

Anderson Localization for the Nonlinear Schrödinger Equation (NLSE): Results and Puzzles

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and SF

Experimental Relevance

Nonlinear Optics

Bose Einstein Condensates (BECs)

Competition between randomness and nonlinearity

The Nonlinear Schroedinger (NLS) Equation

$$i \frac{\partial}{\partial t} \psi = \mathcal{H}_0 \psi + \beta |\psi|^2 \psi$$

1D lattice version

$$\mathcal{H}_0 \psi(x) = -(\psi(x+1) + \psi(x-1)) + \varepsilon(x) \psi(x)$$

1D continuum version

$$\mathcal{H}_0 \psi(x) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \psi(x) + \varepsilon(x) \psi(x)$$

V

random

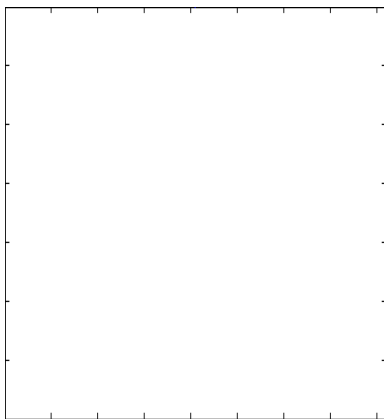


\mathcal{H}_0

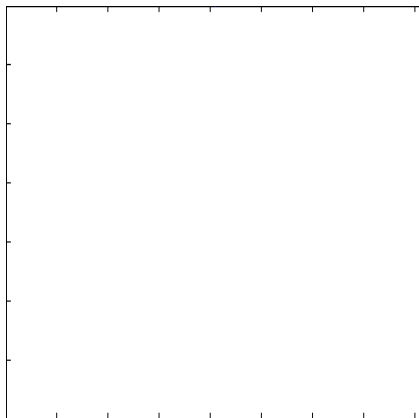
Anderson Model

$$i \frac{\partial \psi(x)}{\partial t} = -(\psi(x+1) + \psi(x-1)) + \varepsilon(x)\psi(x) + \beta |\psi(x)|^2 \psi(x)$$

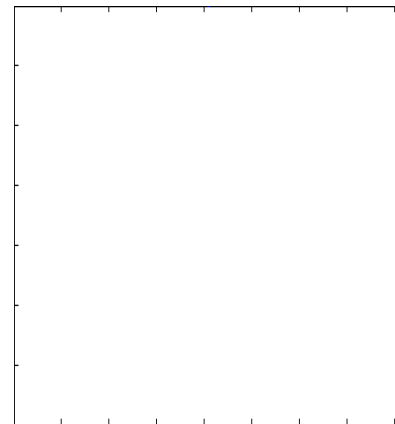
$$\beta = 0 \quad \varepsilon_n = 0$$



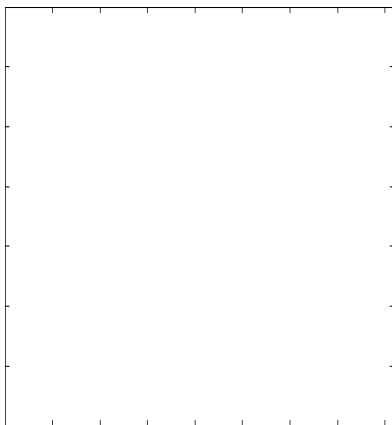
$$\beta = 1 \quad \varepsilon_n = 0$$



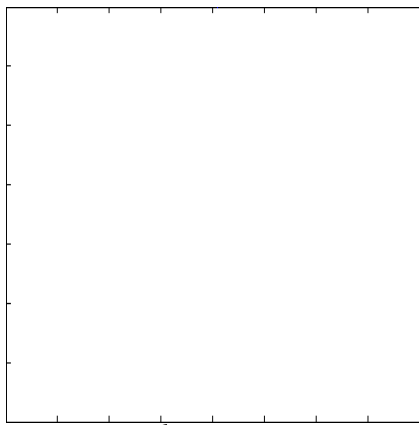
$$\beta = -1 \quad \varepsilon_n = 0$$



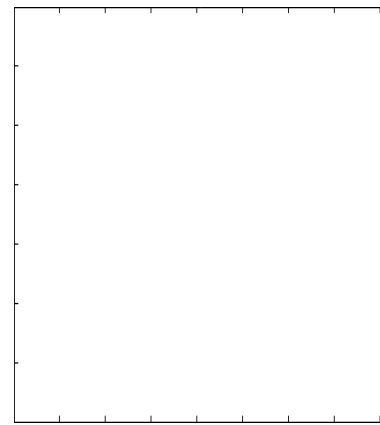
$$\beta = 0 \quad \varepsilon_n \neq 0$$



