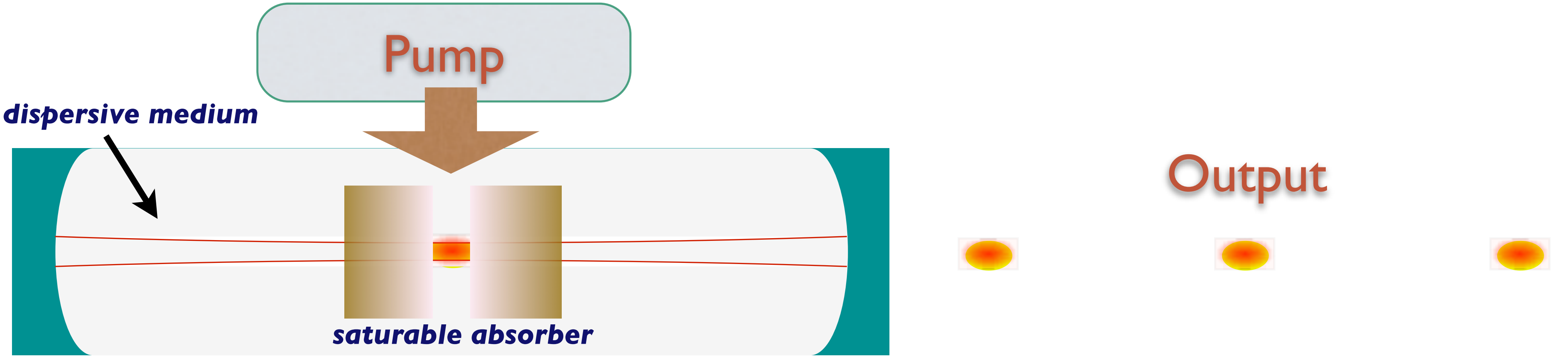

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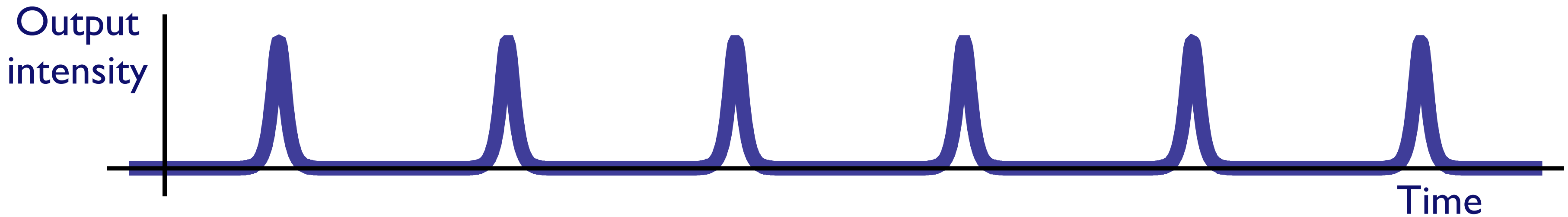
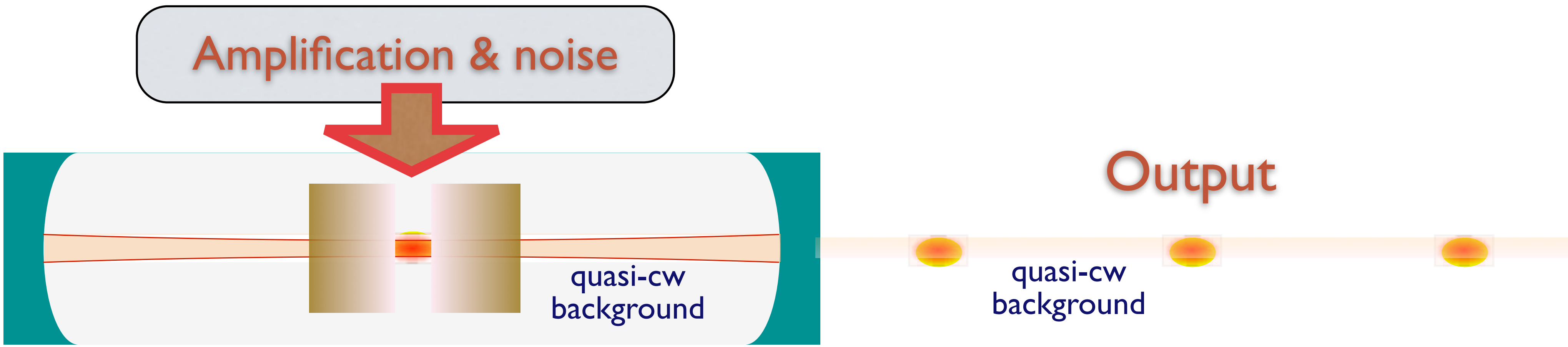
- Ideal output: Periodic pulse train



Noise and fluctuations



- Noisy master equation:

$$\partial_z \psi = (i+\mu)\left(\frac{1}{2}\partial_t^2 \psi + |\psi|^2 \psi\right) + g\psi \boxed{+\epsilon\Gamma(z,t)}$$

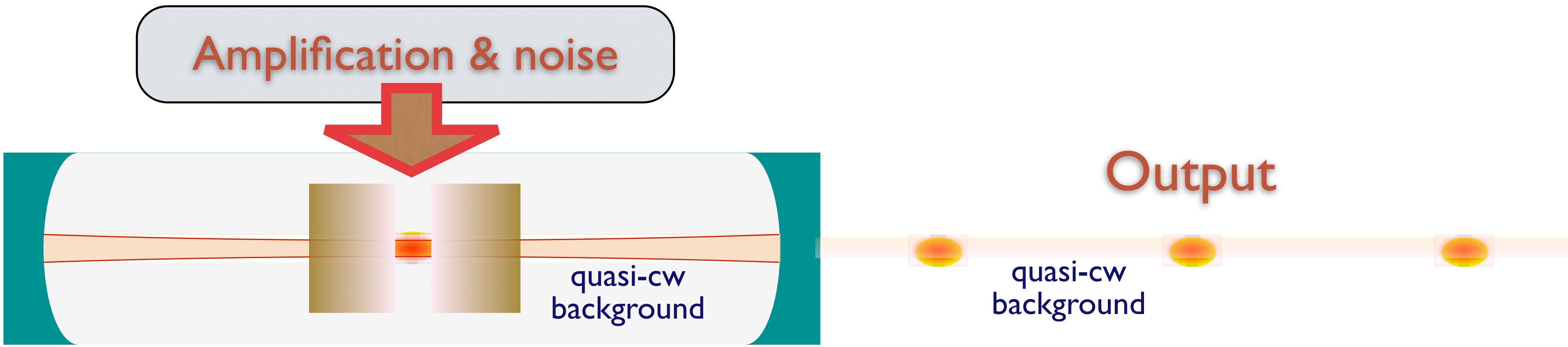
$\epsilon \ll 1$

- Weak Gaussian white noise

$$\langle \epsilon\Gamma(z_1,t_1)\epsilon\Gamma^*(z_2,t_2)\rangle = 2\boxed{\epsilon^2 T} L\delta(z_1-z_2)\delta(t_1-t_2)$$

Noise power
injection rate

Noise and fluctuations



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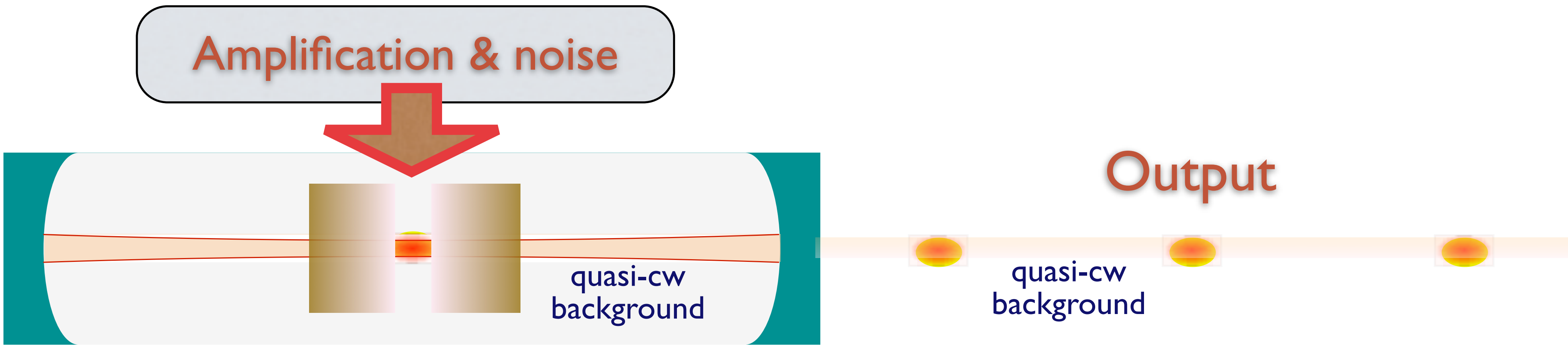
Noise power
injection rate

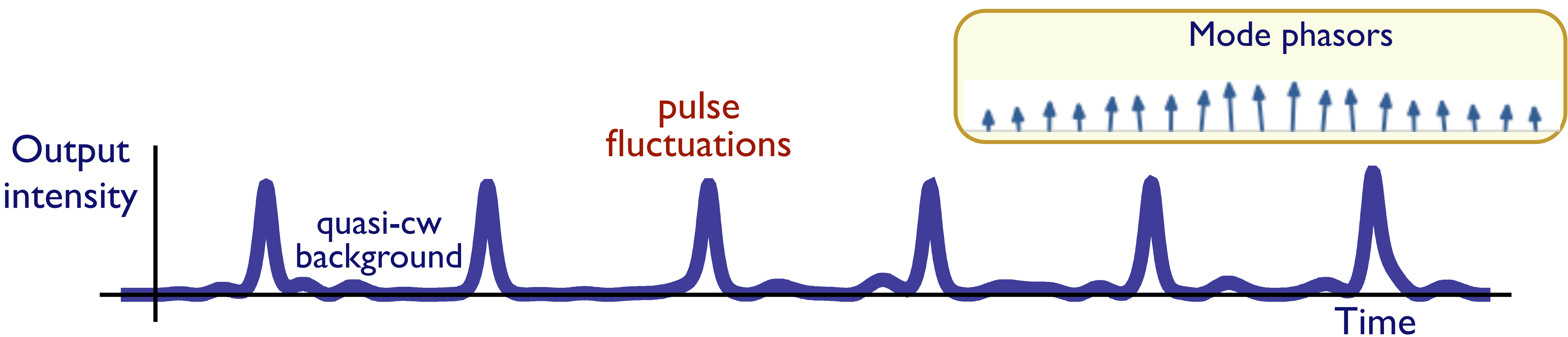
- Waveform is perturbed

$$\boxed{\psi = \psi_s(t,z) + O(\epsilon)}$$

Statistical light-mode dynamics



- Perturbed waveform: $\psi = \underbrace{\psi_s(t,z)}_{\text{Ordered pulse waveform}} + \underbrace{O(\epsilon)}_{\text{Noisy waveform}}$



