

Mode-locked laser pulse fluctuations

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*Acknowledgement: Rafi Weill, Oded Basis, Alex Bekker,
Vladimir Smulakovsky*

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Synopsis

- Mode-locked soliton lasers and noise
- The statistical steady state
- Fluctuations in the steady state
 1. Pulse-continuum interactions
 2. Gain fluctuations
 3. Slow modes of pulse dynamics
 4. Pulse parameter equations of motion
- Autocorrelation and diffusion of pulse parameters

Mode-locked soliton lasers

dispersive medium

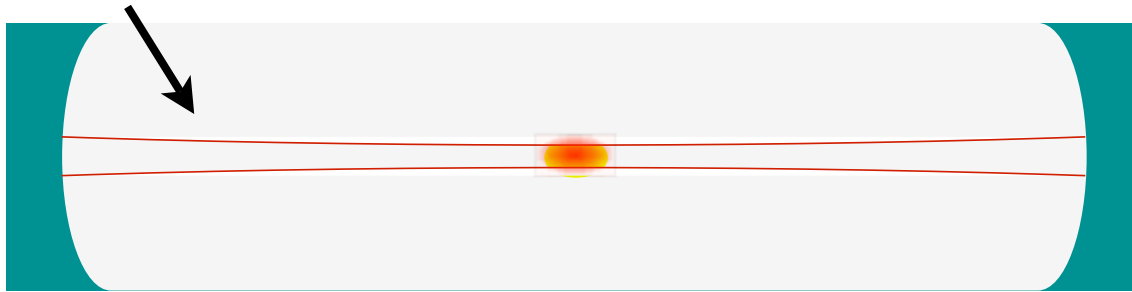


Output



Mode-locked soliton lasers

dispersive medium



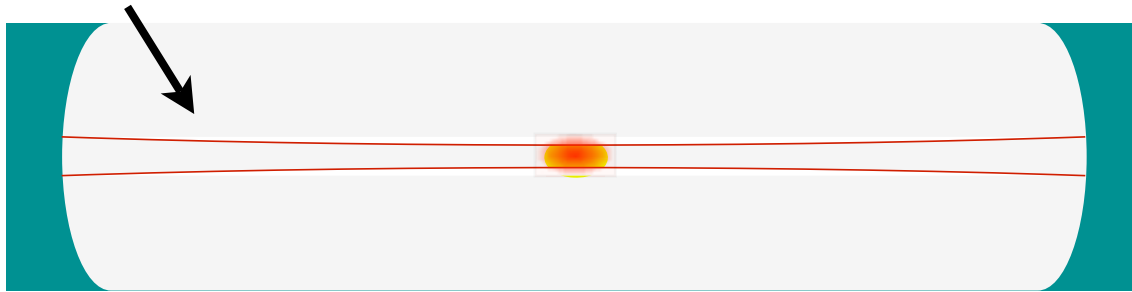
Output



- Ultrashort light pulses $<1\text{ps}$ $\rightarrow$ high intensity, broad bandwidth

Mode-locked soliton lasers

dispersive medium

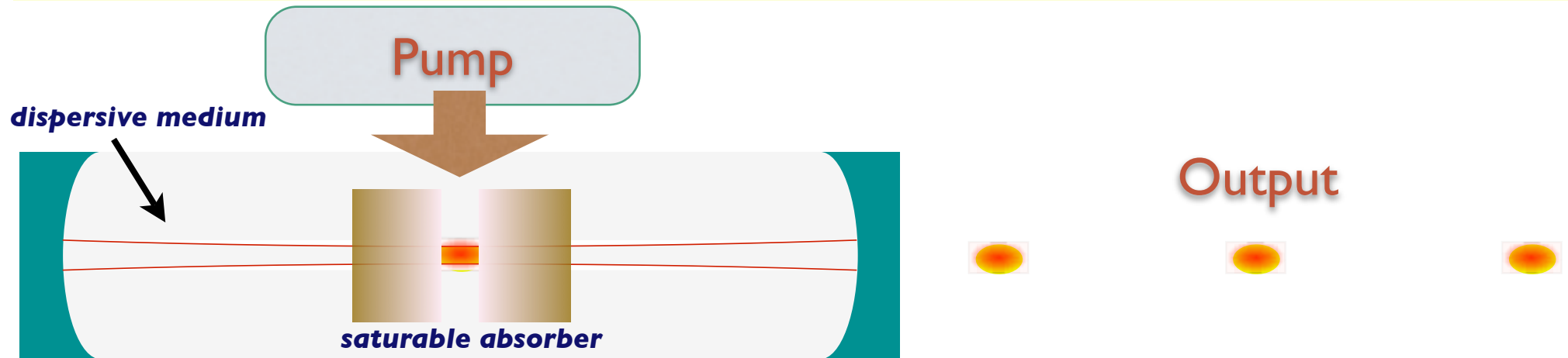


Output



- Ultrashort light pulses $< 1\text{ps}$ $\rightarrow$ high intensity, broad bandwidth
- Dominant dispersive effects:
 - Chromatic dispersion (“anomalous”) — *linear*
 - Kerr effect — *nonlinear*

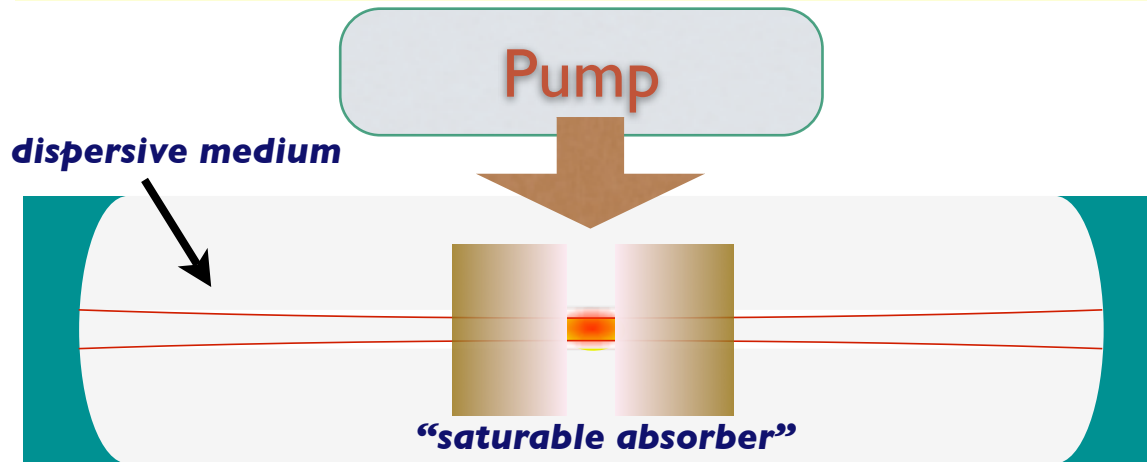
Mode-locked soliton lasers



- Pulse shaping effects:
 - Overall gain & gain filtering — *linear*
 - “Saturable absorption”: Intensity-bleached absorbing element — *nonlinear*
- Relatively weak: proportional to $\mu \ll 1$

Mode-locked soliton lasers

[Haus '75]



- “Master” equation of motion for the field envelope ψ

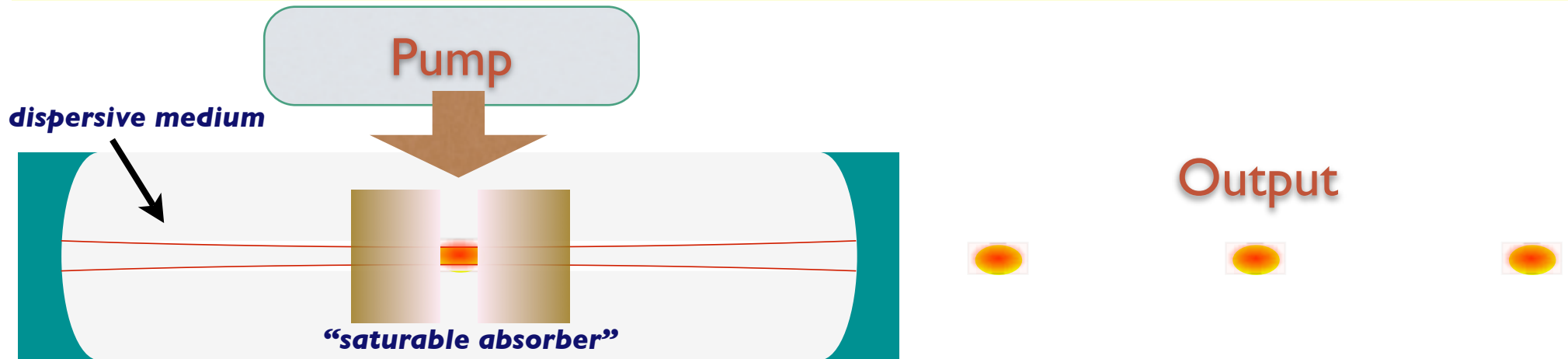
$$\partial_z \psi = (i + \mu) \left(\frac{1}{2} \partial_t^2 \psi + |\psi|^2 \psi \right) + g \psi$$

dispersive effects
(non-dimensionalized)

pulse shaping

overall net gain (<0)

Mode-locked soliton lasers



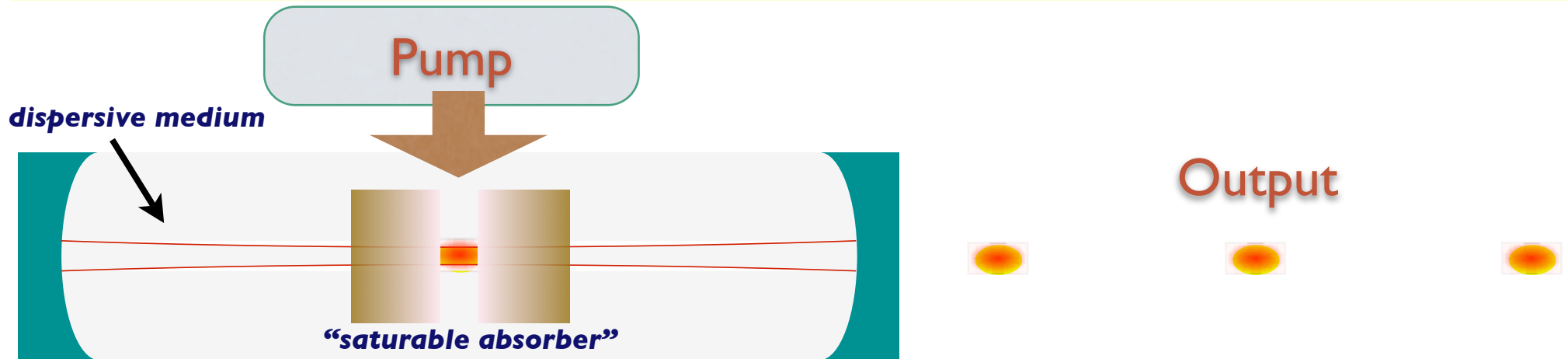
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- Soliton-like pulse: $\psi_s(t, z) = a \operatorname{sech}\left(\frac{t - C(z)}{\tau}\right) e^{i\Phi(t, z)}$

- Parameters: $a = \frac{1}{2}P$ $C = c - \int V dz$
 $\tau = \frac{1}{a}$ $\Phi = \phi + \frac{1}{4}Vt + \frac{1}{8} \int (P^2 + V^2) dz$

Mode-locked soliton lasers



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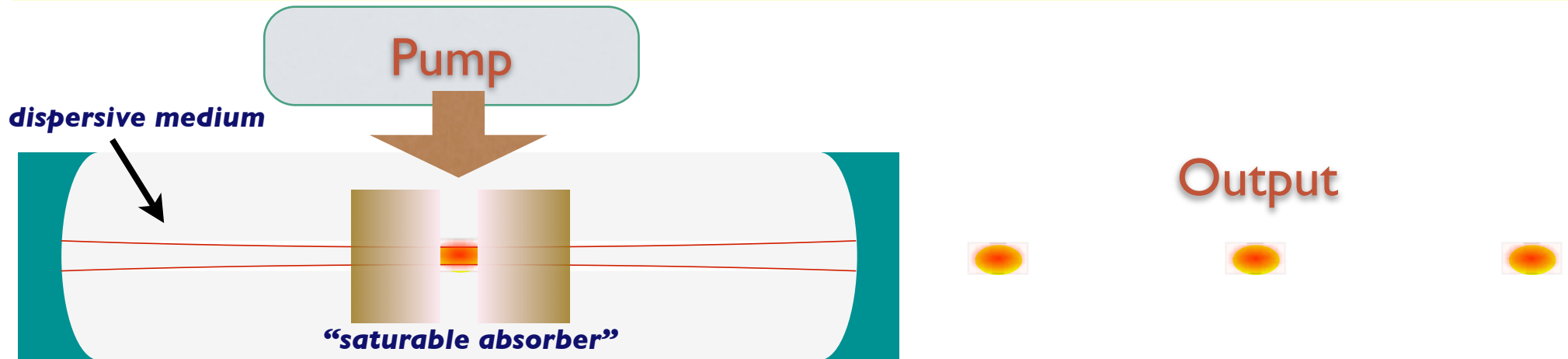
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• Timing
• Phase