$$\beta = 1 \quad \varepsilon_n \neq 0$$

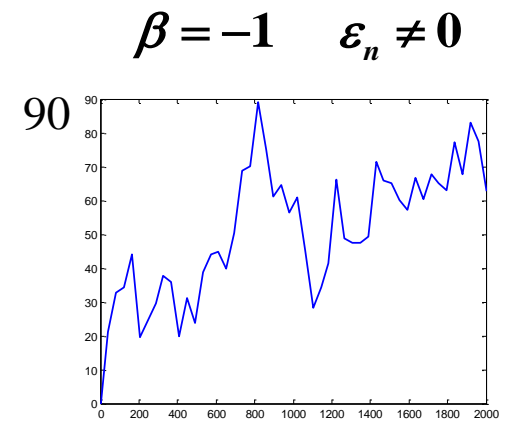
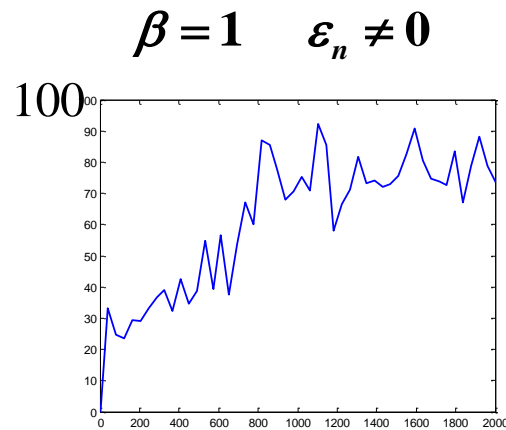
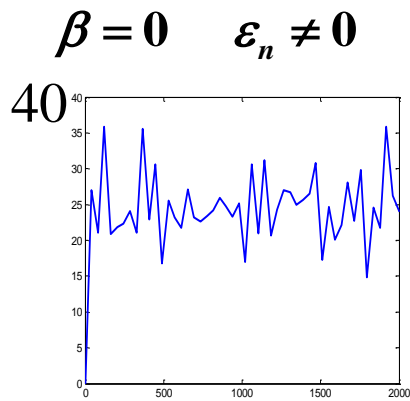
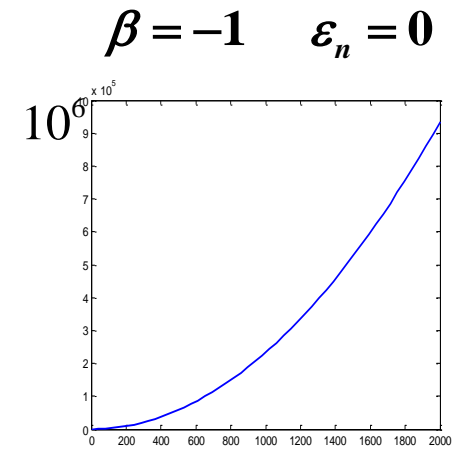
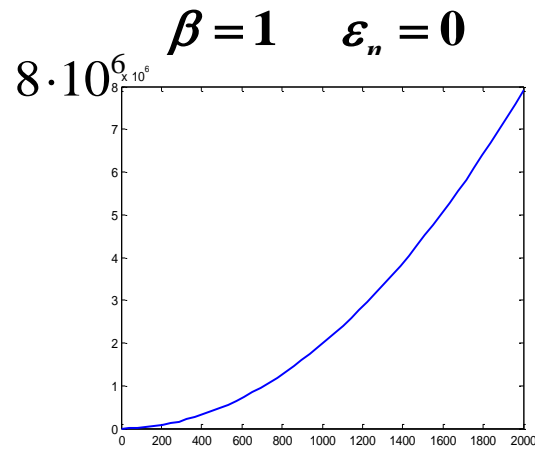
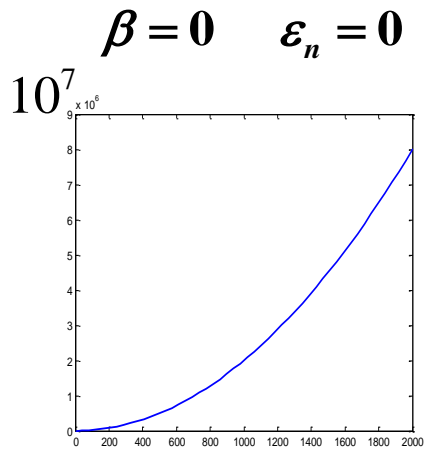