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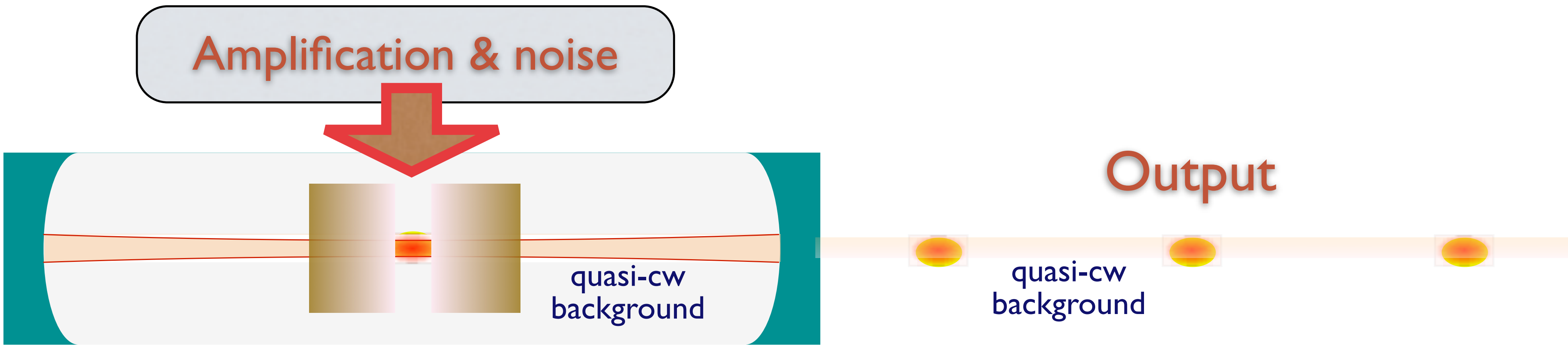

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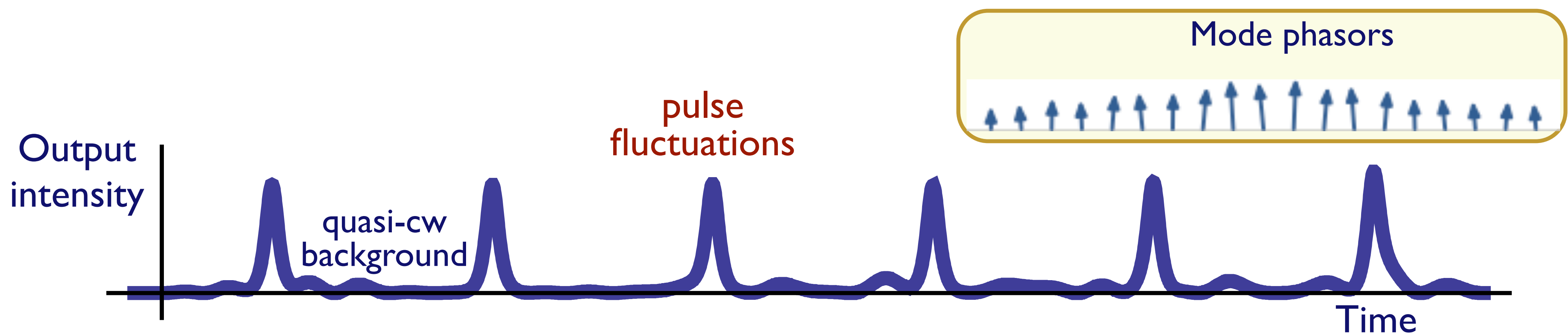
  Ordered pulse waveform — Noisy waveform

  1. Pulse fluctuations



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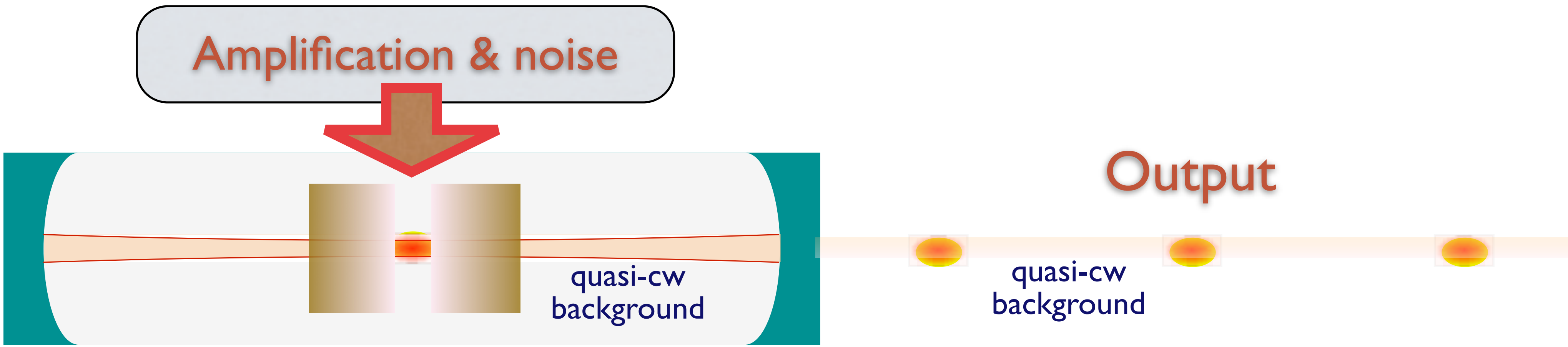

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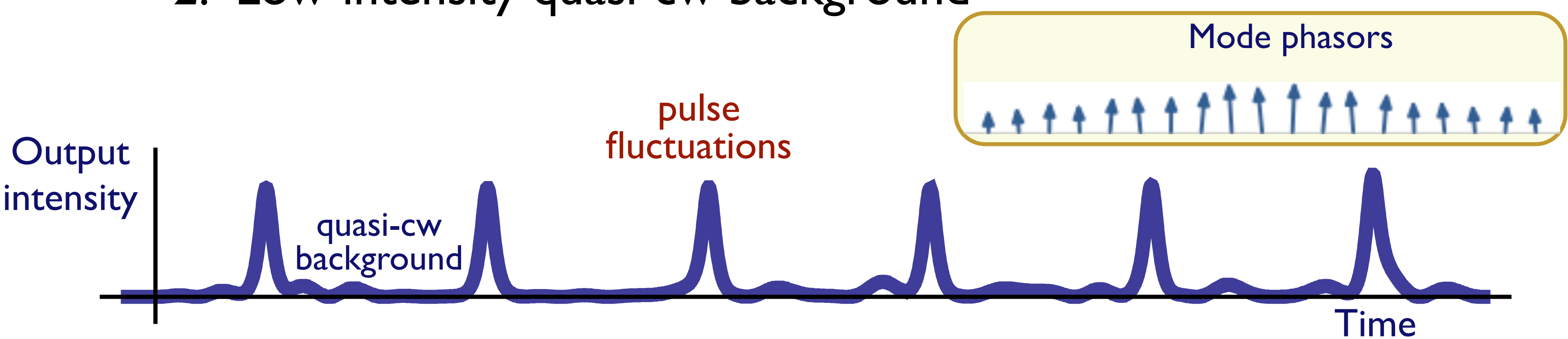
1. Pulse fluctuations
2. Low-intensity quasi-cw background



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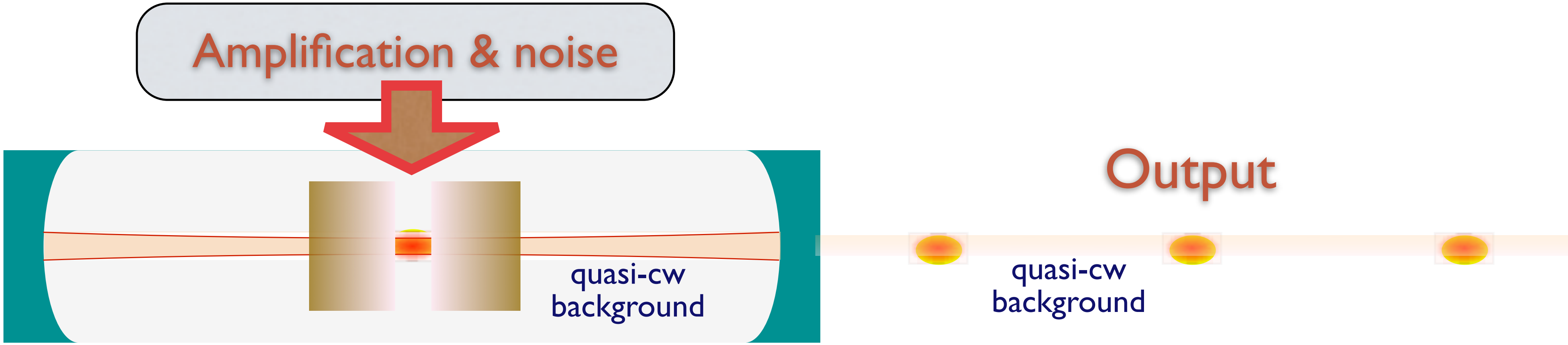
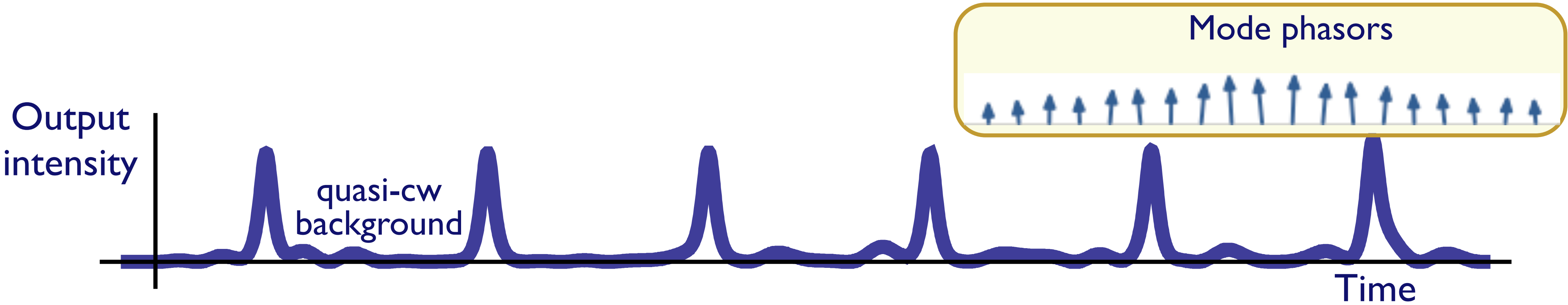


- Pulse waveform $\psi_s + \epsilon\psi_1$: Strong & narrow
- Continuum waveform $\epsilon\psi_c$: Weak & wide