• Power
• Frequency

Mode-locked soliton lasers



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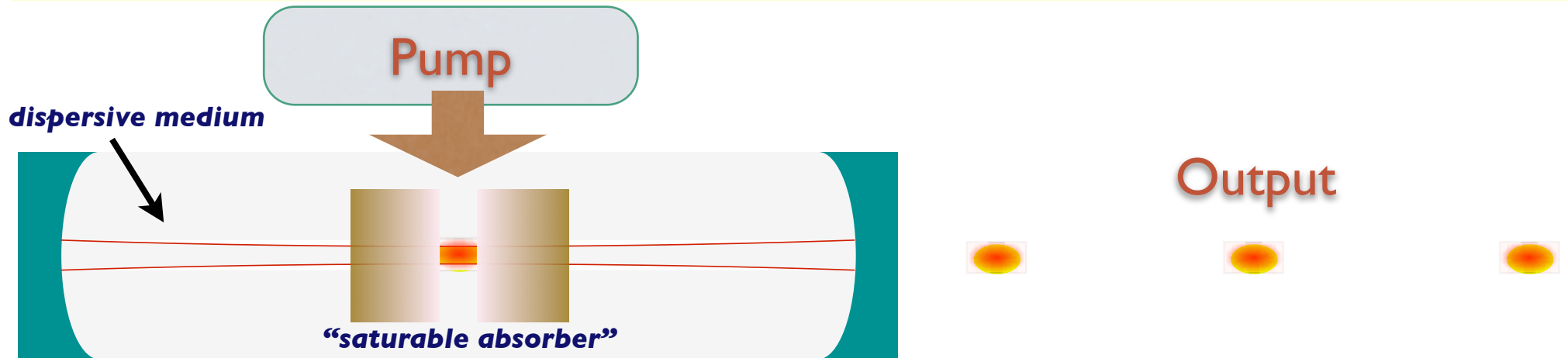
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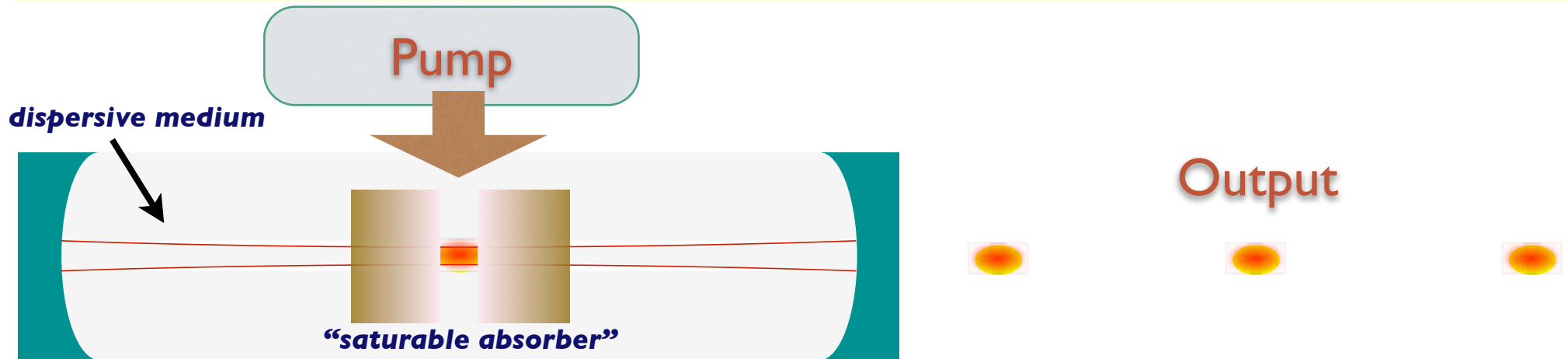
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Mode-locked soliton lasers



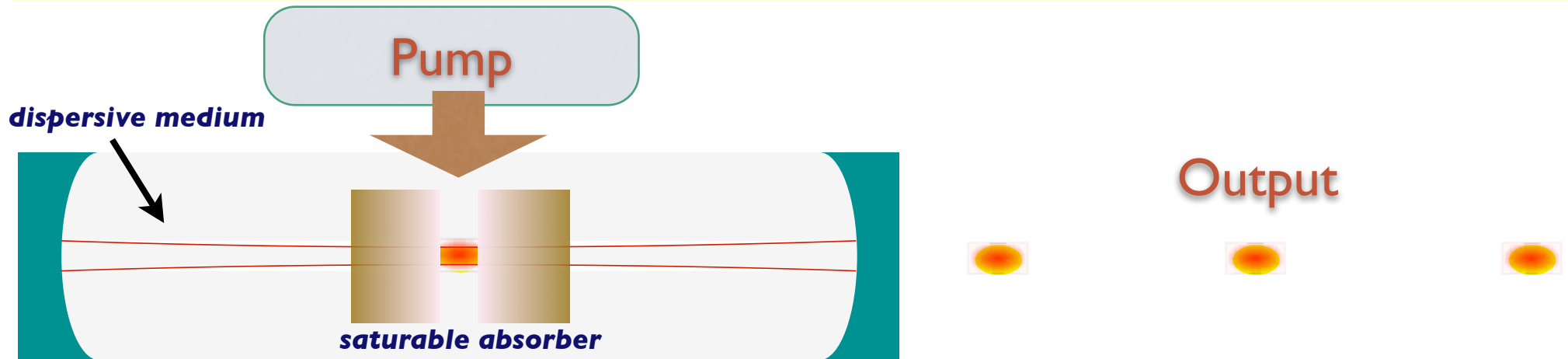
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- Timing
- Phase
- Power
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Mode-locked soliton lasers



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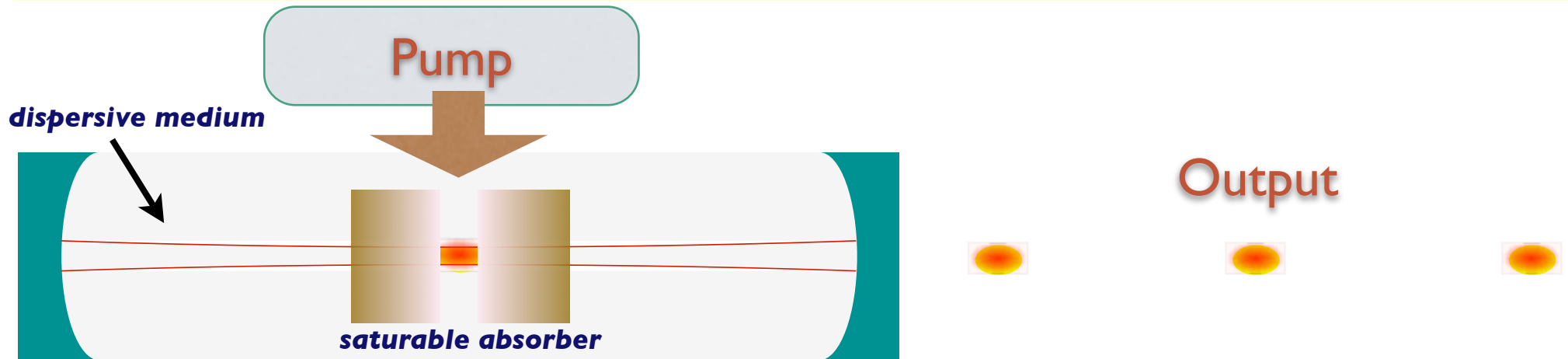
- Pulse parameters:

- Power $P = \sqrt{\frac{8|g|}{\mu}}$ & frequency $V = 0$ fixed

Pulse shaping: Singular perturbation

- Timing c and phase ϕ free — exact symmetries

Mode-locked soliton lasers



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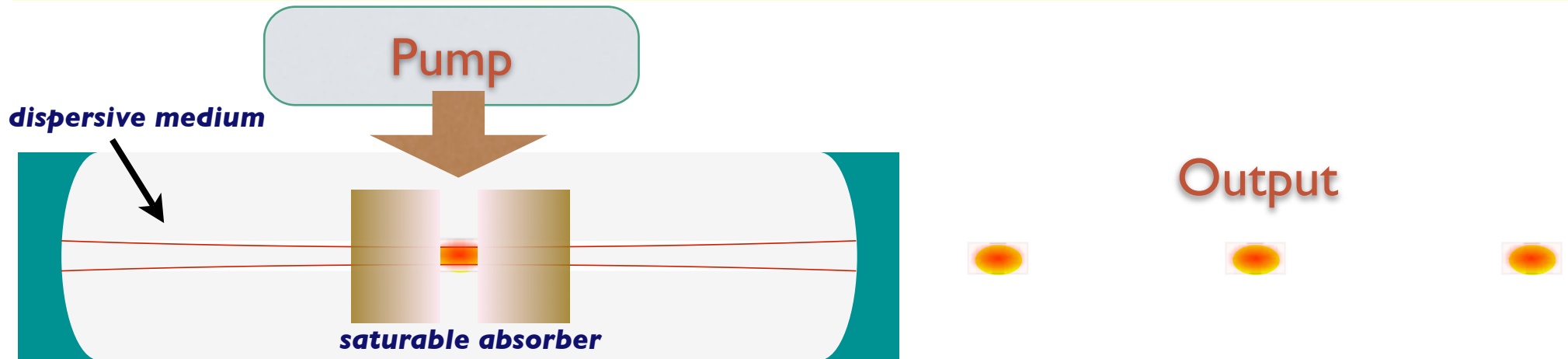
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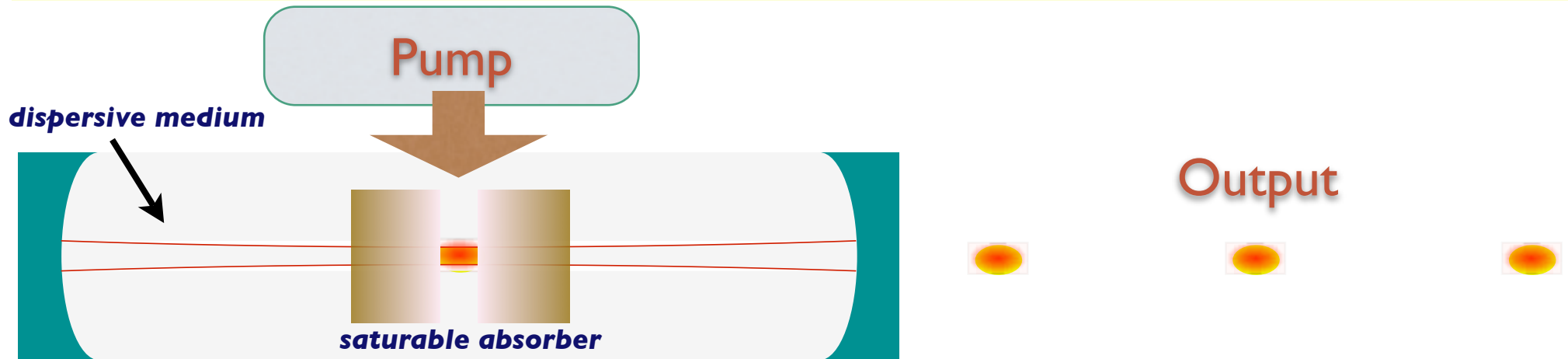
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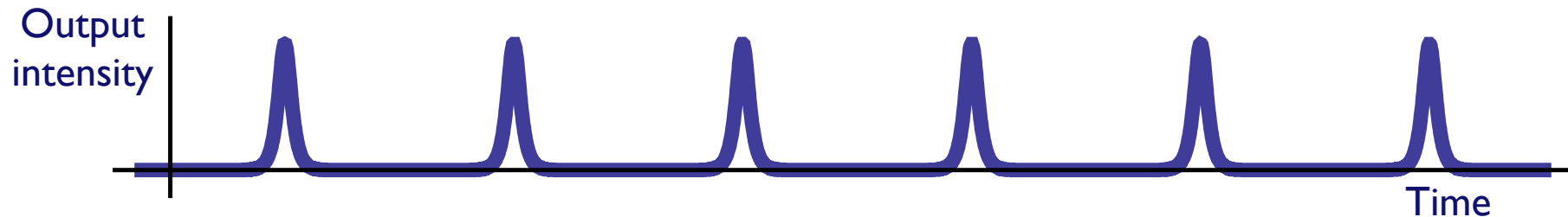
Mode-locked soliton lasers