$$\beta = -1 \quad \varepsilon_n \neq 0$$



$$\xi \approx 6$$

$$m_2(t) = \sum_n x^2 |\psi(x)|^2$$



$\beta = 0 \Rightarrow$ localization

Does Localization Survive the
Nonlinearity???

Does Localization Survive the Nonlinearity???

- **Yes**, if there is spreading the magnitude of the nonlinear term decreases and localization takes over.
- **No**, assume wave-packet width is Δx then the relevant energy spacing is $1/\Delta x$ the perturbation because of the nonlinear term is $\beta|\psi|^2 \approx \beta/\Delta x$ and all depends on β
- **No**, but does not depend on β
- **No**, but it depends on realizations

Does Localization Survive the Nonlinearity?

- **No**, the NLSE is a chaotic dynamical system, **will it remain chaotic for all densities??**
- **No**, but localization asymptotically preserved beyond some front that is logarithmic in time

Numerical Simulations

- In regimes relevant for experiments looks that localization takes place
- Spreading for long time (Shepelyansky, Pikovsky, Molina, Kopidakis, Komineas, Flach, Aubry)
- We do not know the relevant space and time scales
- All results in Split-Step
- No control (but may be correct in some range)
- Supported by various heuristic arguments

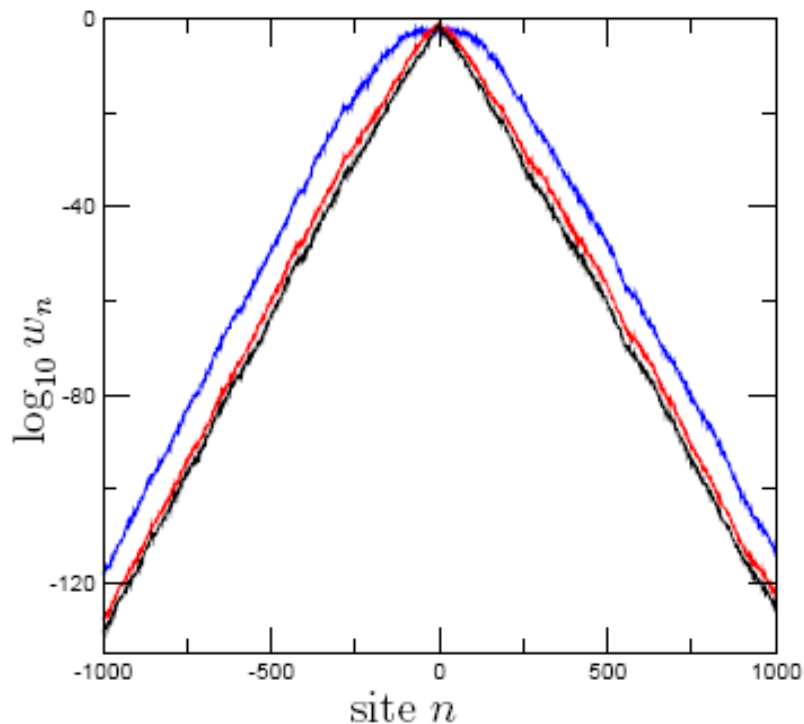


FIG. 2: (color online) Probability distribution w_n over lattice sites n at $W = 4$ for $\beta = 1, t = 10^8$ (top blue/solid curve) and $t = 10^5$ (middle red/gray curve); $\beta = 0, t = 10^5$ (bottom black curve; the order of the curves is given at $n = 500$). At $\beta = 0$ a fit $\ln w_n = -(\gamma|n| + \chi)$ gives $\gamma \approx 0.3, \chi \approx 4$. The values of $\log_{10} w_n$ are averaged over the same disorder realizations as in Fig. 1.