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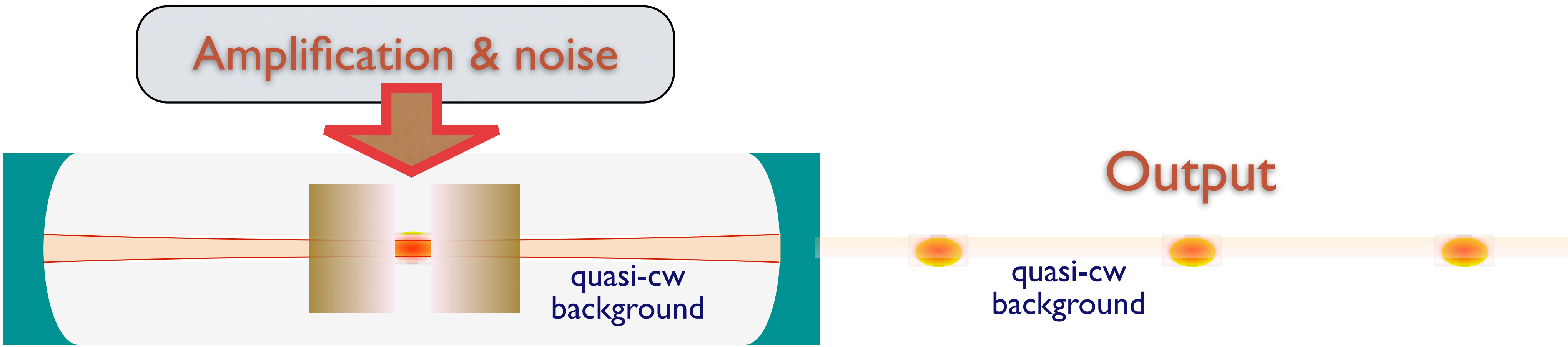

- Pulse waveform $\psi_s + \epsilon\psi_1$: Strong & narrow
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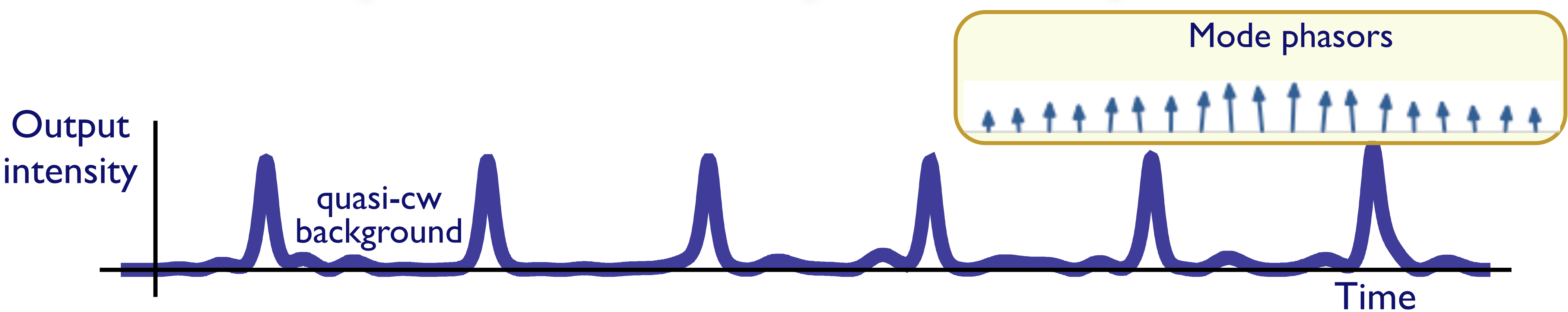
  - Pulse power ~ Continuum power ~ total power



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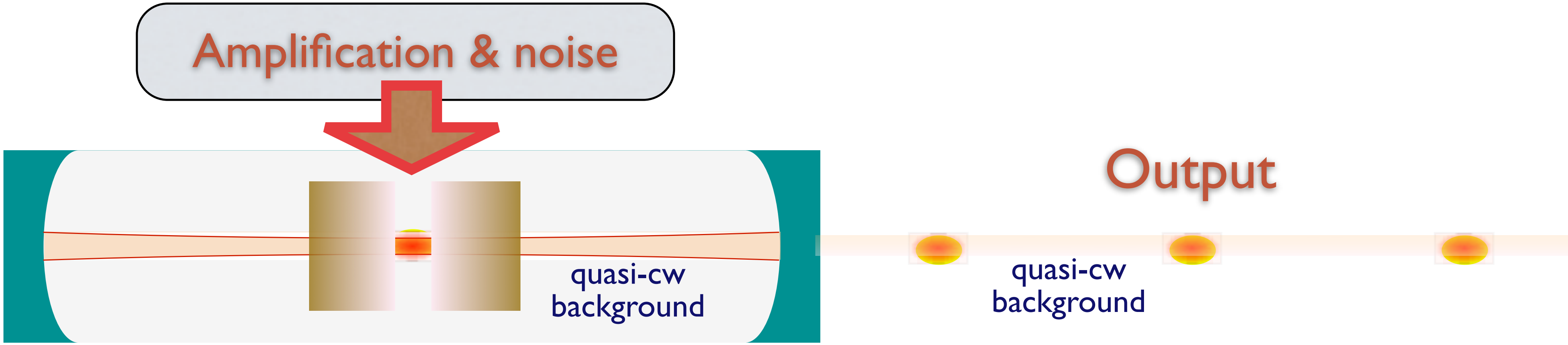

- ~~Pulse waveform $\psi_s + \epsilon\psi_1$: Strong & narrow~~
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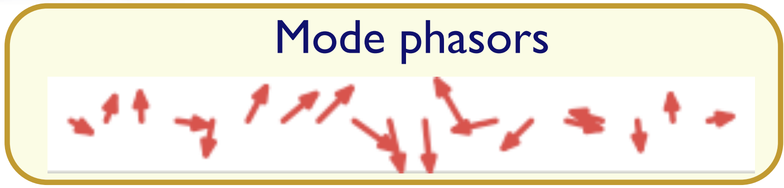
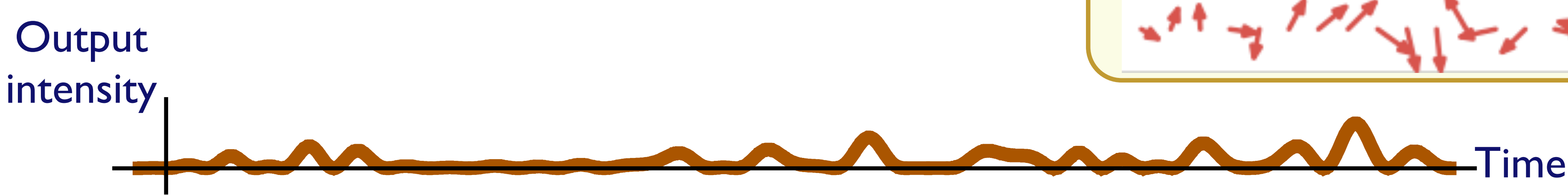
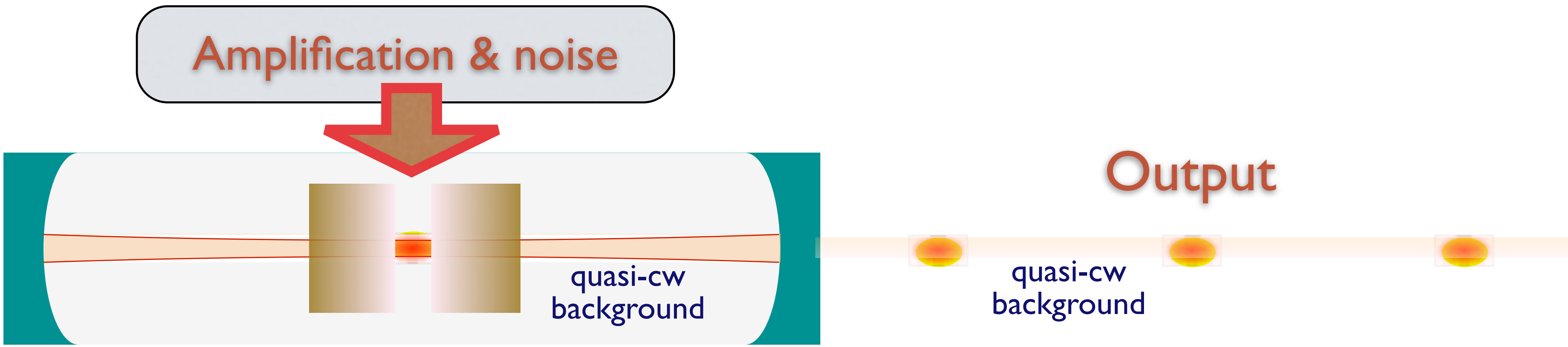
- ~~Pulse power~~ ~ Continuum power ~ total power
- For strong noise, disordering first order transition to cw phase





SLD: The gain balance method

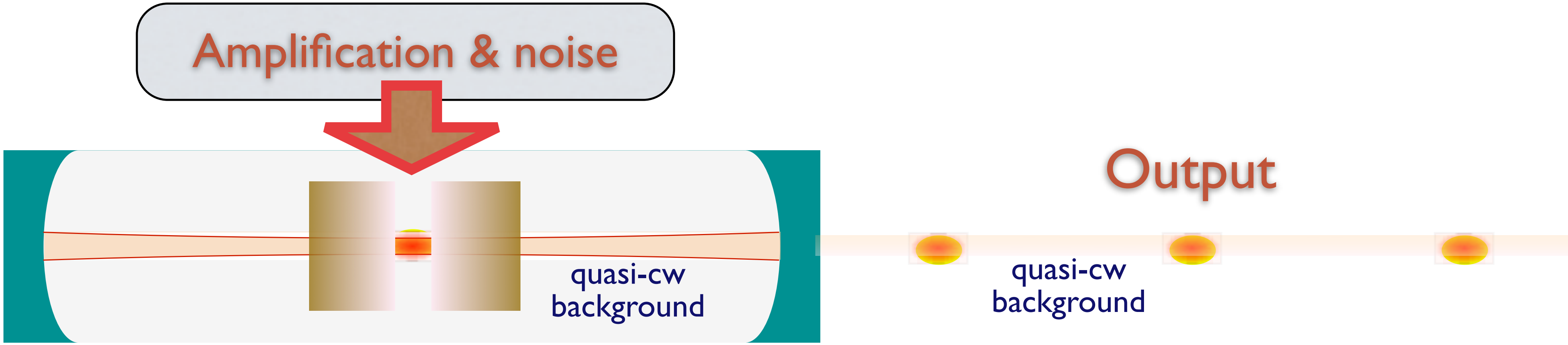


$$\psi_s \cancel{+ \epsilon\psi_1}$$

$$\epsilon\psi_c$$

P $\mathcal{P}$

SLD: The gain balance method



- Pulse waveform $\psi_s \cancel{+ \epsilon\psi_1}$: Strong & narrow → Neglect noise
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- Pulse power P + continuum power = total power $\mathcal{P}$