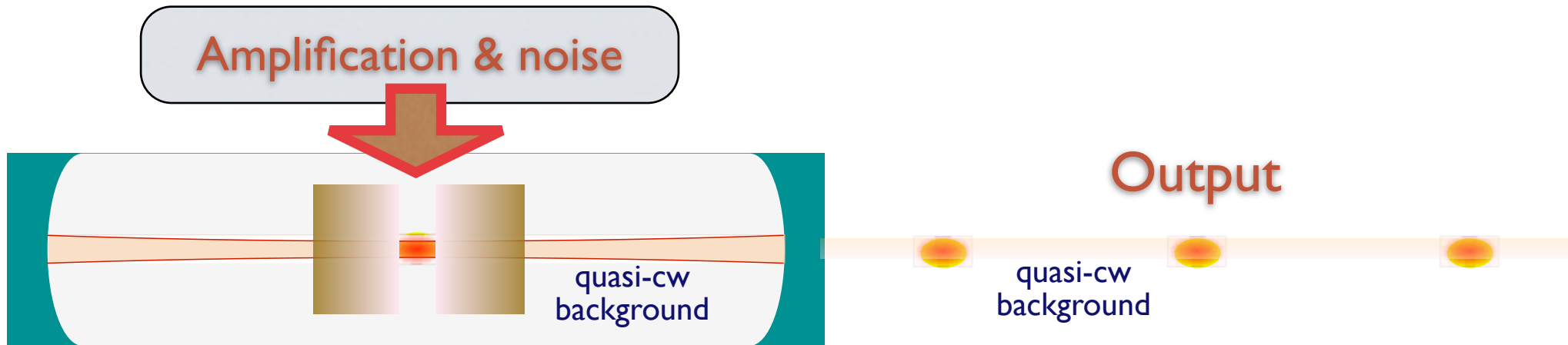
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- Ideal output: Periodic pulse train



Noise and fluctuations



- Noisy master equation:

$$\partial_z \psi = (i + \mu) \left(\frac{1}{2} \partial_t^2 \psi + |\psi|^2 \psi \right) + g\psi + \epsilon \Gamma(z, t)$$

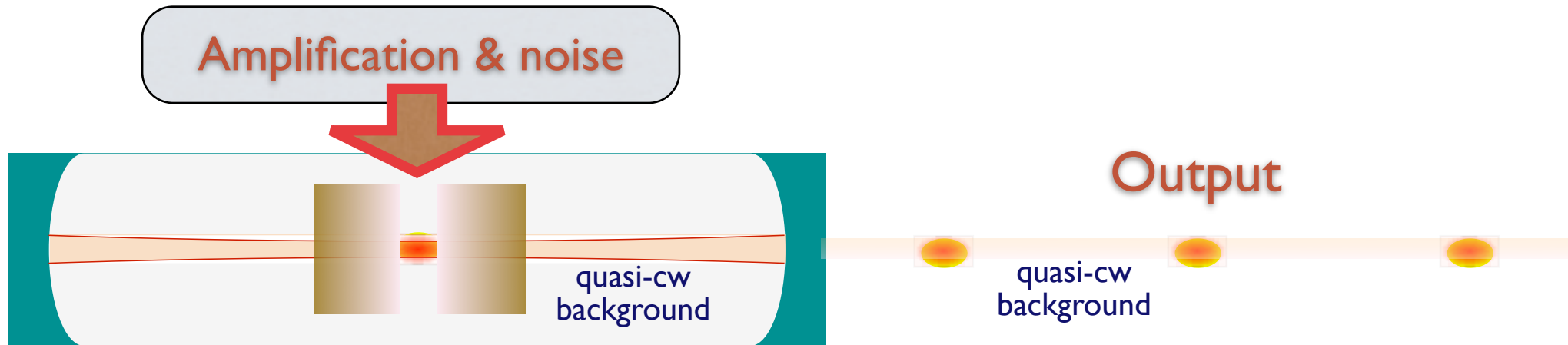
- Weak Gaussian white noise

$$\langle \epsilon \Gamma(z_1, t_1) \epsilon \Gamma^*(z_2, t_2) \rangle = 2\epsilon^2 T L \delta(z_1 - z_2) \delta(t_1 - t_2)$$

Noise power
injection rate

$\epsilon \ll 1$

Noise and fluctuations



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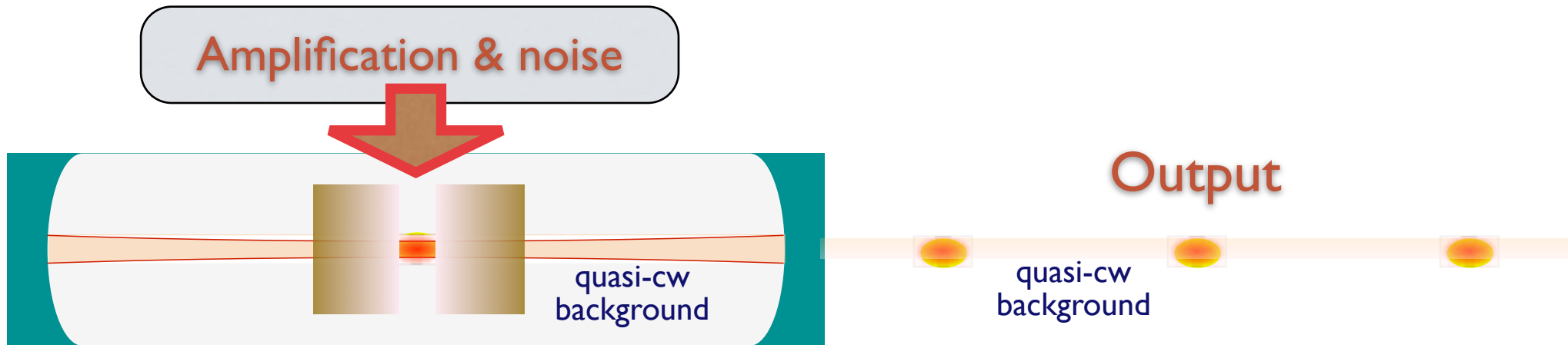
$$\epsilon \ll 1$$

- Waveform is perturbed

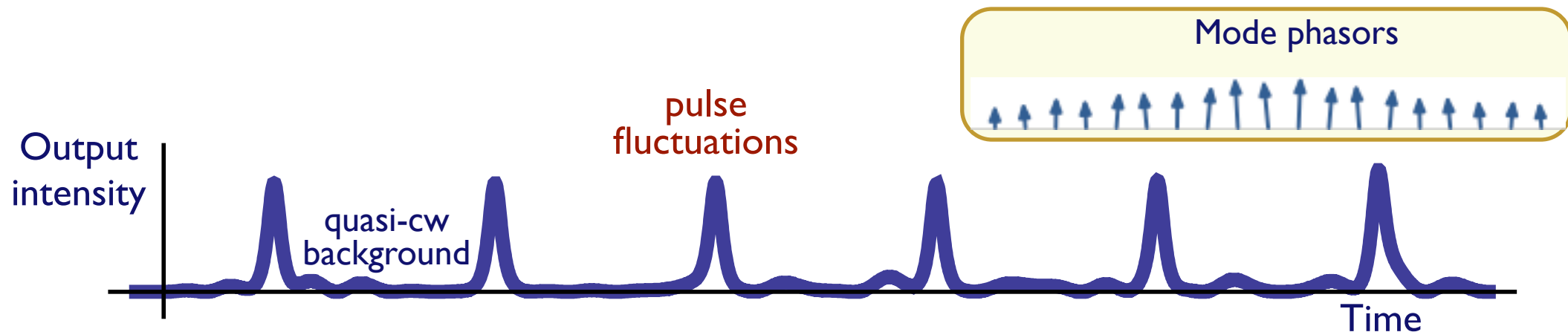
$$\psi = \psi_s(t, z) + O(\epsilon)$$

Noise power
injection rate

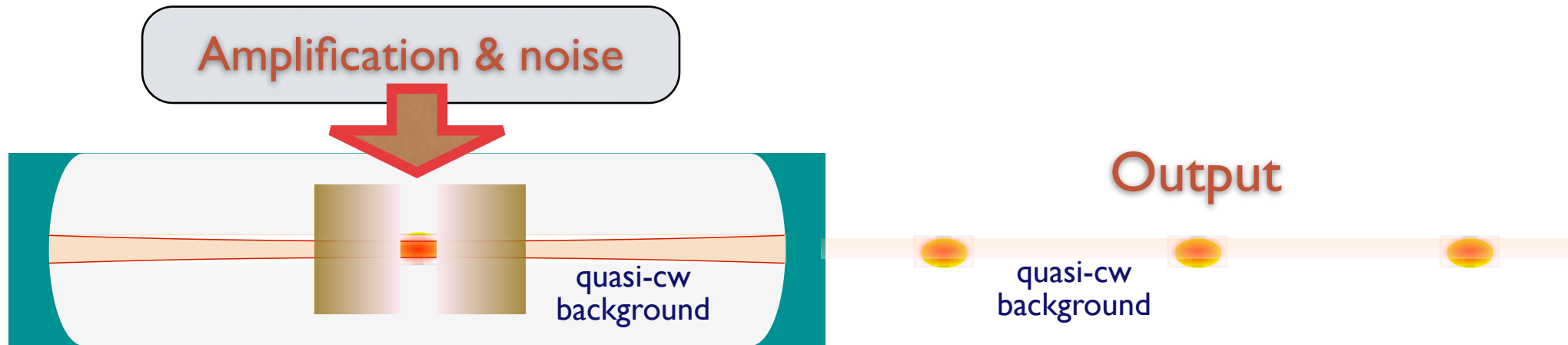
Statistical light-mode dynamics



- Perturbed waveform: $\psi = \underbrace{\psi_s(t, z)}_{\text{Ordered pulse waveform}} + \underbrace{O(\epsilon)}_{\text{Noisy waveform}}$