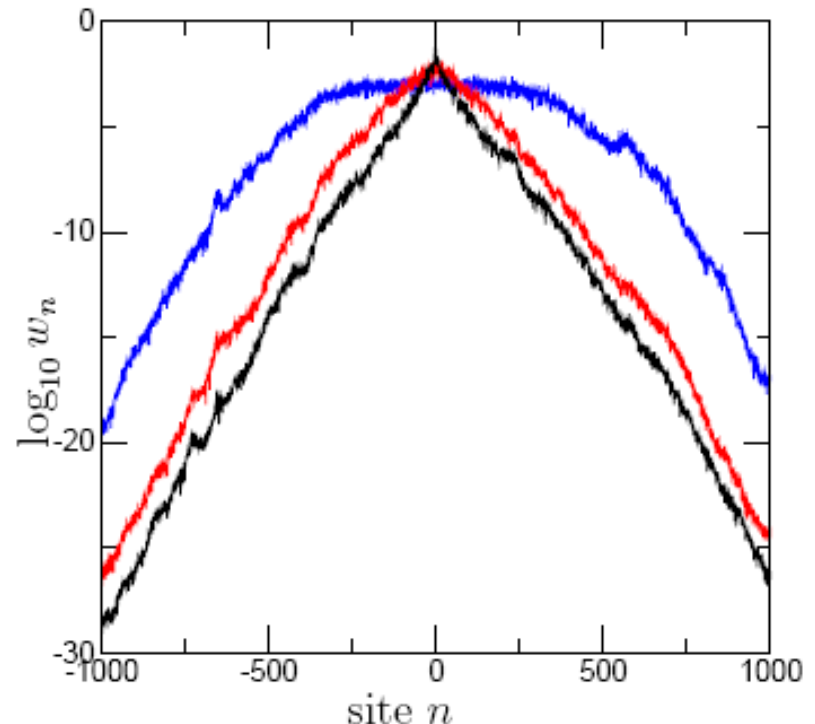


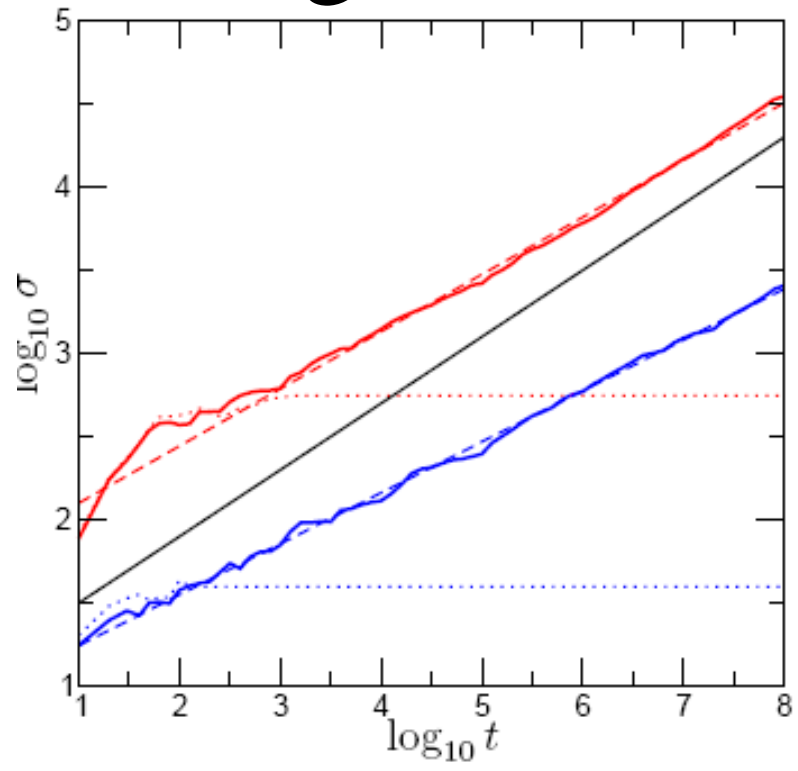
FIG. 3: (color online) Same as in Fig. 2 but with $W = 2$. At $\beta = 0$ a fit $\ln w_n = -(\gamma|n| + \chi)$ gives $\gamma \approx 0.06, \chi \approx -3$. The values of $\ln w_n$ are averaged over the same disorder realizations as in Fig. 1.

spreading sets on. A similar transition occurs for $W = 9$

Slope does not change (contrary to Fermi-Ulam-Pasta)