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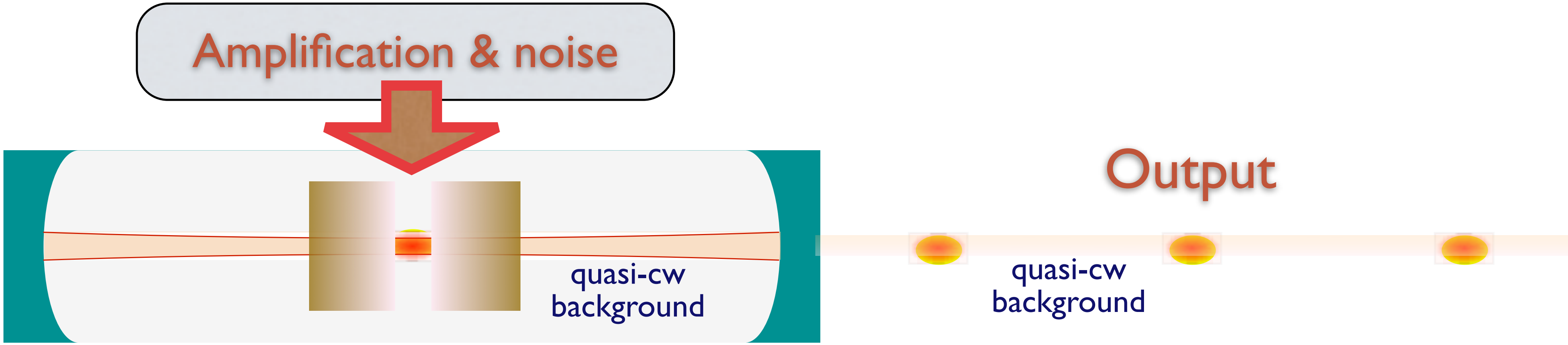

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Net gain g

- Common gain value determines the power distribution between pulse & continuum

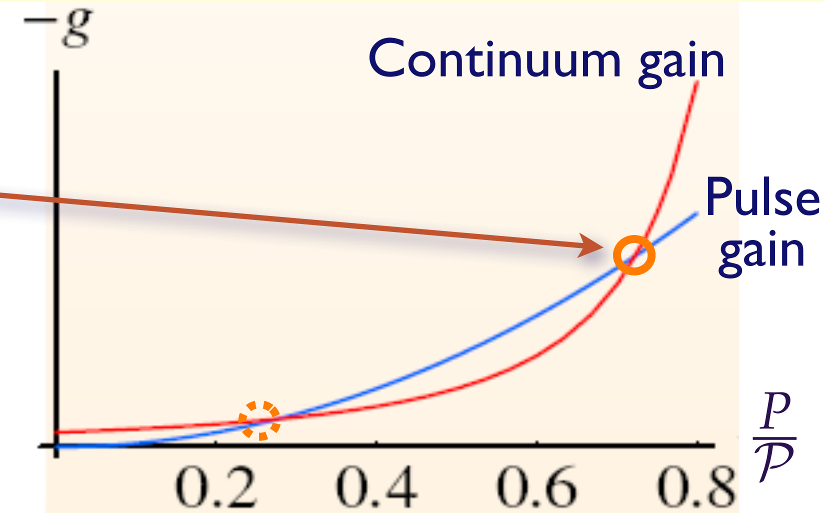
SLD: The gain balance method

- Steady state power distribution

$$\frac{P}{\mathcal{P}} = \frac{1}{2}\left(1 + \sqrt{1 - \frac{\epsilon^2 L^2}{\mu}\frac{8T}{\mathcal{P}^2}}\right)$$

- Thermodynamic limit $L \gg 1$

$$\rightarrow \frac{\epsilon}{\sqrt{\mu}} \ll 1$$



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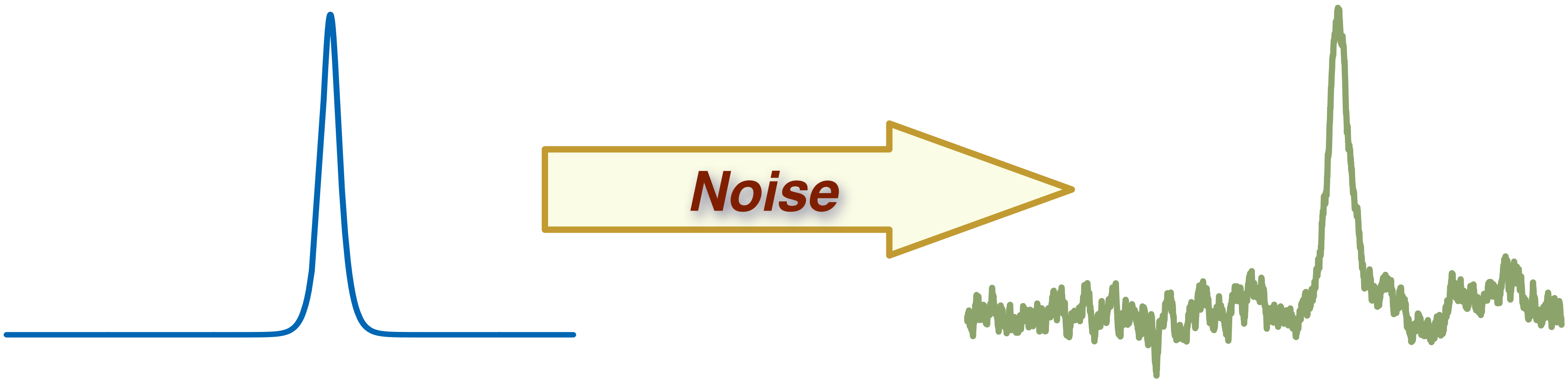
Net gain g

- Mode locking is possible *only* if a consistent power distribution between pulse & continuum exists

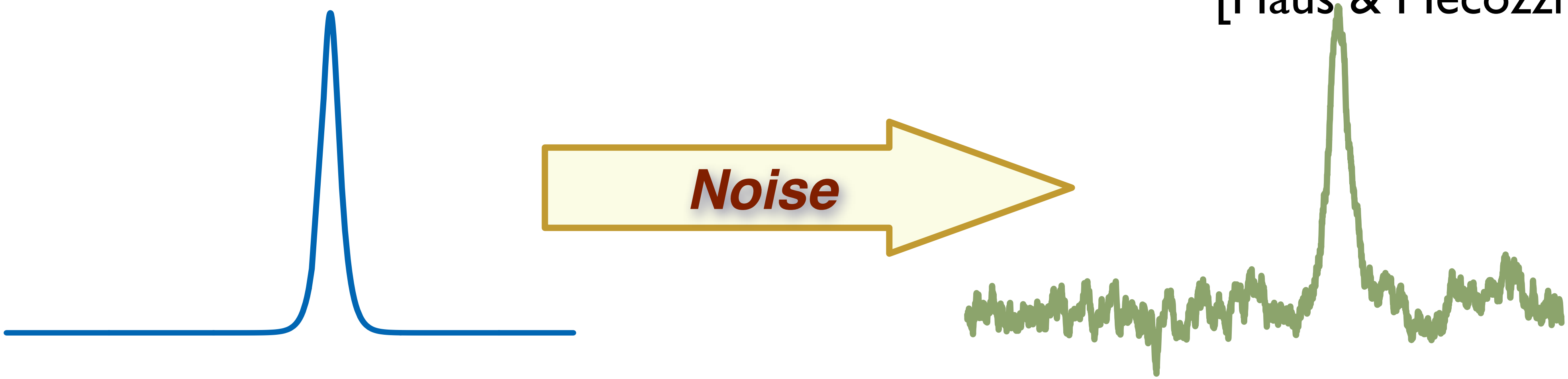
Pulse fluctuations



- Noise causes jitter in pulse parametrs
  - Power P & frequency V fluctuate
  - Timing c and phase ϕ diffuse

Pulse fluctuations

[Haus & Mecozzi '93]



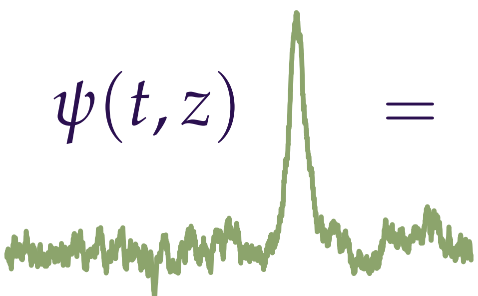
- Question: *What are the statistical properties of the fluctuations?*
- Practical implications:
  - Performance of pulse sources
  - Precision of frequency-comb metrology

SLD beyond thermodynamics

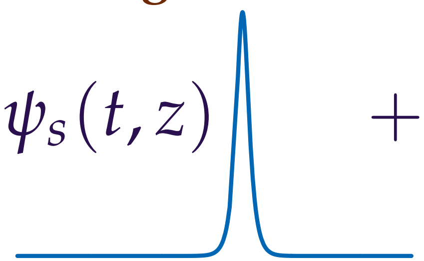
- Idea: Decompose wave form in 3 parts

strong & narrow — weak & *wide* — weak & narrow

$$\psi(t,z) = \psi_s(t,z) + \epsilon\psi_c(t,z) + \epsilon\psi_1(t,z)$$

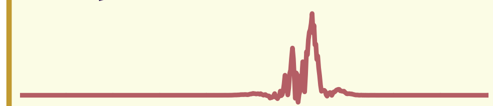


fluctuating waveform,
power = $\mathcal{P}$



soliton-like pulse with
fluctuating parameter
$c(z),\ \phi(z),\ P(z),\ V(z)$

linear continuum,
power ~ $\mathcal{P} - P(z)$



overlap waveform
power $\ll 1$

SLD beyond thermodynamics

- Idea: Decompose wave form in 3 parts

strong & narrow — weak & *wide* — weak & narrow

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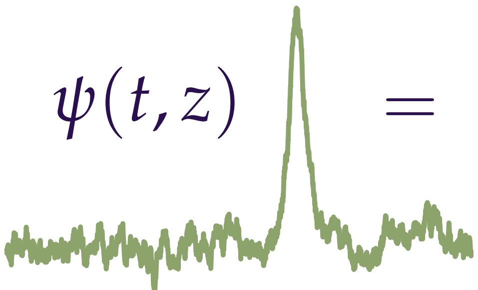
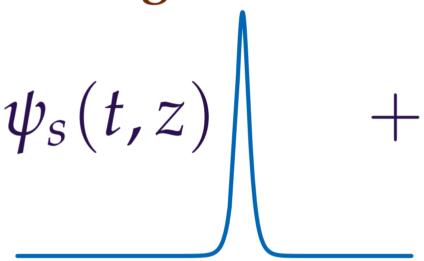