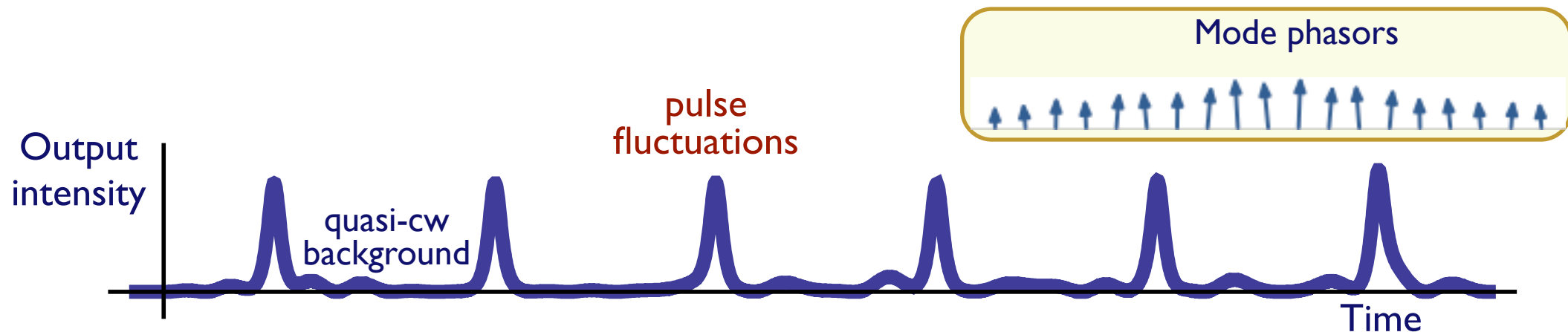


Statistical light-mode dynamics

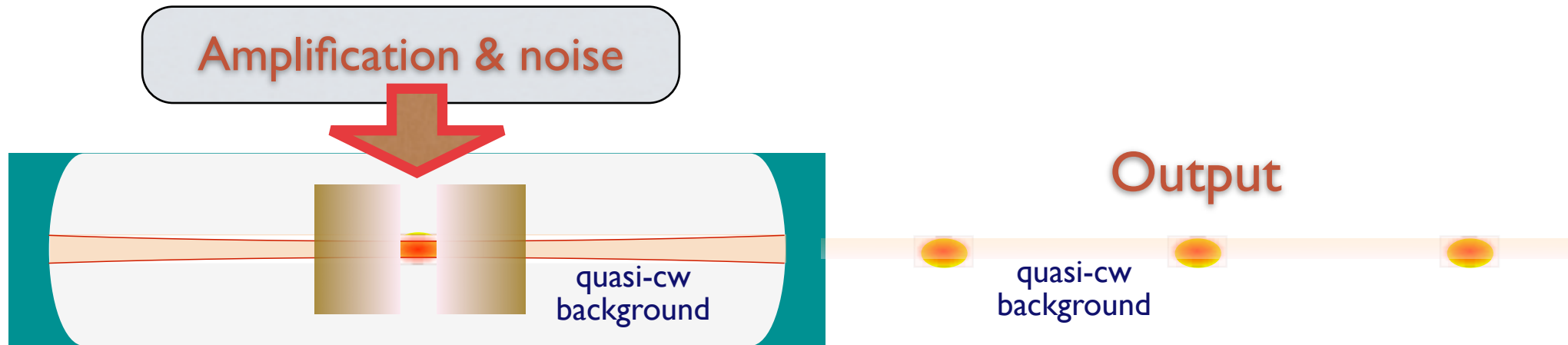


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➔ I. Pulse fluctuations



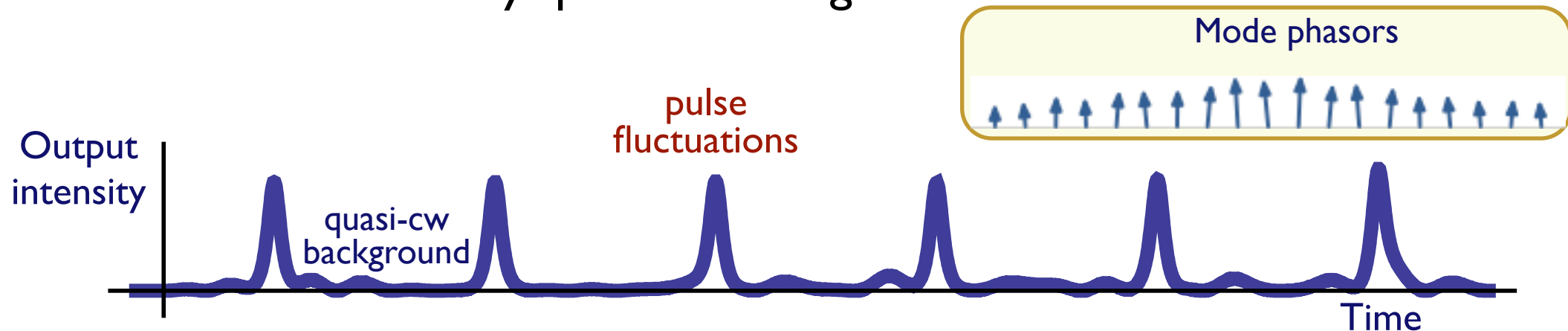
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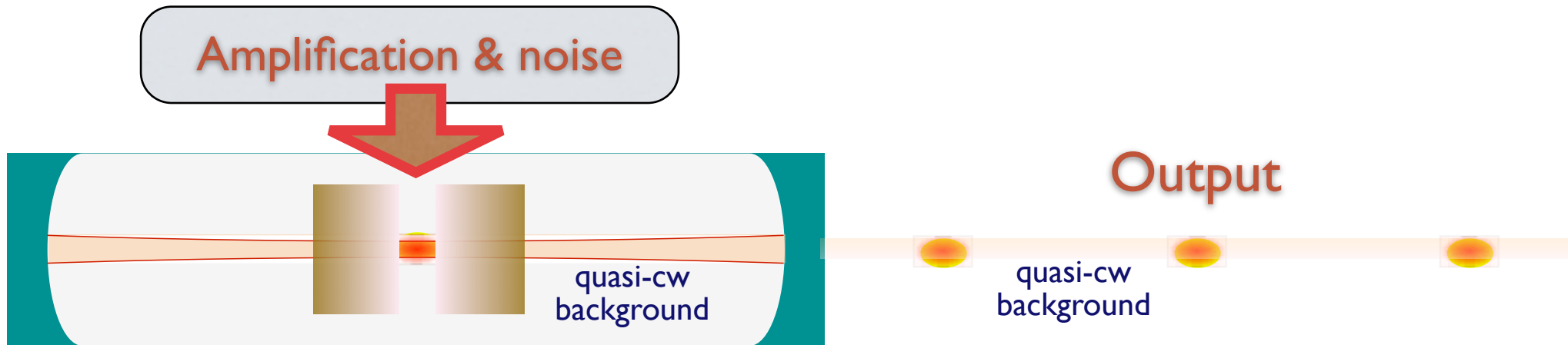
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1. Pulse fluctuations

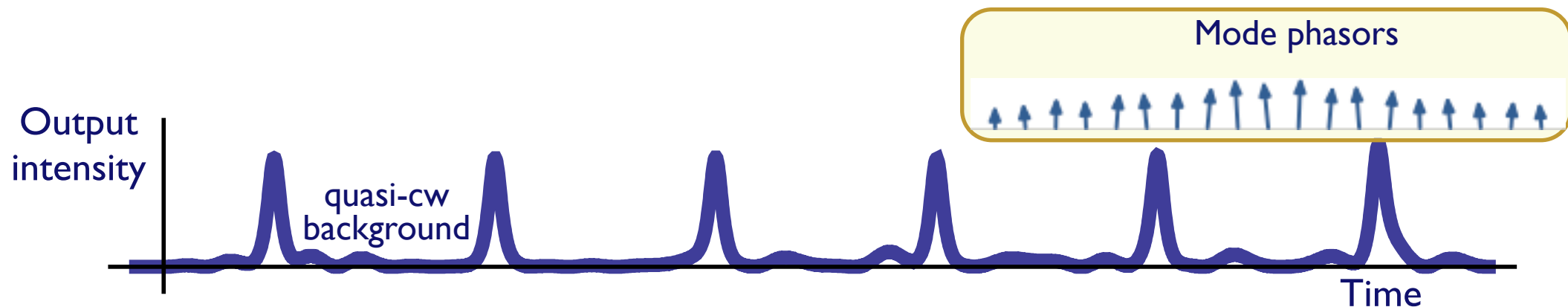
2. Low-intensity quasi-cw background



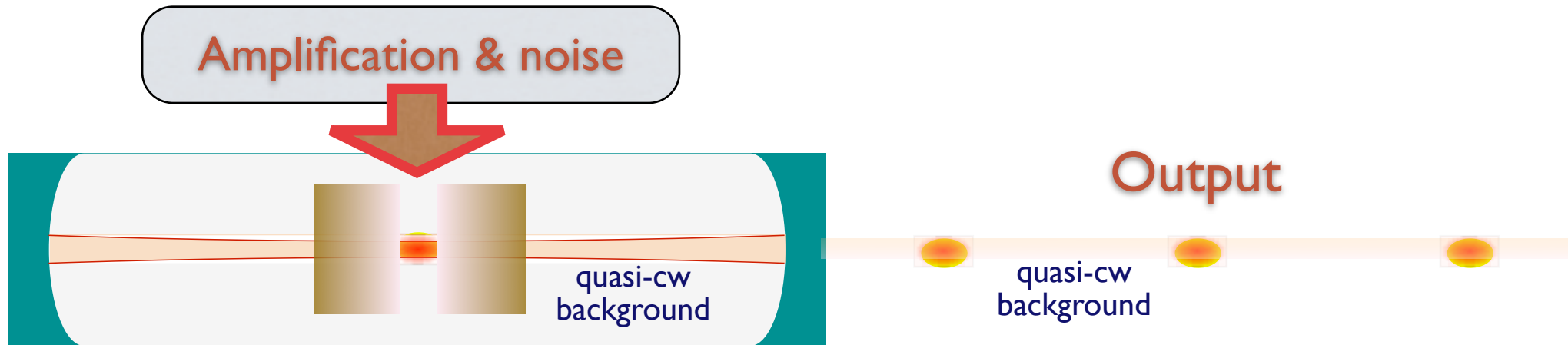
Statistical light-mode dynamics



- Pulse waveform $\psi_s + \epsilon\psi_1$: Strong & narrow
- Continuum waveform $\epsilon\psi_c$: Weak & wide

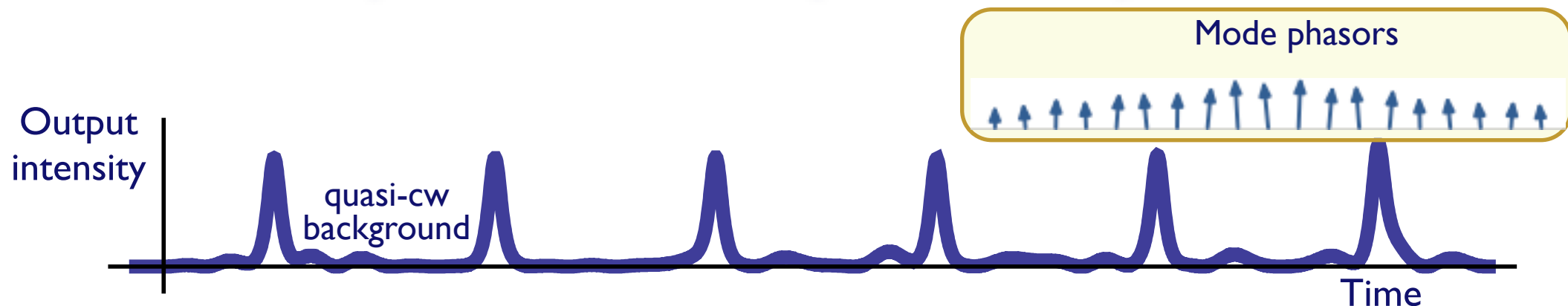


Statistical light-mode dynamics

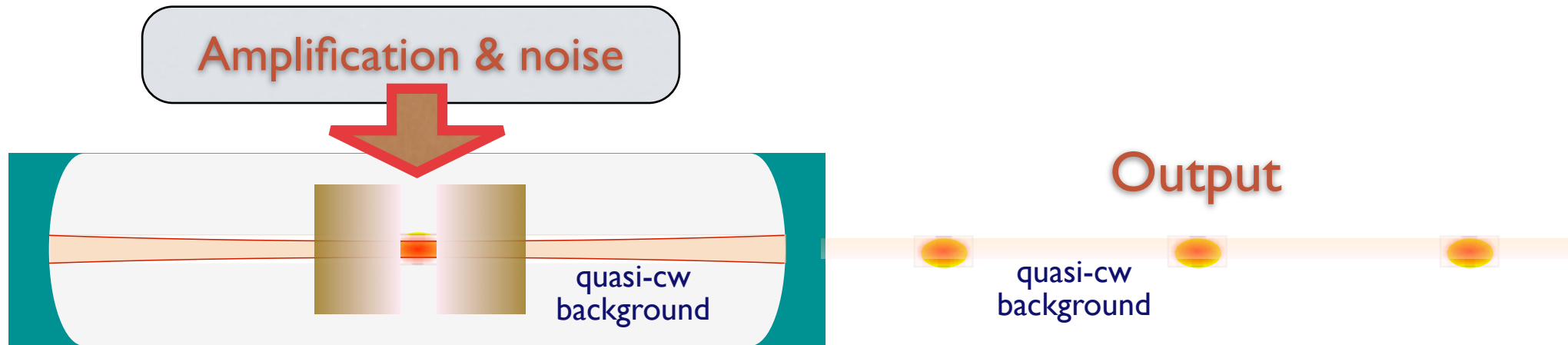


- Pulse waveform $\psi_s + \epsilon\psi_1$: Strong & narrow
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• Pulse power $\sim$ Continuum power $\sim$ total power



Statistical light-mode dynamics



- ~~Pulse waveform $\psi_s + \epsilon\psi_1$: Strong & narrow~~
- Continuum waveform $\epsilon\psi_c$: Weak & wide
- ~~Pulse power~~ ~ Continuum power ~ total power
- For **strong noise**, disordering first order transition to **cw** phase