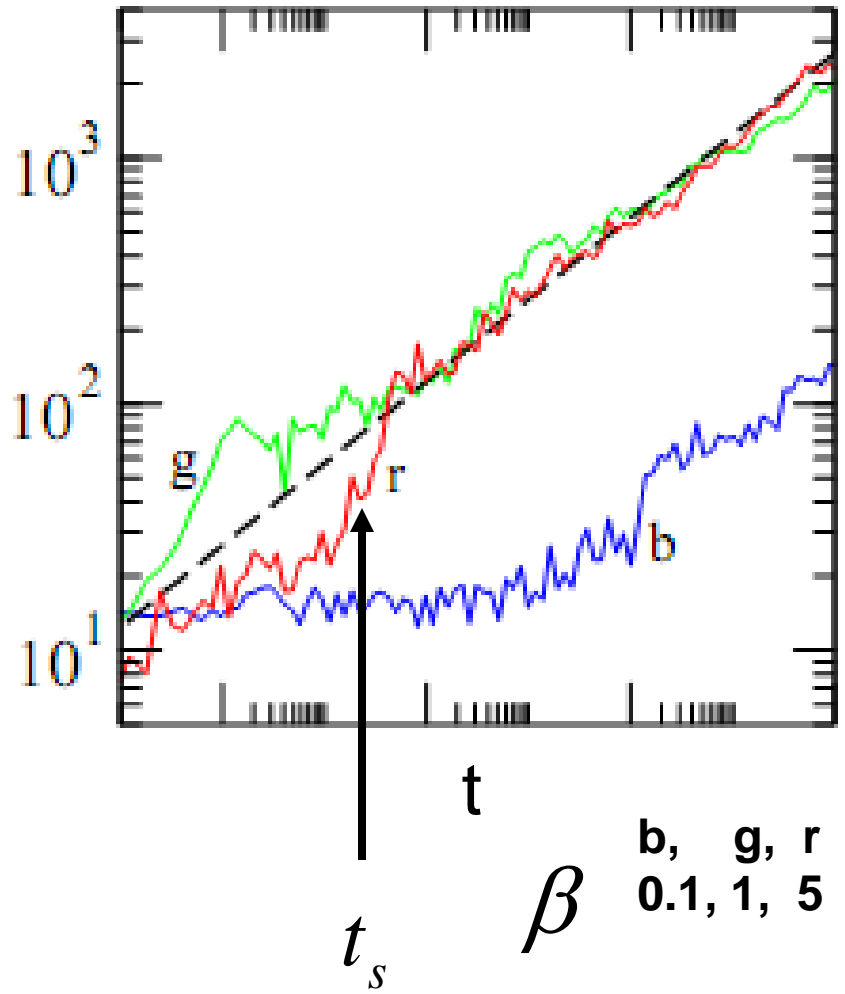
Pikovsky, Shepelyansky

$$\log \langle x^2 \rangle$$



S.Flach, D.Krimer and S.Skokos

$$\log \langle x^2 \rangle$$



Effective Noise Theories

- D. Shepeyansky and A. Pikovsky
- Ch. Skokos, D.O. Krimer, S. Komineas and S. Flach

$$\psi(x, t) = \sum_m c_m(t) e^{-iE_m t} u_m(x)$$

$$i \frac{\partial}{\partial t} c_n = \beta \sum_{m_1, m_2, m_3} V_n^{m_1, m_2, m_3} c_{m_1}^* c_{m_2} c_{m_3} e^{i(E_n + E_{m_1} - E_{m_2} - E_{m_3})t}$$

Overlap
$$V_n^{m_1, m_2, m_3} = \sum_x u_n(x) u_{m_1}(x) u_{m_2}(x) u_{m_3}(x)$$

$$\left| V_n^{m_1 m_2 m_3} \right| \leq [\text{const}] e^{-\frac{1}{3}\gamma(|x_n - x_{m_1}| + |x_n - x_{m_2}| + |x_n - x_{m_3}|)}$$

of the range of the localization length ξ

$$i \frac{\partial}{\partial t} c_n = \beta \sum_{m_1, m_2, m_3} V_n^{m_1, m_2, m_3} c_{m_1}^* c_{m_2} c_{m_3} e^{i(E_n + E_{m_1} - E_{m_2} - E_{m_3})t}$$