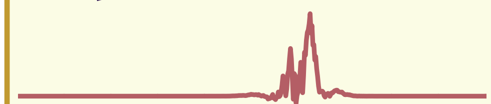
fluctuating waveform, power = $\mathcal{P}$



soliton-like pulse with fluctuating parameter

$c(z),\ \phi(z),\ P(z),\ V(z)$

linear continuum, power ~ $\mathcal{P} - P(z)$



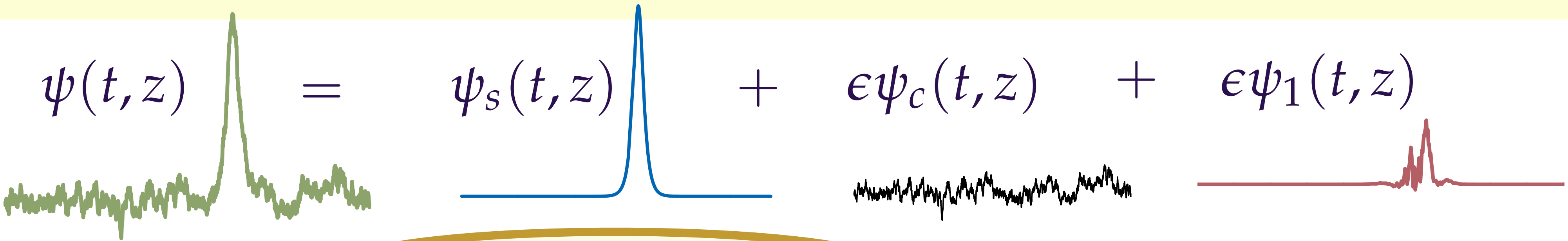
overlap waveform power $\ll 1$

- Negligible in steady-state
- Important for fluctuations

- Main goal: Calculate statistics of pulse paramaters

SLD beyond thermodynamics

$$\psi(t,z) = \psi_s(t,z) + \epsilon\psi_c(t,z) + \epsilon\psi_1(t,z)$$



$c(z),\ \phi(z),\ P(z),\ V(z)$

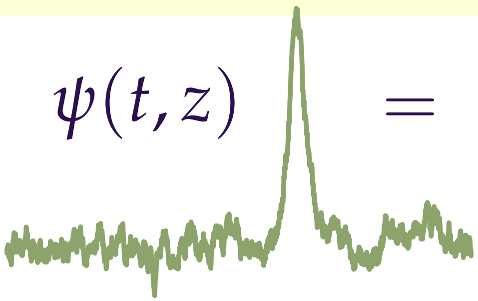
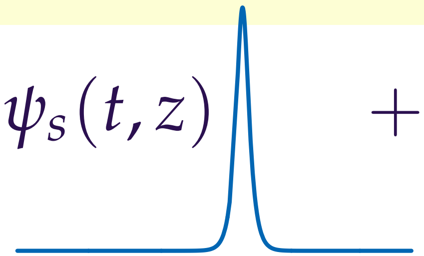
- New qualitative effects:

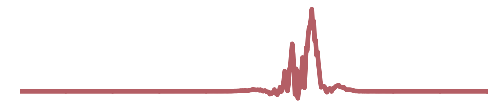
1. **Oscillations** in power & frequency correlation functions

2. **Enhancement** of phase diffusion rate

1. The perturbation equation

$$\psi(t,z) = \psi_s(t,z) + \epsilon\psi_c(t,z) + \epsilon\psi_1(t,z)$$



- Let $P = P_0 + \frac{\epsilon}{\sqrt{\mu}} p(z)$, $V = \frac{\epsilon}{\sqrt{\mu}} v(z)$, $c \longrightarrow \frac{\epsilon}{\sqrt{\mu}} c(z)$, $\phi \longrightarrow \frac{\epsilon}{\sqrt{\mu}} \phi(z)$

steady-state pulse power

fluctuations small parameter

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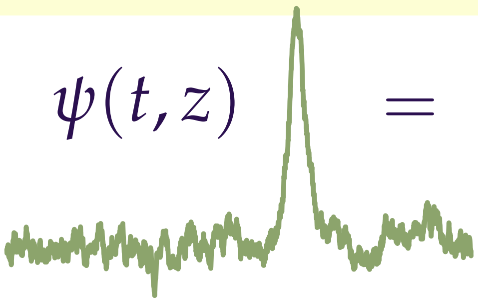
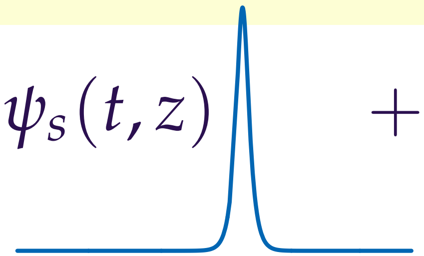

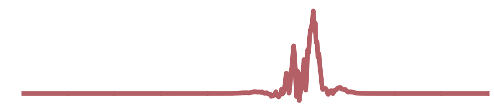

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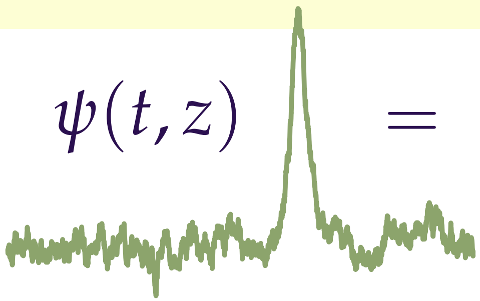
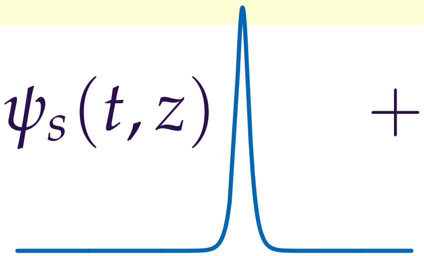
parameter set

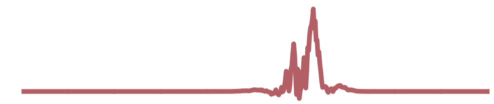
gain fluctuations

- Nonlinearity-generated forcing by overlap of ψ_s & ψ_c

2. Gain fluctuations

$$\psi(t,z) = \psi_s(t,z) + \epsilon\psi_c(t,z) + \epsilon\psi_1(t,z)$$



- Double role of net gain in the steady state:

1. Determines the pulse & continuum power $\boxed{g_0 = -\frac{\mu}{8}P_0}$

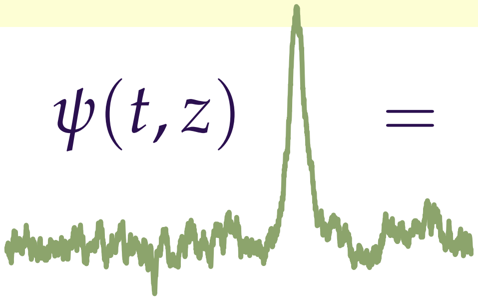
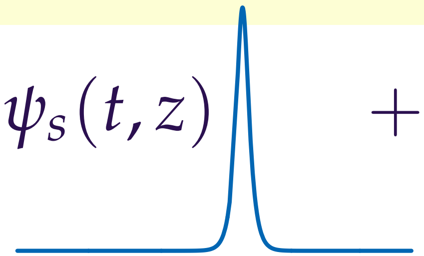
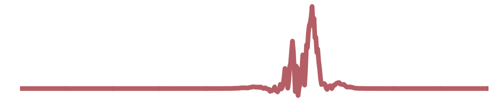
   Gain balance

2. Determined by the overall power $\boxed{g_0 = g(\mathcal{P})}$

   Gain saturation

2. Gain fluctuations

$$\psi(t,z) = \psi_s(t,z) + \epsilon\psi_c(t,z) + \epsilon\psi_1(t,z)$$

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1. Random shuffle of power between pulse and continuum

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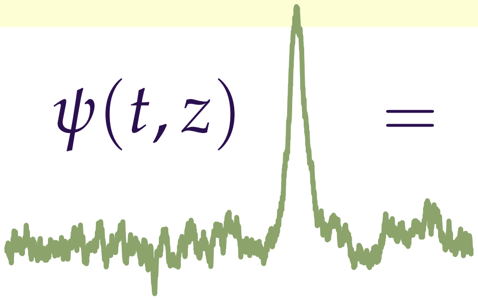
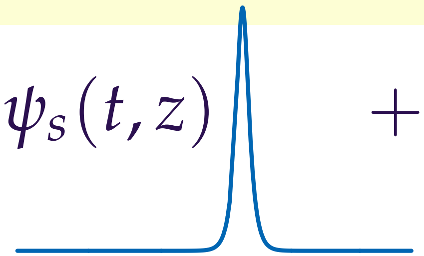

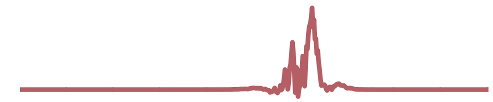

- Double role of net gain in the steady state:

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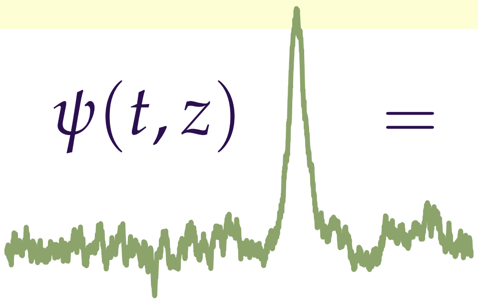 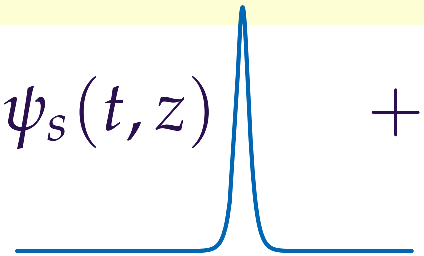  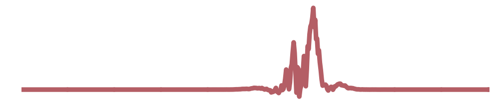

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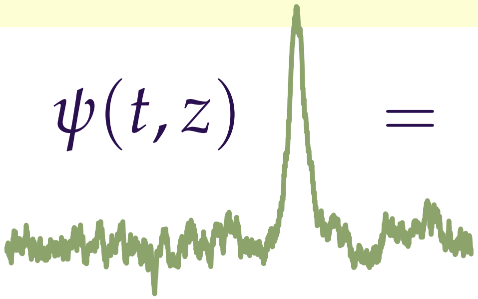
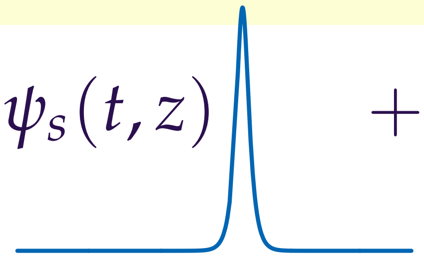
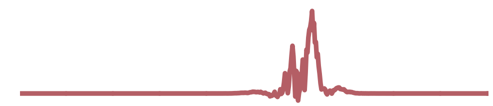
- Linearized master equation for ψ_1

$$\partial_z\psi_1 + \frac{1}{\sqrt{\mu}}\partial_z\vec{x}\cdot\nabla_x\psi_s = L\psi_1 - i\tfrac{1}{2}\sqrt{\mu}v(z)\partial_t\psi_s + g_1\psi_s + f$$