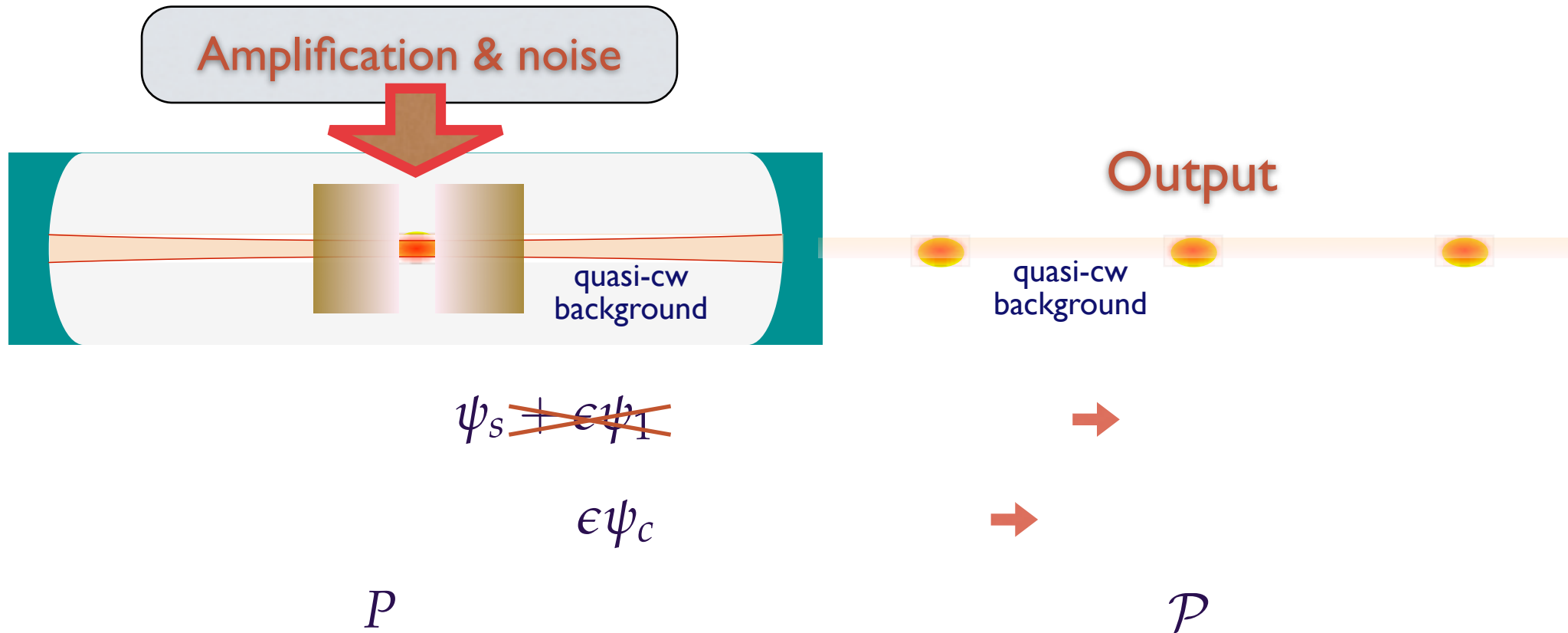
Output
intensity



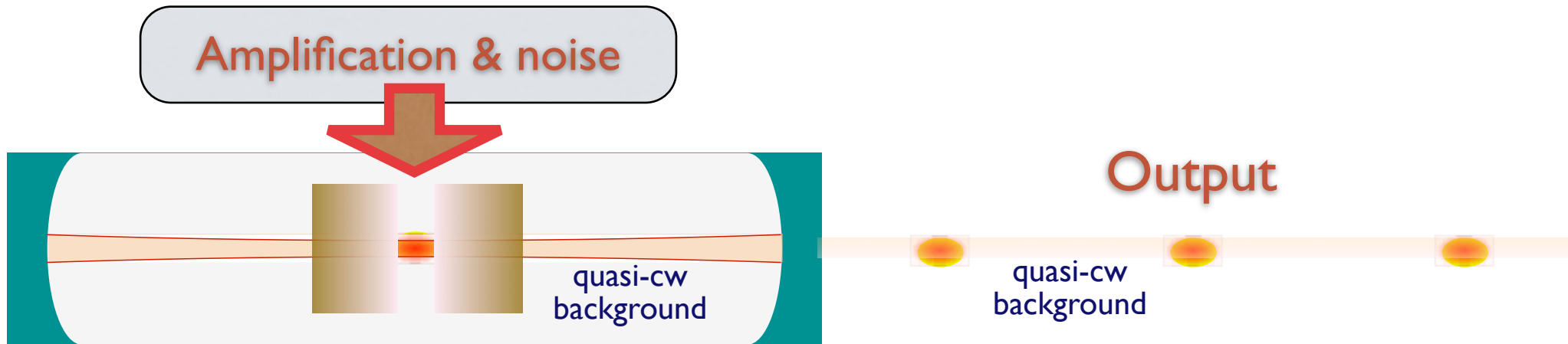
Mode phasors



SLD: The gain balance method

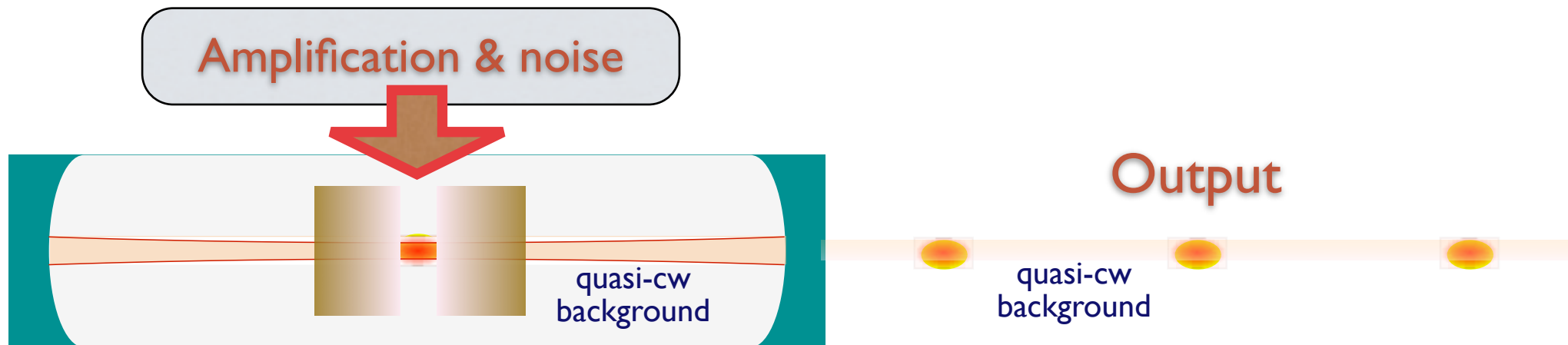


SLD: The gain balance method



- Pulse waveform $\psi_s \neq \epsilon\psi_1$: Strong & narrow $\rightarrow$ Neglect noise
- Continuum waveform $\epsilon\psi_c$: Weak & wide $\rightarrow$ Neglect nonlinearity
- Pulse power P + continuum power = total power $\mathcal{P}$

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$\uparrow$ $\uparrow$
Net gain g

- Common gain value determines the power distribution between pulse & continuum

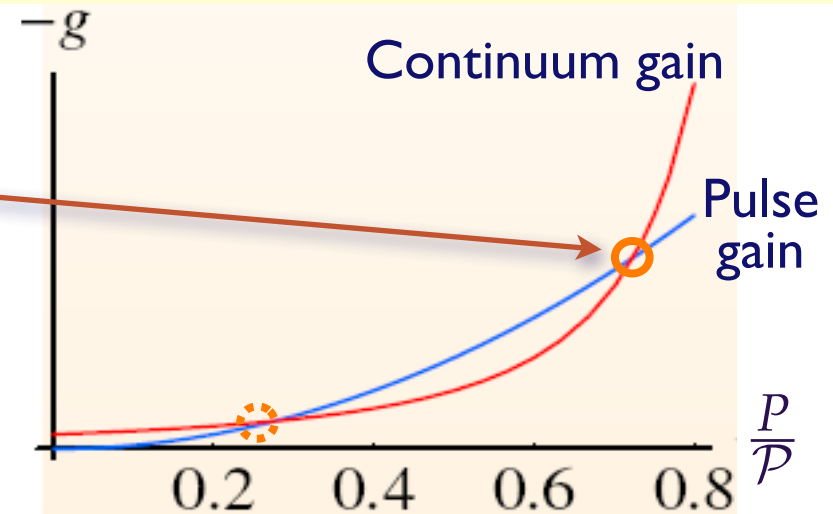
SLD: The gain balance method

- Steady state power distribution

$$\frac{P}{\mathcal{P}} = \frac{1}{2} \left(1 + \sqrt{1 - \frac{\epsilon^2 L^2}{\mu} \frac{8T}{\mathcal{P}^2}} \right)$$

- Thermodynamic limit $L \gg 1$

$$\frac{\epsilon}{\sqrt{\mu}} \ll 1$$

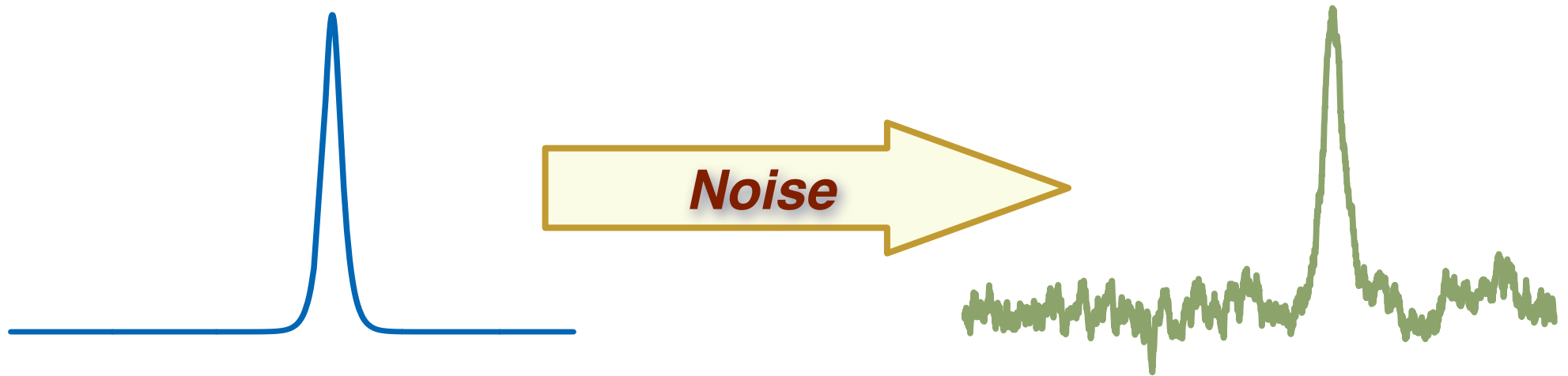


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- Mode locking is possible *only* if a consistent power distribution between pulse & continuum exists

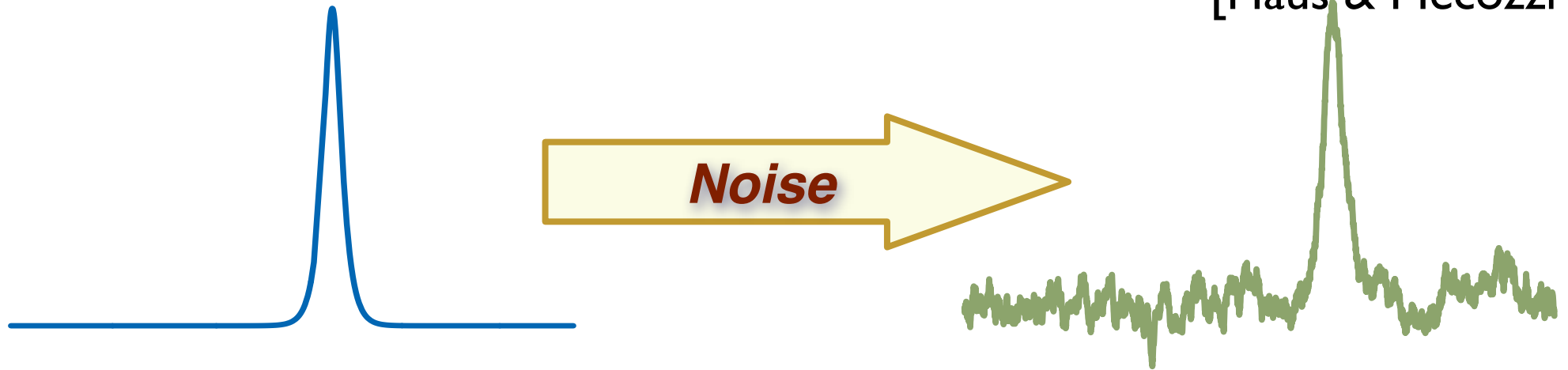
Pulse fluctuations



- Noise causes jitter in pulse parameters
 - Power P & frequency V fluctuate
 - Timing c and phase ϕ diffuse

Pulse fluctuations

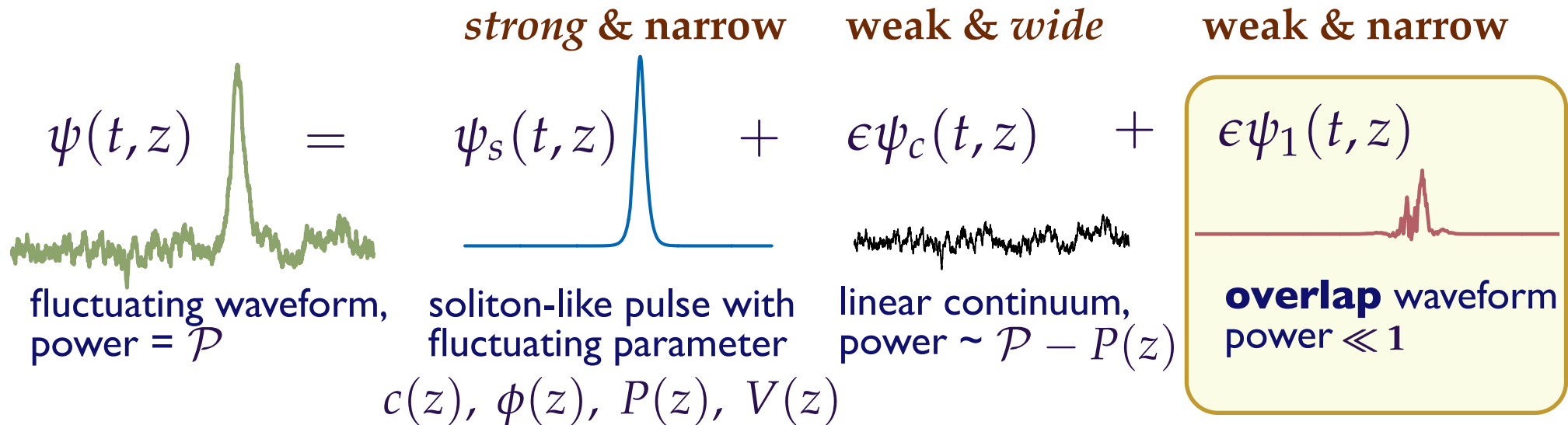
[Haus & Mecozi '93]



- Question: *What are the statistical properties of the fluctuations?*
- Practical implications:
 - Performance of pulse sources
 - Precision of frequency-comb metrology