Assume $|c_{m_1}^2| \approx |c_{m_2}^2| \approx |c_{m_3}^2| \approx \rho$ initially $|c_n^2| \ll \rho$

$$i \frac{\partial}{\partial t} c_n \approx P \beta \rho^{3/2} f(t) \quad f(t) \quad \text{Random uncorrelated}$$

$$\langle x^2 \rangle \propto t^{1/3}$$

Scaling Properties of Chaos

Arkady Pikovsky

$$i \frac{d}{dt} \psi(x) = -J (\psi(x+1) + \psi(x-1)) + \varepsilon(x) \psi(x) + |\psi(x)|^2 \psi(x)$$

x integer $1 \leq x \leq L$

$\psi(x)$ Are dynamical variables

Initial data, nearly homogeneous spreading in space

Growth of deviations

$$\delta\psi(t) \approx \delta\psi(t=0) e^{\lambda t} \quad \lambda \text{ Largest Lyapunov exponent}$$

$\lambda > 0 \longrightarrow$ Chaos

Is it possible that chaos disappears?

Divide chain into intervals of length L_0 Number of intervals $\frac{L}{L_0}$

Assuming independence, if intervals large enough $L_0 \gg \xi$
 The probability to be regular:

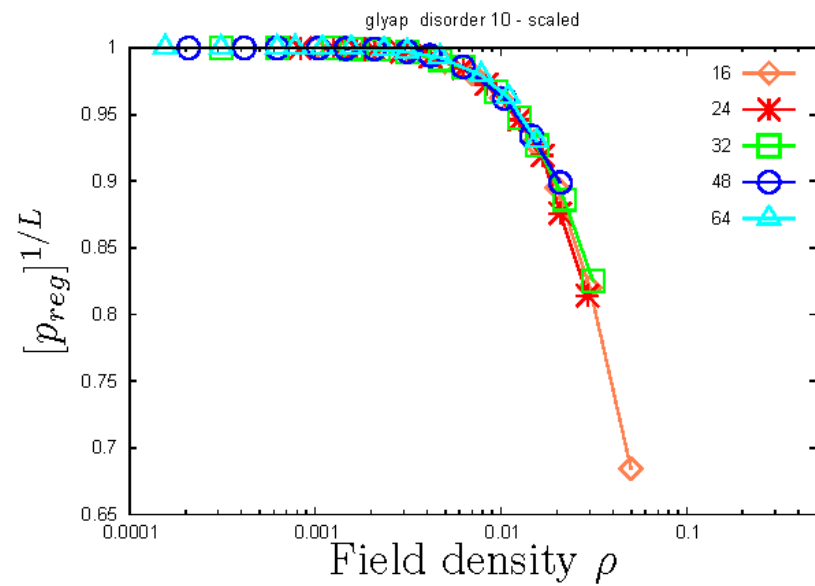
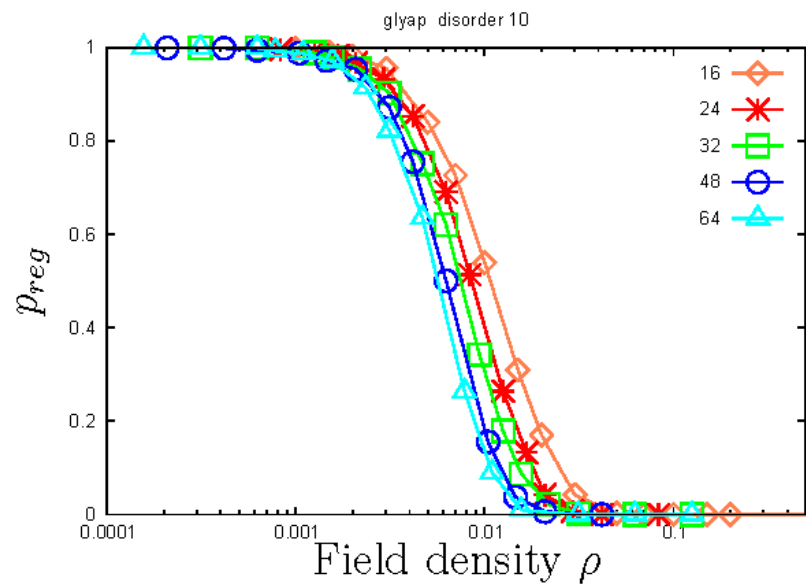
$$p_{reg}(W, \rho, L) = p_{reg}(W, \rho, L_0)^{L/L_0}$$