2. Gain fluctuations

$$\psi(t,z) = \psi_s(t,z) + \epsilon\psi_c(t,z) + \epsilon\psi_1(t,z)$$



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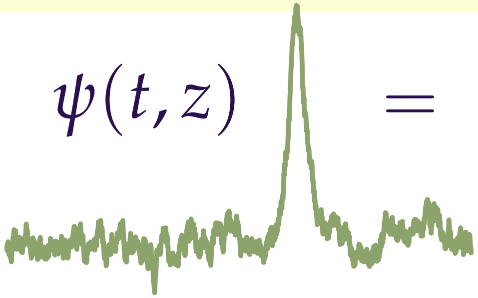
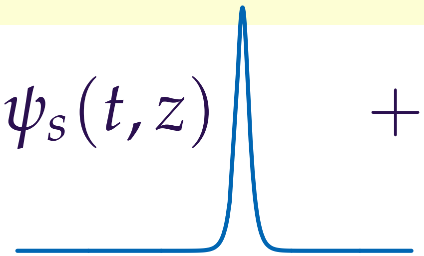
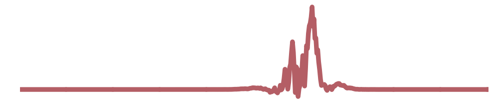
- Fluctuation conserve overall power

$$\boxed{g_1\mathcal{P} = -\frac{\sqrt{\mu}P_0^2}{4}p - \int dz\,\psi_s^*\left(\Gamma + L(\psi_c + \psi_1)\right)}$$

3. The slow modes

$$\psi(t,z) = \psi_s(t,z) + \epsilon\psi_c(t,z) + \epsilon\psi_1(t,z)$$



- Perturbation equation including gain fluctuations:

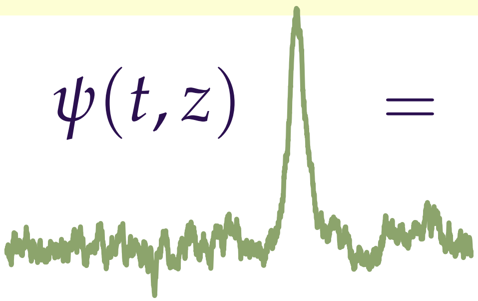
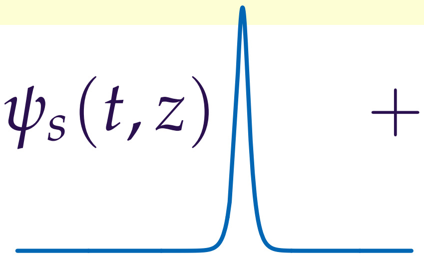
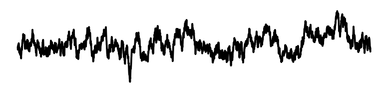
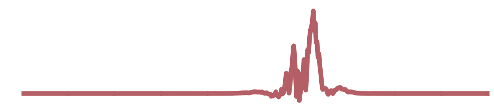
$$\partial_z\psi_1 + \frac{1}{\sqrt{\mu}}\partial_z\vec{x}\cdot\nabla_x\psi_s = \tilde{L}\psi_1 - i\frac{1}{2}\sqrt{\mu}v(z)\partial_t\psi_s + \tilde{f}$$

Linear operator including
rank-1 gain fluctuations term

Forcing including
gain fluctuations

3. The slow modes

$$\psi(t,z) = \psi_s(t,z) + \epsilon\psi_c(t,z) + \epsilon\psi_1(t,z)$$

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$$\partial_z \psi_1 + \frac{1}{\sqrt{\mu}}\partial_z \vec{x} \cdot \nabla_x \psi_s = \tilde{L}\psi_1 - i\frac{1}{2}\sqrt{\mu}v(z)\partial_t\psi_s + \tilde{f}$$

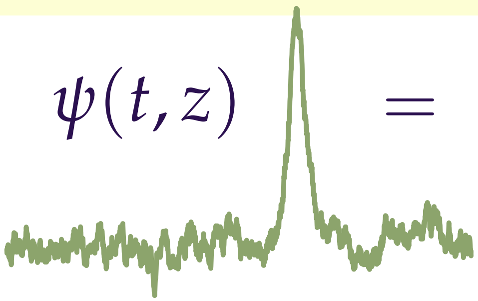
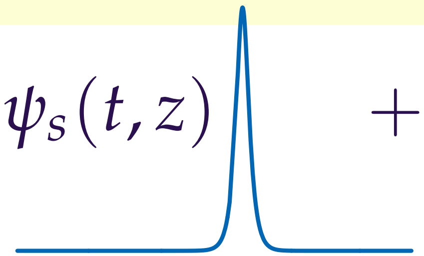
Linear operator including rank-1 gain fluctuations term

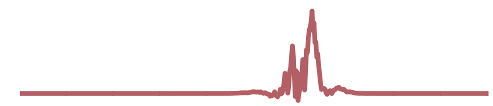
Forcing including gain fluctuations

- Remaining arbitrariness: A slight shift in $\vec{x}$ can be absorbed in ψ_1, $\psi_s(\vec{x}) + \epsilon\psi_1 = \psi_s(\vec{x} + \delta\vec{x}) + \epsilon(\psi_1 + \delta\psi_1)$

3. The slow modes

$$\psi(t,z) \quad = \quad \psi_s(t,z) \quad + \quad \epsilon\psi_c(t,z) \quad + \quad \epsilon\psi_1(t,z)$$



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Linear operator including rank-1 gain fluctuations term

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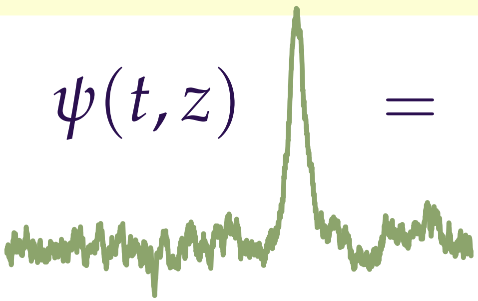
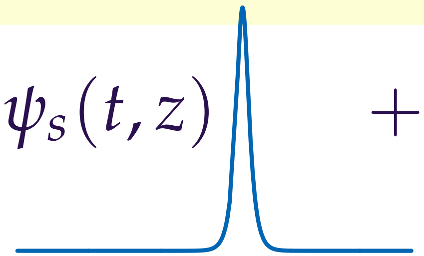
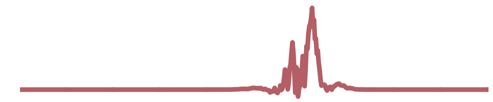
- *Q*: How to define the pulse parameters?
- *A*: Let ψ_1 lie outside 4-dimensional slow eigen-space of $\tilde{L}$

3. The slow modes

$$\psi(t,z) \quad = \quad \psi_s(t,z) \quad + \quad \epsilon\psi_c(t,z) \quad + \quad \epsilon\psi_1(t,z)$$



- Perturbation equation including gain fluctuations:

$$\partial_z\psi_1 + \frac{1}{\sqrt{\mu}}\partial_z\vec{x}\cdot\nabla_x\psi_s = \tilde{L}\psi_1 - i\tfrac{1}{2}\sqrt{\mu}v(z)\partial_t\psi_s + \tilde{f}$$

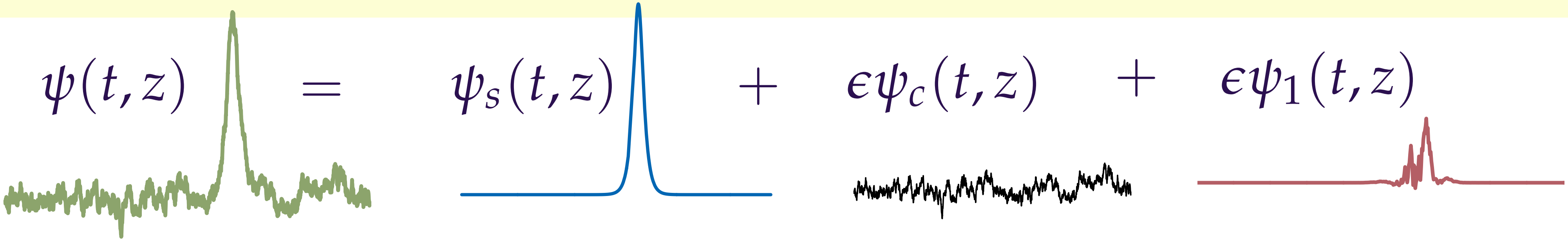
- *Recall:* linearized NLS has 4-d zero eigen-space:

timing, phase:
annihilated by L_{NLS}

frequency, power:
annihilated by L^2_{NLS}

3. The slow modes

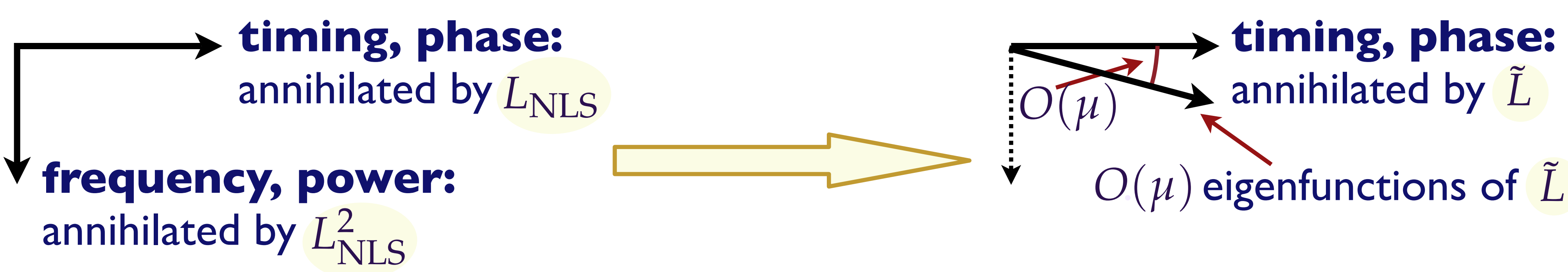
$$\psi(t,z) = \psi_s(t,z) + \epsilon\psi_c(t,z) + \epsilon\psi_1(t,z)$$



- Perturbation equation including gain fluctuations:

$$\partial_z \psi_1 + \frac{1}{\sqrt{\mu}} \partial_z \vec{x} \cdot \nabla_x \psi_s = \tilde{L}\psi_1 - i\tfrac{1}{2}\sqrt{\mu}\, v(z)\partial_t \psi_s + \tilde{f}$$