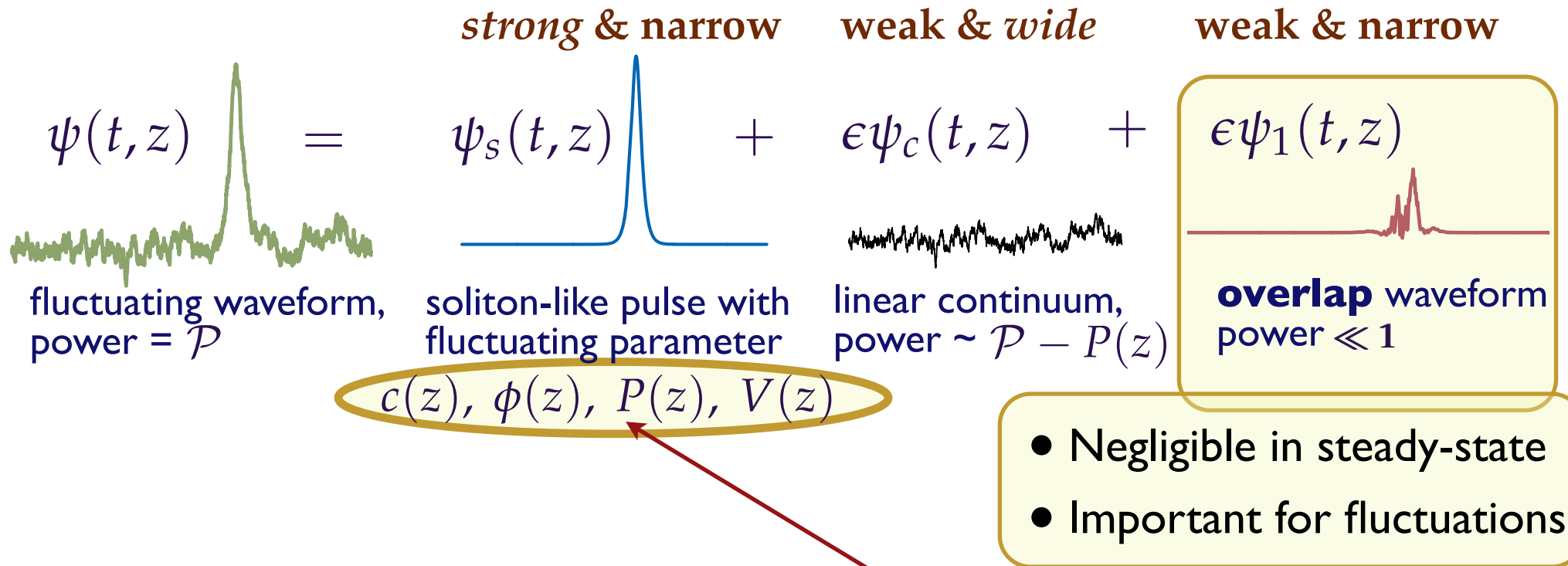
SLD beyond thermodynamics

- Idea: Decompose wave form in 3 parts



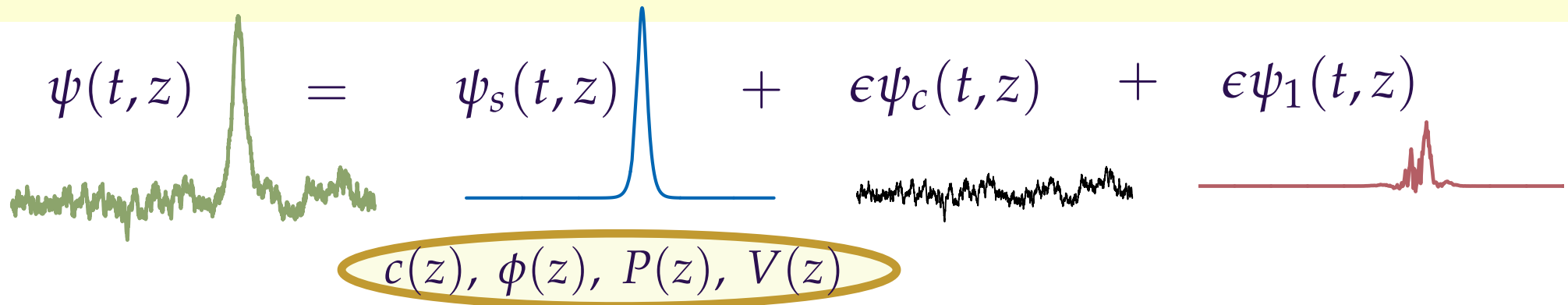
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- Idea: Decompose wave form in 3 parts



- Main goal: Calculate statistics of pulse parameters

SLD beyond thermodynamics

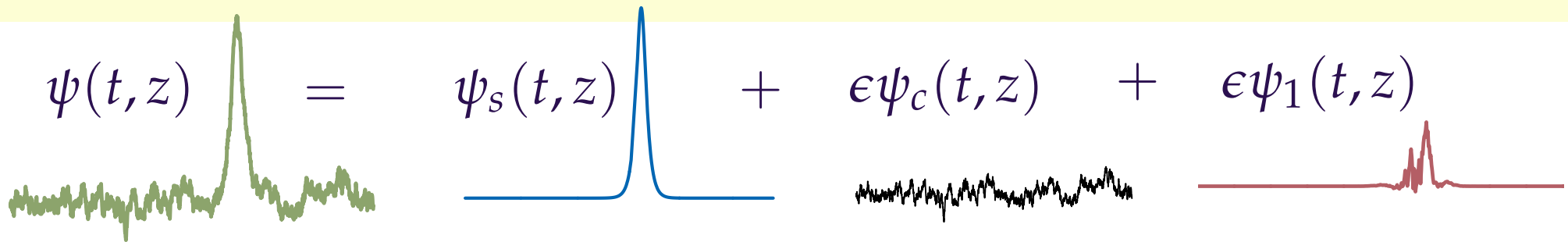
$$\psi(t, z) = \psi_s(t, z) + \epsilon\psi_c(t, z) + \epsilon\psi_1(t, z)$$


The diagram illustrates the decomposition of a signal $\psi(t, z)$ into three components: a noisy green signal, a sharp blue peak, and a noisy red signal. The blue peak is labeled with a yellow oval containing the text $c(z), \phi(z), P(z), V(z)$.

- New qualitative effects:

1. **Oscillations** in power & frequency correlation functions
2. **Enhancement** of phase diffusion rate

I. The perturbation equation

$$\psi(t, z) = \psi_s(t, z) + \epsilon \psi_c(t, z) + \epsilon \psi_1(t, z)$$


- Let $P = \underbrace{P_0}_{\text{steady-state pulse power}} + \frac{\epsilon}{\sqrt{\mu}} p(z)$, $V = \frac{\epsilon}{\sqrt{\mu}} v(z)$, $c \rightarrow \underbrace{\frac{\epsilon}{\sqrt{\mu}} c(z)}_{\text{fluctuations small parameter}}$, $\phi \rightarrow \frac{\epsilon}{\sqrt{\mu}} \phi(z)$

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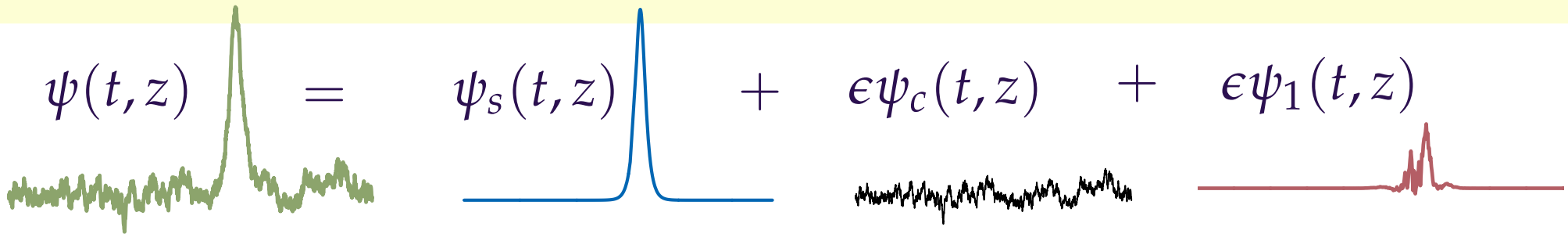
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- Linearized master equation for ψ_1

$$\partial_z \psi_1 + \frac{1}{\sqrt{\mu}} \partial_z \underbrace{\vec{x}}_{\text{parameter set}} \cdot \nabla_x \psi_s = L \psi_1 - i \frac{1}{2} \sqrt{\mu} v(z) \partial_t \psi_s + \underbrace{g_1}_{\text{gain fluctuations}} \psi_s + \underbrace{f}$$

- Nonlinearity-generated forcing by overlap of ψ_s & ψ_c

2. Gain fluctuations



The diagram illustrates the decomposition of a laser signal $\psi(t, z)$ into its steady-state component $\psi_s(t, z)$ and two fluctuation components $\epsilon\psi_c(t, z)$ and $\epsilon\psi_1(t, z)$. The signal $\psi(t, z)$ is represented by a green waveform with a sharp peak and noise. The steady-state component $\psi_s(t, z)$ is a blue waveform with a single sharp peak. The fluctuation components $\epsilon\psi_c(t, z)$ and $\epsilon\psi_1(t, z)$ are represented by black and red waveforms, respectively, showing noise and small fluctuations.

$$\psi(t, z) = \psi_s(t, z) + \epsilon\psi_c(t, z) + \epsilon\psi_1(t, z)$$

- Double role of net gain in the steady state:

1. Determines the pulse & continuum power $g_0 = -\frac{\mu}{8}P_0$
Gain balance
2. Determined by the overall power $g_0 = g(\mathcal{P})$
Gain saturation

2. Gain fluctuations

$$\psi(t, z) = \psi_s(t, z) + \epsilon\psi_c(t, z) + \epsilon\psi_1(t, z)$$

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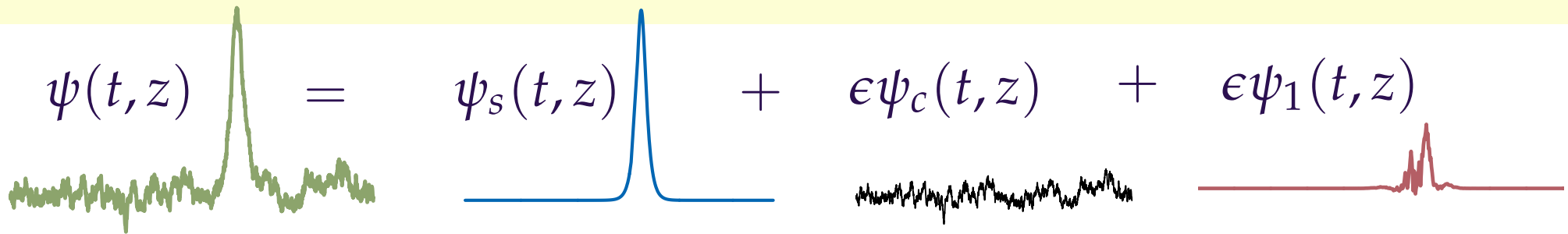
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Gain saturation

- Gain fluctuations $g = g_0 + \epsilon g_1$ arise from

1. Random shuffle of power between pulse and continuum
2. Fluctuations of overall power

2. Gain fluctuations

$$\psi(t, z) = \psi_s(t, z) + \epsilon\psi_c(t, z) + \epsilon\psi_1(t, z)$$


The diagram illustrates the decomposition of a signal $\psi(t, z)$ into three components: a steady-state signal $\psi_s(t, z)$, a continuum signal $\epsilon\psi_c(t, z)$, and a fluctuation signal $\epsilon\psi_1(t, z)$. The steady-state signal is represented by a green noisy trace with a sharp peak. The continuum signal is represented by a blue trace with a sharp peak. The fluctuation signal is represented by a black noisy trace. The total signal is represented by a red trace with a sharp peak.

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Gain balance

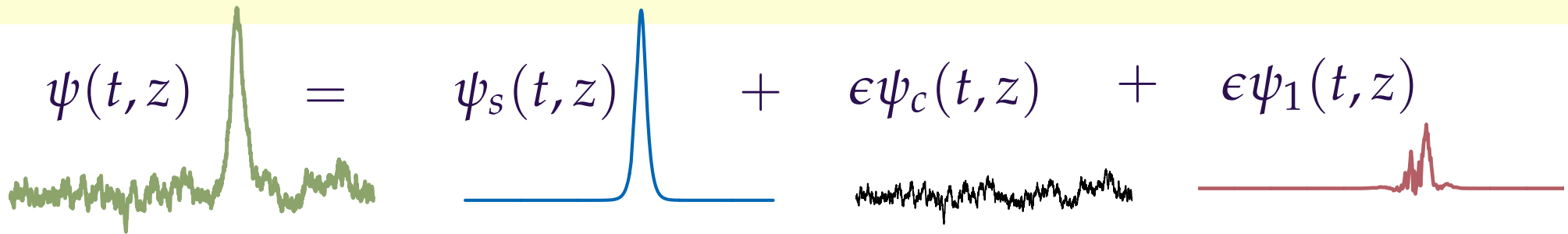
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2. ~~Fluctuations of overall power~~ ← Assume deep saturation

2. Gain fluctuations

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The diagram illustrates the decomposition of a signal $\psi(t, z)$ into three components: a signal $\psi_s(t, z)$, a noise component $\epsilon\psi_c(t, z)$, and a fluctuation component $\epsilon\psi_1(t, z)$. The signal $\psi(t, z)$ is shown as a green waveform with a sharp peak. The signal $\psi_s(t, z)$ is shown as a blue waveform with a sharp peak. The noise component $\epsilon\psi_c(t, z)$ is shown as a black waveform with a noisy baseline. The fluctuation component $\epsilon\psi_1(t, z)$ is shown as a red waveform with a sharp peak.