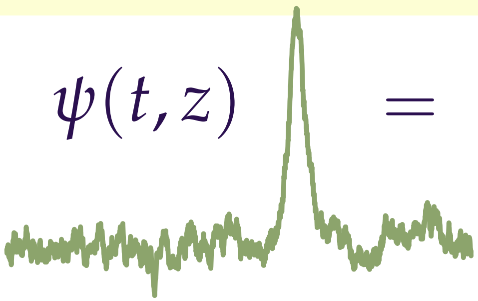
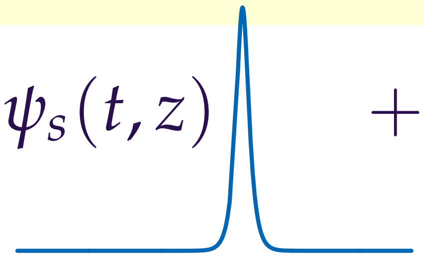
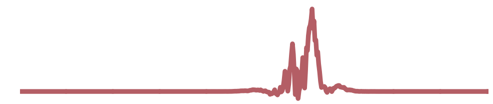
- *Recall:* linearized NLS has 4-d zero eigen-space:



- Degeneracy is half-lifted by $\tilde{L} = L_{\mathrm{NLS}} + O(\mu)$

4. Pulse parameter dynamics

$$\psi(t,z) = \psi_s(t,z) + \epsilon\psi_c(t,z) + \epsilon\psi_1(t,z)$$

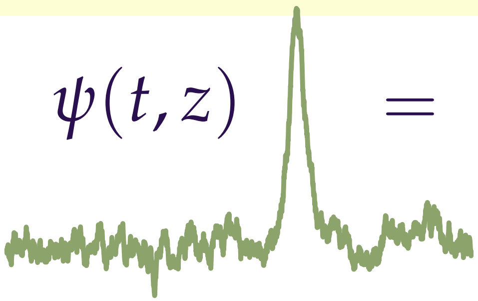
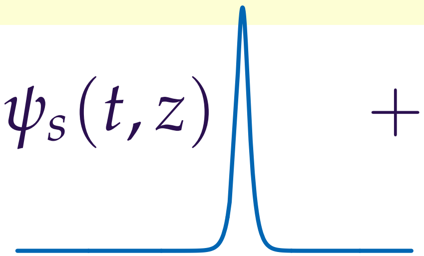
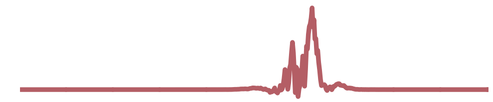
- Perturbation equation:

$$\partial_z\psi_1 + \frac{1}{\sqrt{\mu}}\partial_z\vec{x}\cdot\nabla_x\psi_s = \tilde{L}\psi_1 - i\tfrac{1}{2}\sqrt{\mu}v(z)\partial_t\psi_s + \tilde{f}$$

- *where:* ψ_1 lies outside 4-dimensional slow eigen-space of $\tilde{L}$

$\langle q_n, \psi_1\rangle = 0$ where $q_1,\ldots,q_4$ are slow *left* $\tilde{L}$ eigenfunctions

4. Pulse parameter dynamics

$\psi(t,z)$  $=$ $\psi_s(t,z)$  $+$ $\epsilon\psi_c(t,z)$  $+$ $\epsilon\psi_1(t,z)$ 

- Perturbation equation:

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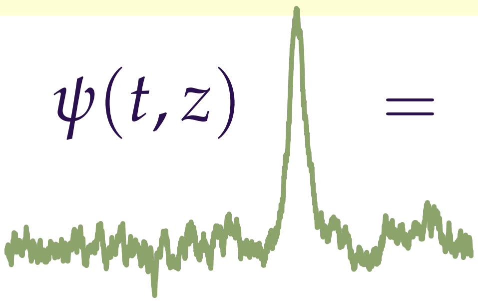
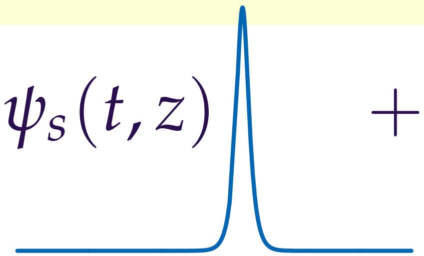
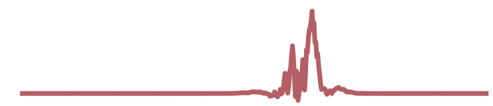
$\langle q_n, \psi_1\rangle = 0$ where $q_1, \ldots, q_4$ are slow *left* $\tilde{L}$ eigenfunctions

- $\langle q_n, \cdot\rangle$ projection of the perturbation equation yields (linear combination of) parameter equations of motion

4. Pulse parameter dynamics

$$\psi(t,z) \quad = \quad \psi_s(t,z) \quad + \quad \epsilon\psi_c(t,z) \quad + \quad \epsilon\psi_1(t,z)$$

- $\langle q_n, \cdot \rangle$ projection of the perturbation equation yields (linear combination of) parameter equations of motion:

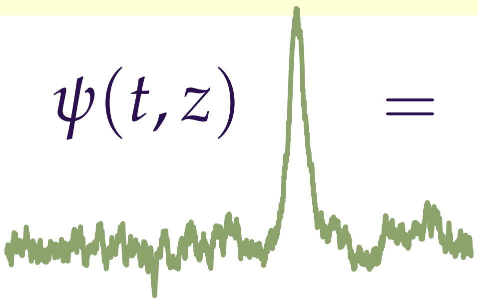
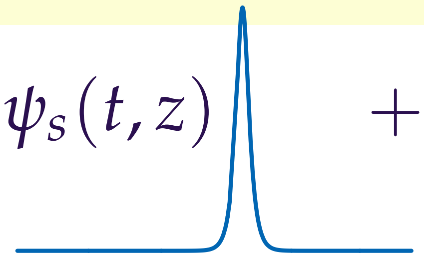
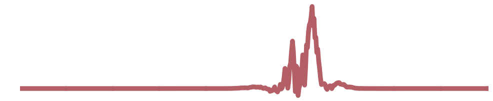
Power:	$\partial_z p = -\mu \frac{P_0^3}{4\mathcal{P}} p - \sqrt{\mu} f_p$
Phase:	$\partial_z \phi = \sqrt{\mu} f_\phi$
Frequency:	$\partial_z v = -\mu \frac{P_0^2}{6} v - \sqrt{\mu} f_v$
Timing:	$\partial_z c = \sqrt{\mu} f_c$

Restoring terms

Random forcing

4. Pulse parameter dynamics

$$\psi(t,z) \quad = \quad \psi_s(t,z) \quad + \quad \epsilon\psi_c(t,z) \quad + \quad \epsilon\psi_1(t,z)$$

- $\langle q_n, \cdot \rangle$ projection of the perturbation equation yields (linear combination of) parameter equations of motion:

$$\begin{aligned}
&\text{Power:} && \partial_z p = -\mu \frac{P_0^3}{4\mathcal{P}} p - \sqrt{\mu} f_p \\
&\text{Phase:} && \partial_z \phi = \sqrt{\mu} f_\phi \\
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\end{aligned}$$

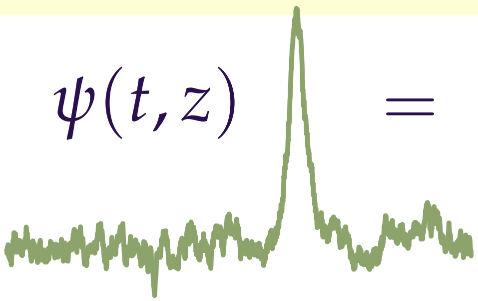
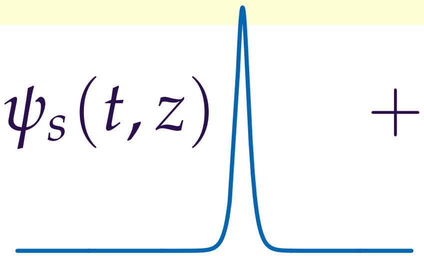
Bounded fluctuations

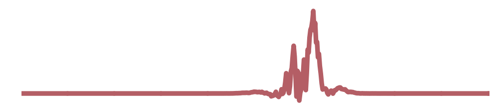
Restoring terms

Random forcing

4. Pulse parameter dynamics

$$\psi(t,z) \quad = \quad \psi_s(t,z) \quad + \quad \epsilon\psi_c(t,z) \quad + \quad \epsilon\psi_1(t,z)$$



- $\langle q_n, \cdot \rangle$ projection of the perturbation equation yields (linear combination of) parameter equations of motion:

Power:	$\partial_z p = -\mu \frac{P_0^3}{4\mathcal{P}} p - \sqrt{\mu} f_p$
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Bounded fluctuations

Diffusion

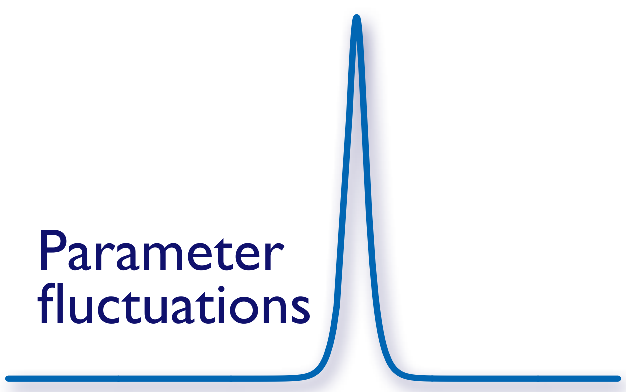
Restoring terms

Random forcing

Results: Autocorrelation

- Pulse frequency: $$\langle v_{t+\tau} v_t^* \rangle = \frac{T}{P_0}\left(e^{-\frac{\mu P_0^2}{6}|\tau|} + \pi \int dk k^2 I_k(\tau)\right)$$

Direct term: exponential damping



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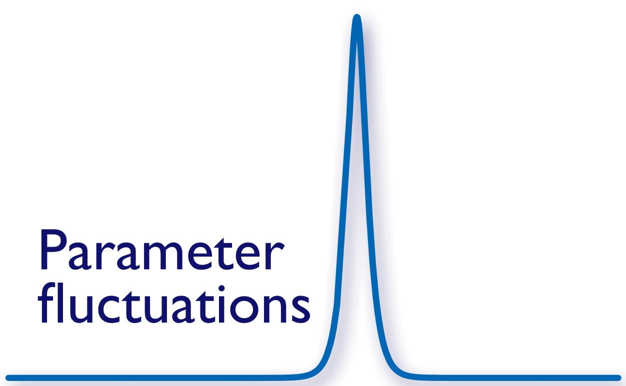

$$\boxed{I_k(\tau) = \frac{\mathrm{sech}^2(\pi k/2)}{k^2+1} e^{-\frac{\mu P_0^2(k^2+1)}{8}|\tau|} \cos\left(\frac{P_0^2(k^2+1)}{8}\tau\right)}$$