- Gain fluctuations $g = g_0 + \epsilon g_1$ arise from

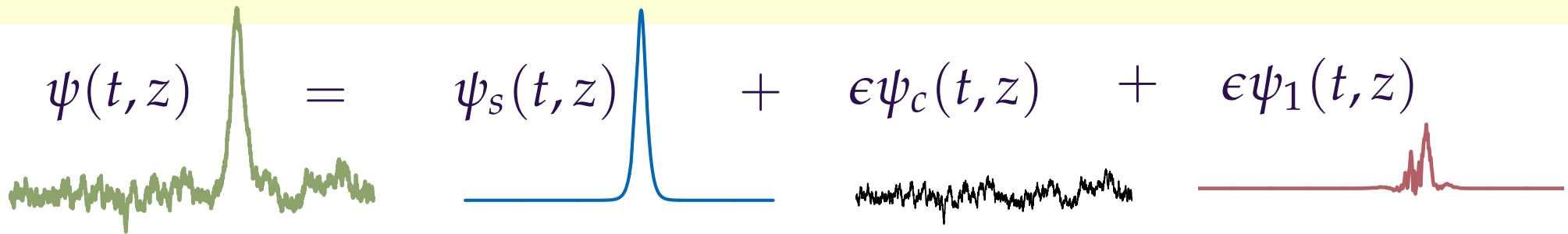
1. Random shuffle of power between pulse and continuum

2. ~~Fluctuations of overall power~~ ← Assume deep saturation

- Linearized master equation for ψ_1

$$\partial_z \psi_1 + \frac{1}{\sqrt{\mu}} \partial_z \vec{x} \cdot \nabla_x \psi_s = L \psi_1 - i \frac{1}{2} \sqrt{\mu} v(z) \partial_t \psi_s + g_1 \psi_s + f$$

2. Gain fluctuations



The diagram illustrates the decomposition of a signal $\psi(t, z)$ into three components: a signal $\psi_s(t, z)$, a noise component $\epsilon\psi_c(t, z)$, and a fluctuation component $\epsilon\psi_1(t, z)$. The signal $\psi(t, z)$ is shown as a green waveform with a sharp peak. The signal $\psi_s(t, z)$ is shown as a blue waveform with a sharp peak. The noise component $\epsilon\psi_c(t, z)$ is shown as a black waveform with a noisy baseline. The fluctuation component $\epsilon\psi_1(t, z)$ is shown as a red waveform with a sharp peak.

$$\psi(t, z) = \psi_s(t, z) + \epsilon\psi_c(t, z) + \epsilon\psi_1(t, z)$$

- Gain fluctuations $g = g_0 + \epsilon g_1$ arise from

1. Random shuffle of power between pulse and continuum

2. ~~Fluctuations of overall power~~ ← Assume deep saturation

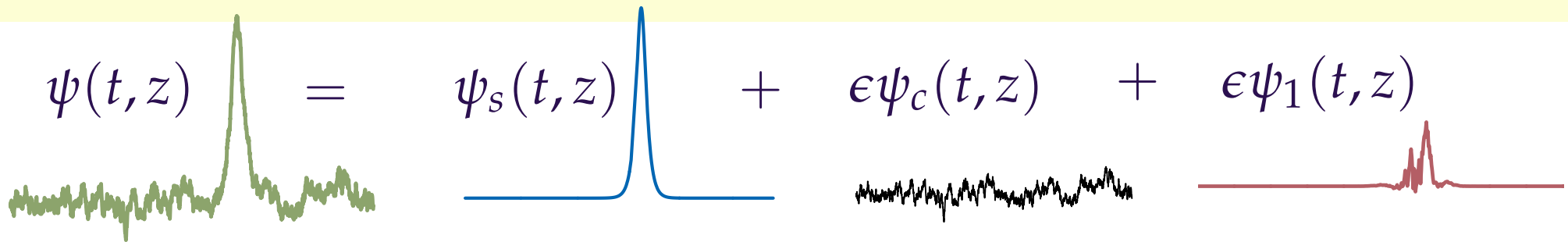
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- Fluctuation conserve overall power

$$g_1 \mathcal{P} = -\frac{\sqrt{\mu} P_0^2}{4} p - \int dz \psi_s^* (\Gamma + L(\psi_c + \psi_1))$$

3. The slow modes

$$\psi(t, z) = \psi_s(t, z) + \epsilon \psi_c(t, z) + \epsilon \psi_1(t, z)$$


- Perturbation equation including gain fluctuations:

$$\partial_z \psi_1 + \frac{1}{\sqrt{\mu}} \partial_z \vec{x} \cdot \nabla_x \psi_s = \tilde{L} \psi_1 - i \frac{1}{2} \sqrt{\mu} v(z) \partial_t \psi_s + \tilde{f}$$

Linear operator including rank-1 gain fluctuations term

Forcing including gain fluctuations

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Linear operator including rank-1 gain fluctuations term
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- Remaining arbitrariness: A slight shift in $\vec{x}$ can be

absorbed in ψ_1 ,

$$\psi_s(\vec{x}) + \epsilon \psi_1 = \psi_s(\vec{x} + \delta \vec{x}) + \epsilon (\psi_1 + \delta \psi_1)$$

3. The slow modes

$$\psi(t, z) = \psi_s(t, z) + \epsilon \psi_c(t, z) + \epsilon \psi_1(t, z)$$

The diagram illustrates the decomposition of a signal $\psi(t, z)$ into three components: a slow mode $\psi_s(t, z)$, a fast mode $\epsilon \psi_c(t, z)$, and a perturbation $\epsilon \psi_1(t, z)$. The slow mode is represented by a blue line with a single sharp peak. The fast mode is represented by a black line with high-frequency noise. The perturbation is represented by a red line with a small peak. The total signal $\psi(t, z)$ is shown as a green line with a peak and noise.

- Perturbation equation including gain fluctuations:

$$\partial_z \psi_1 + \frac{1}{\sqrt{\mu}} \partial_z \vec{x} \cdot \nabla_x \psi_s = \tilde{L} \psi_1 - i \frac{1}{2} \sqrt{\mu} v(z) \partial_t \psi_s + \tilde{f}$$

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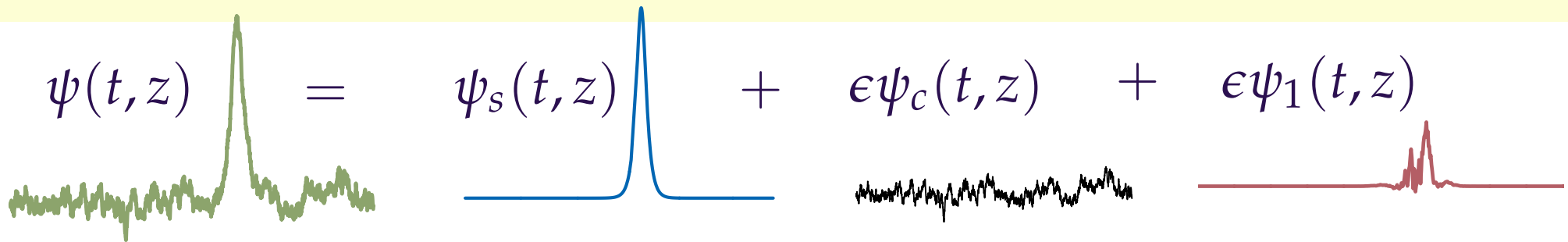
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$$\text{absorbed in } \psi_1, \quad \psi_s(\vec{x}) + \epsilon \psi_1 = \psi_s(\vec{x} + \delta \vec{x}) + \epsilon(\psi_1 + \delta \psi_1)$$

The diagram shows the equation $\psi_s(\vec{x}) + \epsilon \psi_1 = \psi_s(\vec{x} + \delta \vec{x}) + \epsilon(\psi_1 + \delta \psi_1)$. Red arrows point from $\delta \vec{x}$ and $\delta \psi_1$ to the text "absorbed in ψ_1 ".

- Q: How to define the pulse parameters?
- A: Let ψ_1 lie outside 4-dimensional slow eigen-space of $\tilde{L}$

3. The slow modes



The diagram illustrates the decomposition of a wave packet $\psi(t, z)$ into its slow and fast components. On the left, a green plot shows a noisy signal with a sharp central peak, representing the total wave packet. This is followed by an equals sign. To the right of the equals sign are four terms: a blue plot of a smooth, narrow peak representing the slow mode $\psi_s(t, z)$; a plus sign; a black plot of a noisy signal representing the fast mode $\epsilon\psi_c(t, z)$; another plus sign; and a red plot of a noisy signal with a small peak representing the perturbation $\epsilon\psi_1(t, z)$.

$$\psi(t, z) = \psi_s(t, z) + \epsilon\psi_c(t, z) + \epsilon\psi_1(t, z)$$

- Perturbation equation including gain fluctuations:

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- Recall: linearized NLS has 4-d zero eigen-space:

→ **timing, phase:**
annihilated by L_{NLS}

↓ **frequency, power:**
annihilated by L_{NLS}^2

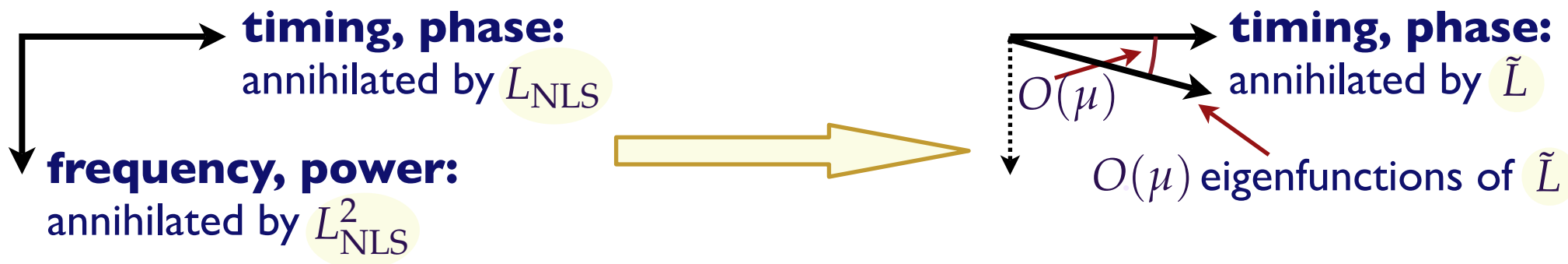
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- Recall: linearized NLS has 4-d zero eigen-space:



- Degeneracy is half-lifted by $\tilde{L} = L_{\text{NLS}} + O(\mu)$

4. Pulse parameter dynamics

$$\psi(t, z) = \psi_s(t, z) + \epsilon\psi_c(t, z) + \epsilon\psi_1(t, z)$$

- Perturbation equation:

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- where: ψ_1 lies outside 4-dimensional slow eigen-space of $\tilde{L}$

$\langle q_n, \psi_1 \rangle = 0$ where $q_1, \dots, q_4$ are slow left $\tilde{L}$ eigenfunctions

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$\langle q_n, \psi_1 \rangle = 0$ where $q_1, \dots, q_4$ are slow left $\tilde{L}$ eigenfunctions

- $\langle q_n, \cdot \rangle$ projection of the perturbation equation yields (linear combination of) parameter equations of motion

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Power:	$\partial_z p = -\mu \frac{p_0^3}{4\mathcal{P}} p - \sqrt{\mu} f_p$
Phase:	$\partial_z \phi = \sqrt{\mu} f_\phi$
Frequency:	$\partial_z v = -\mu \frac{p_0^2}{6} v - \sqrt{\mu} f_v$
Timing:	$\partial_z c = \sqrt{\mu} f_c$

Restoring
terms

Random
forcing

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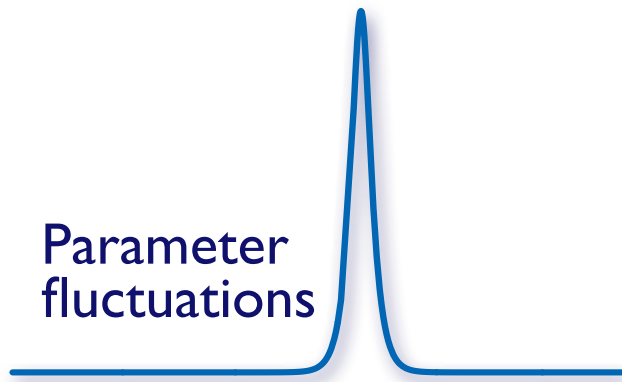
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Frequency:	$\partial_z v = -\mu \frac{p_0^2}{6} v - \sqrt{\mu} f_v$	Diffusion
Timing:	$\partial_z c = \sqrt{\mu} f_c$	

Restoring terms
Random forcing

Results: Autocorrelation

- Pulse frequency: $\langle v_{t+\tau} v_t^* \rangle = \frac{T}{P_0} \left(e^{-\frac{\mu P_0^2}{6} |\tau|} + \pi \int dk k^2 I_k(\tau) \right)$