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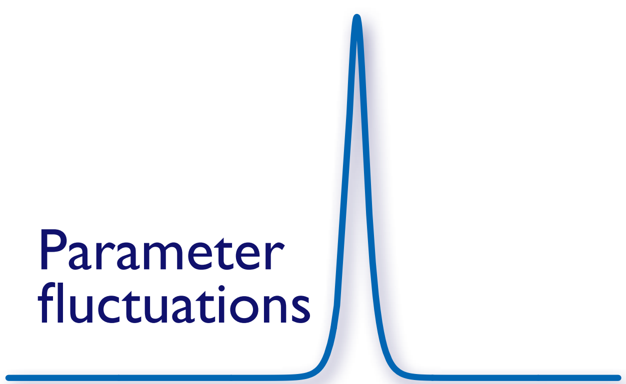

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Continuum enhancement factor

Results: Autocorrelation

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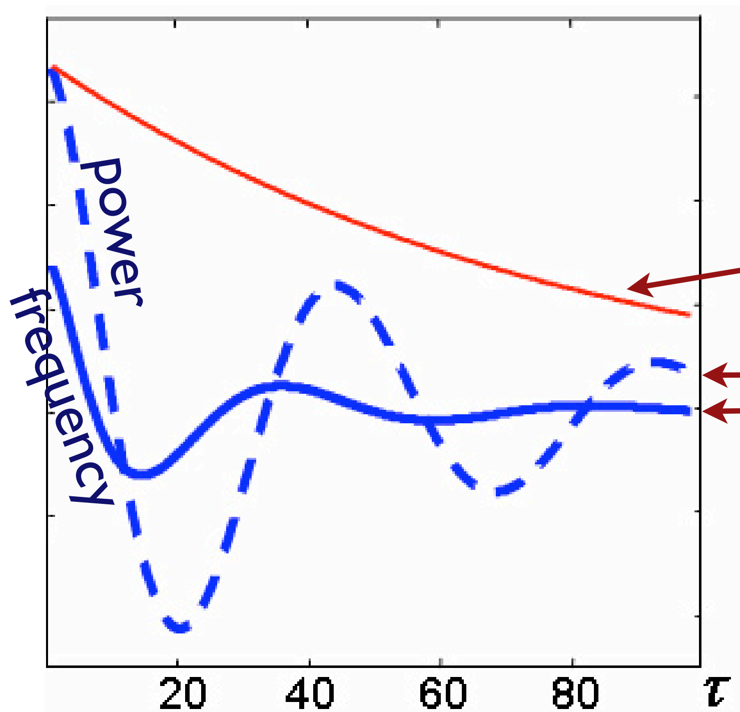

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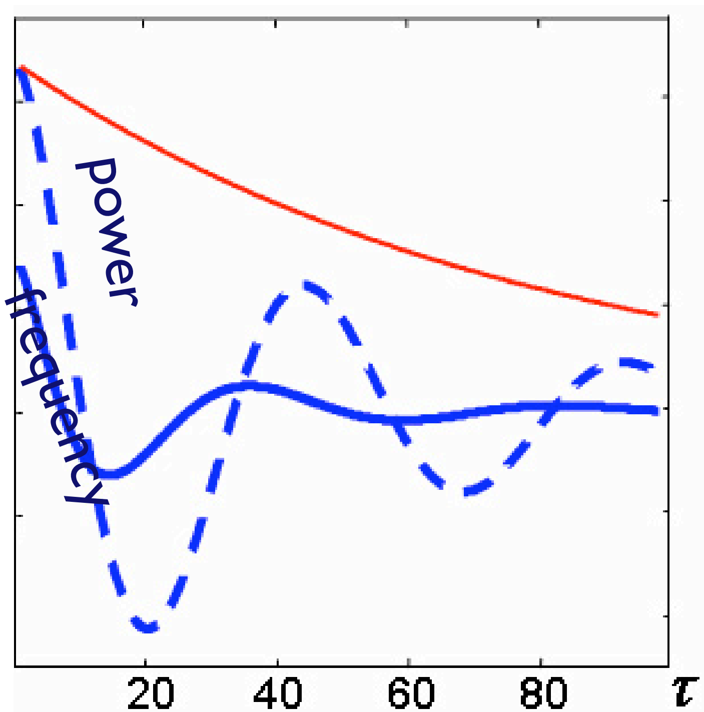
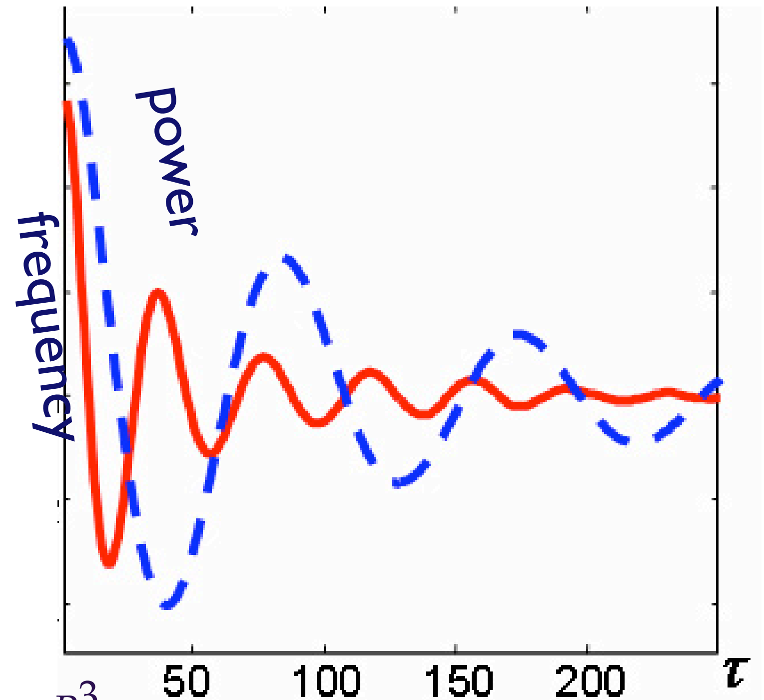


Theory

Numerics

Alternative parameter definition



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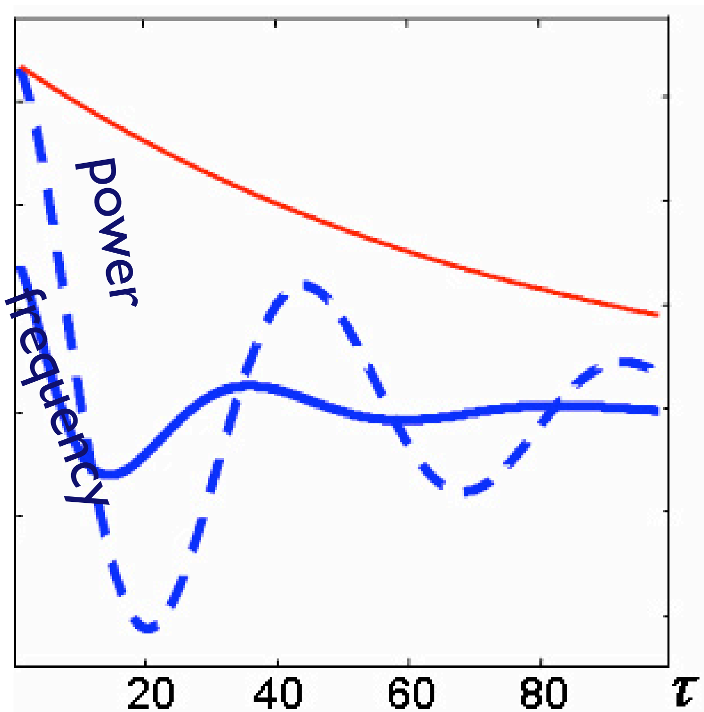

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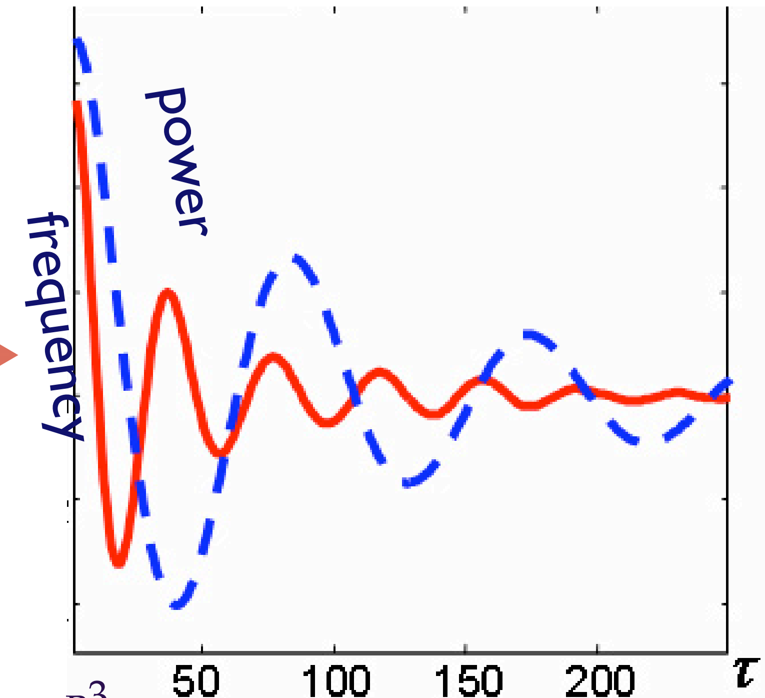

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Nonlinear phase shift

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Pulse shaping rate

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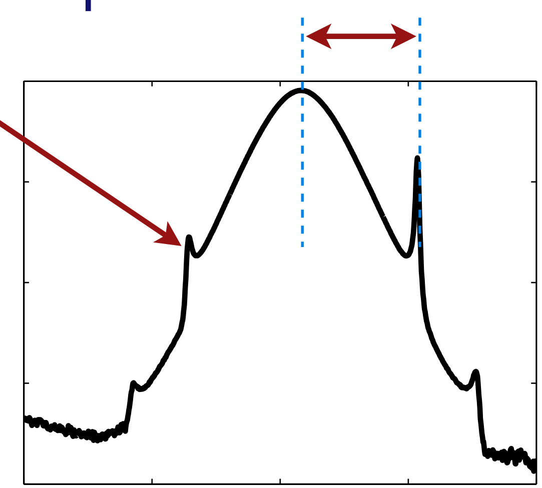
- Compare with *pulse-generated continuum*: Kelly sidebands



Results: Diffusion

Parameter equations

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$$C = c - \int V dz$$

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Frequency- & power-fluctuations driven diffusion

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Frequency- & power-
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- Phase jitter $\langle \Phi(z)^2 \rangle = \boxed{\frac{T\mathcal{P}^2}{P_0^3}} z$ enhanced by $\left(\frac{\mathcal{P}}{P_0} \right)^2$

Diffusion constants

Summary & conclusions

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    1. Steady-state linear continuum
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- Restoring terms suppressed by continuum: *enhanced* phase jitter

Outlook

- Laser noise has ***triple*** effect
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Thank you!