Direct term:
exponential
damping

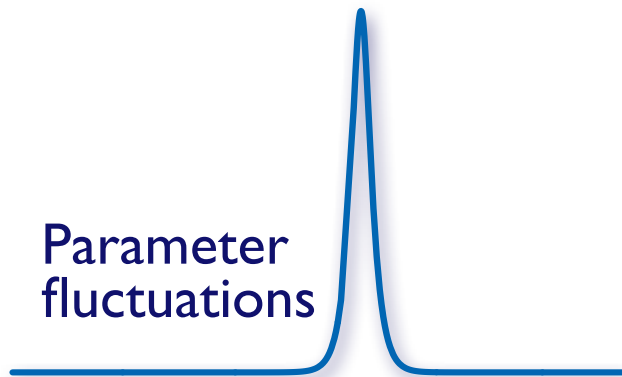
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Continuum term:
Damped oscillations

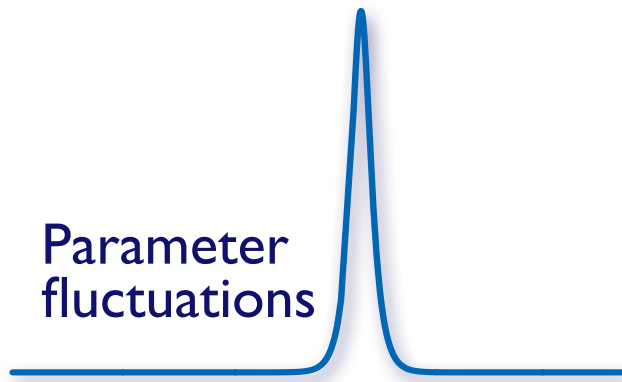


$$I_k(\tau) = \frac{\text{sech}^2(\pi k/2)}{k^2+1} e^{-\frac{\mu P_0^2(k^2+1)}{8} |\tau|} \cos\left(\frac{P_0^2(k^2+1)}{8} \tau\right)$$

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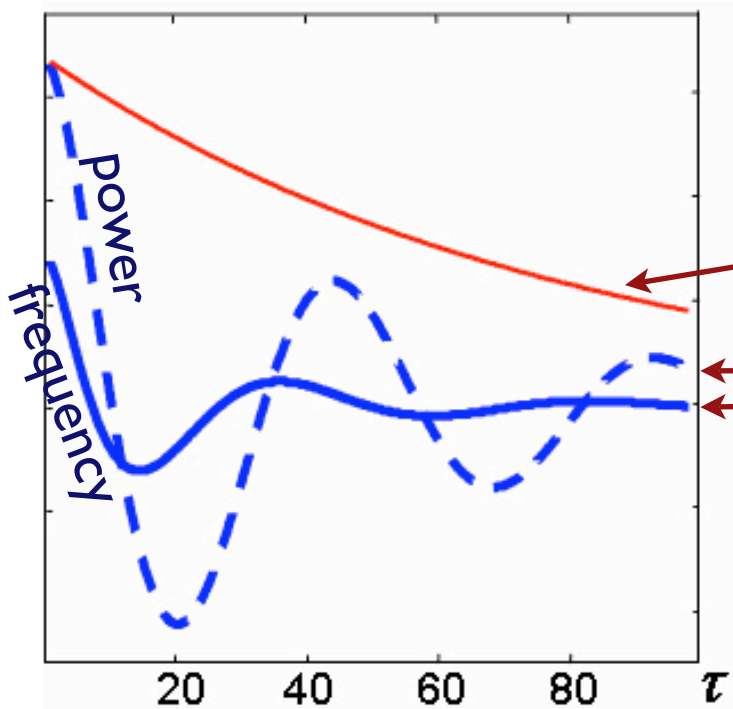
- Pulse power:

$$\langle p_{t+\tau} p_t^* \rangle = \frac{2T}{P_0} \left(\frac{P}{P_0} e^{-\frac{\mu P_0^3}{4P} |\tau|} + \frac{\pi}{2} \int dk I_k(\tau) \right)$$

Continuum
enhancement factor

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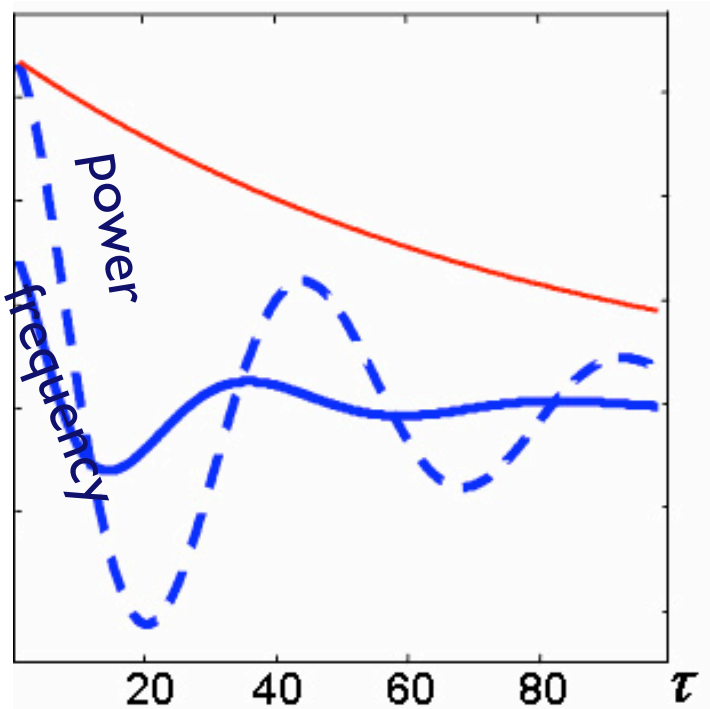
Continuum term:
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- Pulse power: $\langle p_{t+\tau} p_t^* \rangle = \frac{2T}{P_0} \left(\left(\frac{\mathcal{P}}{P_0} \right) e^{-\frac{\mu P_0^3}{4\mathcal{P}} |\tau|} + \frac{\pi}{2} \int dk I_k(\tau) \right)$

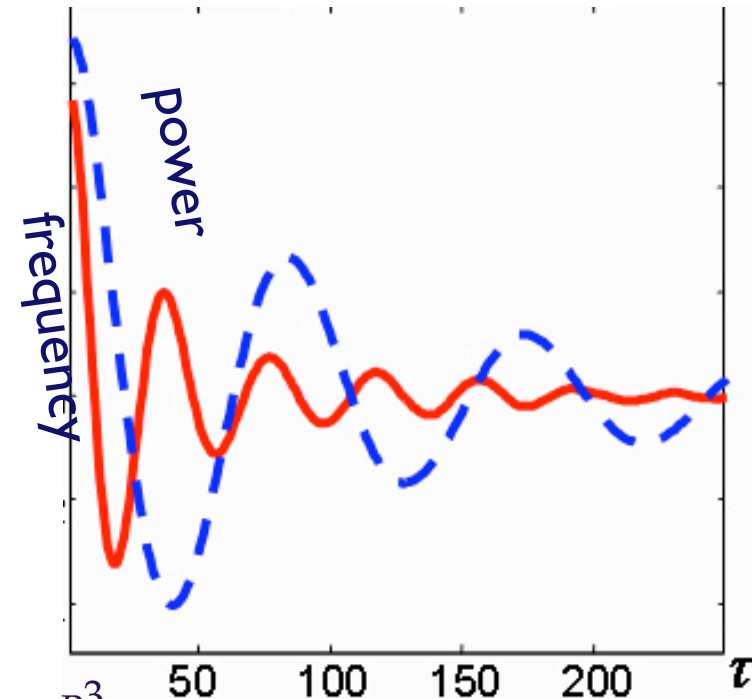
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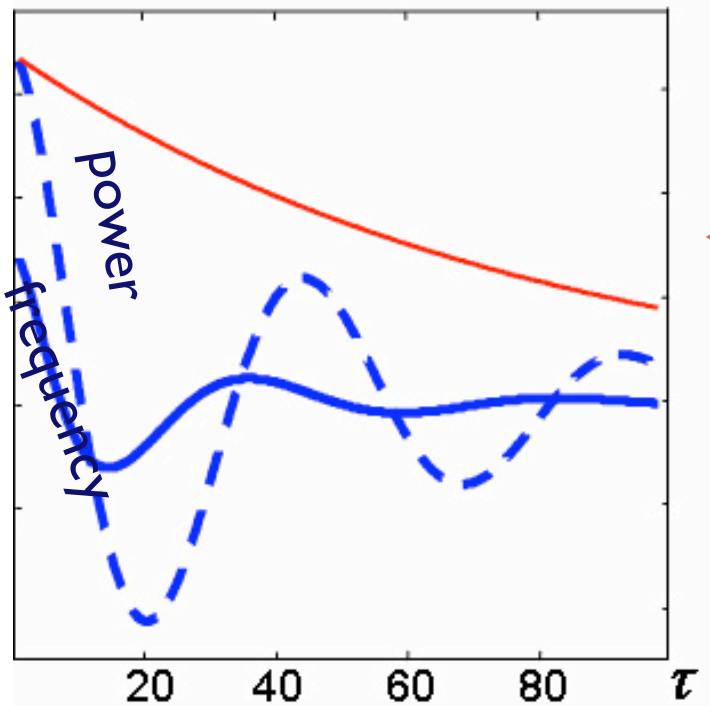
Theory
Numerics
Alternative
parameter
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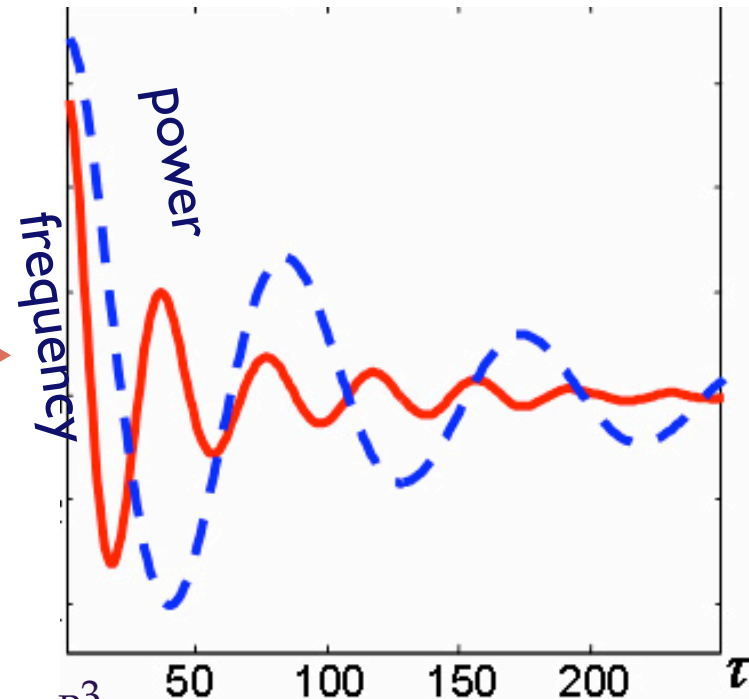
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← Theory

Numerics →

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Pulse
shaping rate

Nonlinear
phase shift

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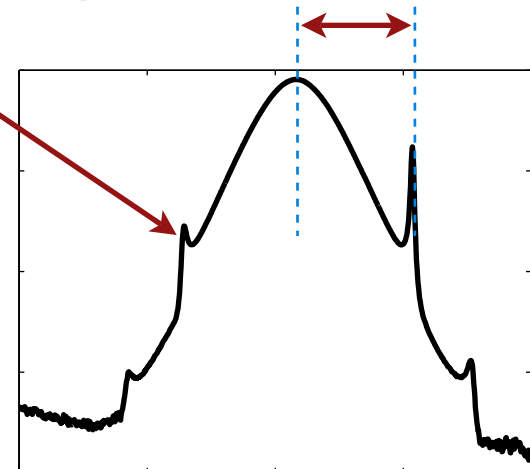
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- Compare with *pulse-generated continuum*: Kelly sidebands



Results: Diffusion

Parameter equations

$$\partial_z p = -\mu \frac{p_0^3}{4\mathcal{P}} p - \sqrt{\mu} f_p$$

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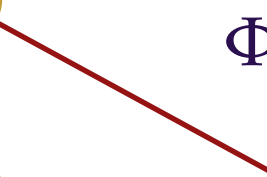
$$\partial_z c = \sqrt{\mu} f_c$$

Pulse timing & phase

$$C = c - \int V dz$$

$$\Phi = \phi + \frac{1}{4} V t + \frac{1}{8} \int (P^2 + V^2) dz$$

Direct
diffusion



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- Timing jitter $\langle C(z)^2 \rangle = 12 \frac{T}{P_0^3} z$ = Haus-Mecozzi jitter

- Phase jitter $\langle \Phi(z)^2 \rangle = \frac{T P^2}{P_0^3} z$ enhanced by $\left(\frac{P}{P_0} \right)^2$

Diffusion constants

Summary & conclusions

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- Laser noise has dual effect
 1. Steady-state linear continuum
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- Restoring terms suppressed by continuum:
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Outlook

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Thank you!