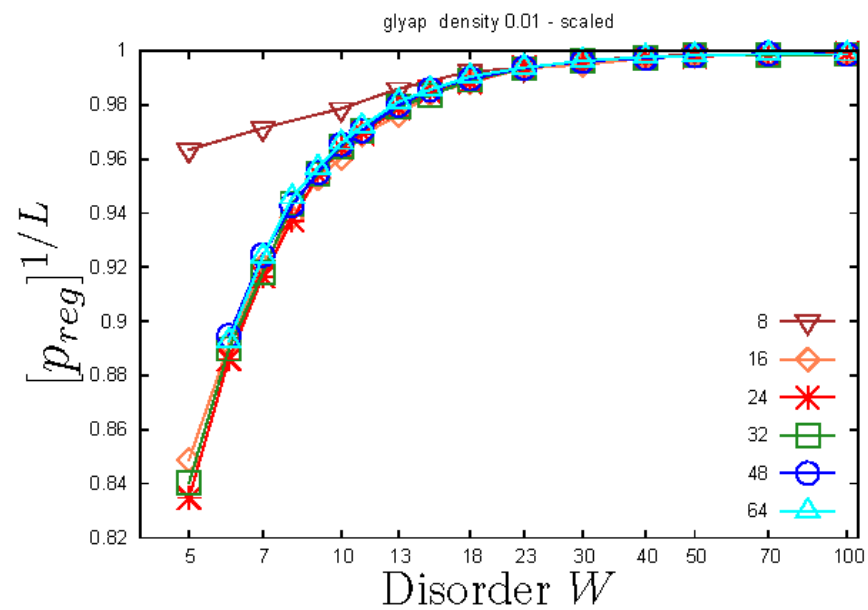
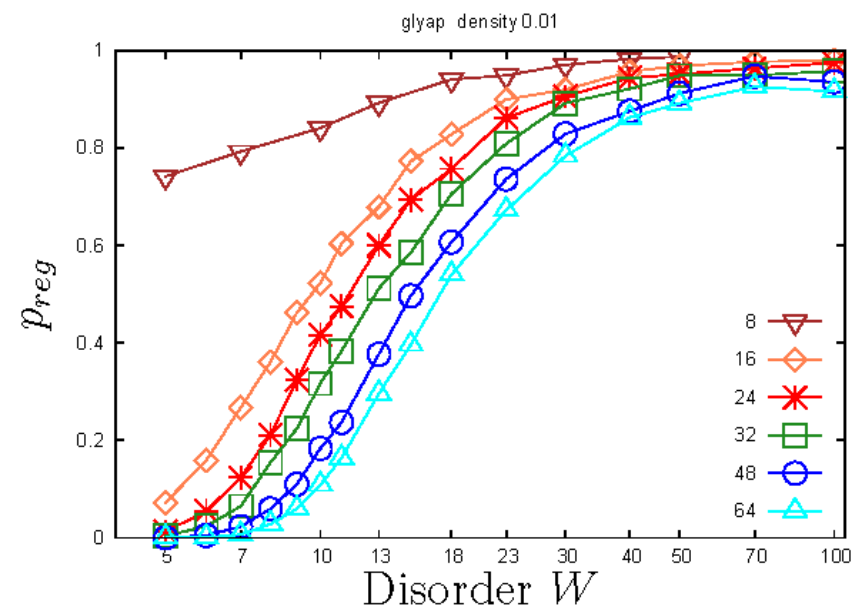
Regularity=all orbits regular

density $\rho = \frac{1}{L} \sum_{x=1}^L |\psi(x)|^2$

$$\bar{p}_{reg}(W, \rho) \equiv p_{reg}(W, \rho, L)^{1/L} = p_{reg}(W, \rho, L_0)^{1/L_0}$$

independent of L





Scaling


Define $Q \equiv \frac{\bar{p}_{reg}}{1 - \bar{p}_{reg}} \rightarrow \bar{p}_{reg} = \frac{1}{1 + 1/Q}$

Scaling function $Q = \frac{1}{W^a} q\left(\frac{\rho}{W^b}\right)$

$$\bar{p}_{reg} = \frac{1}{1 + 1/Q} \approx 1 - \frac{1}{Q} \approx 1 - \rho^{2.25}$$

$$\rho = \frac{1}{L} \sum_{x=1}^L |\psi(x)|^2$$

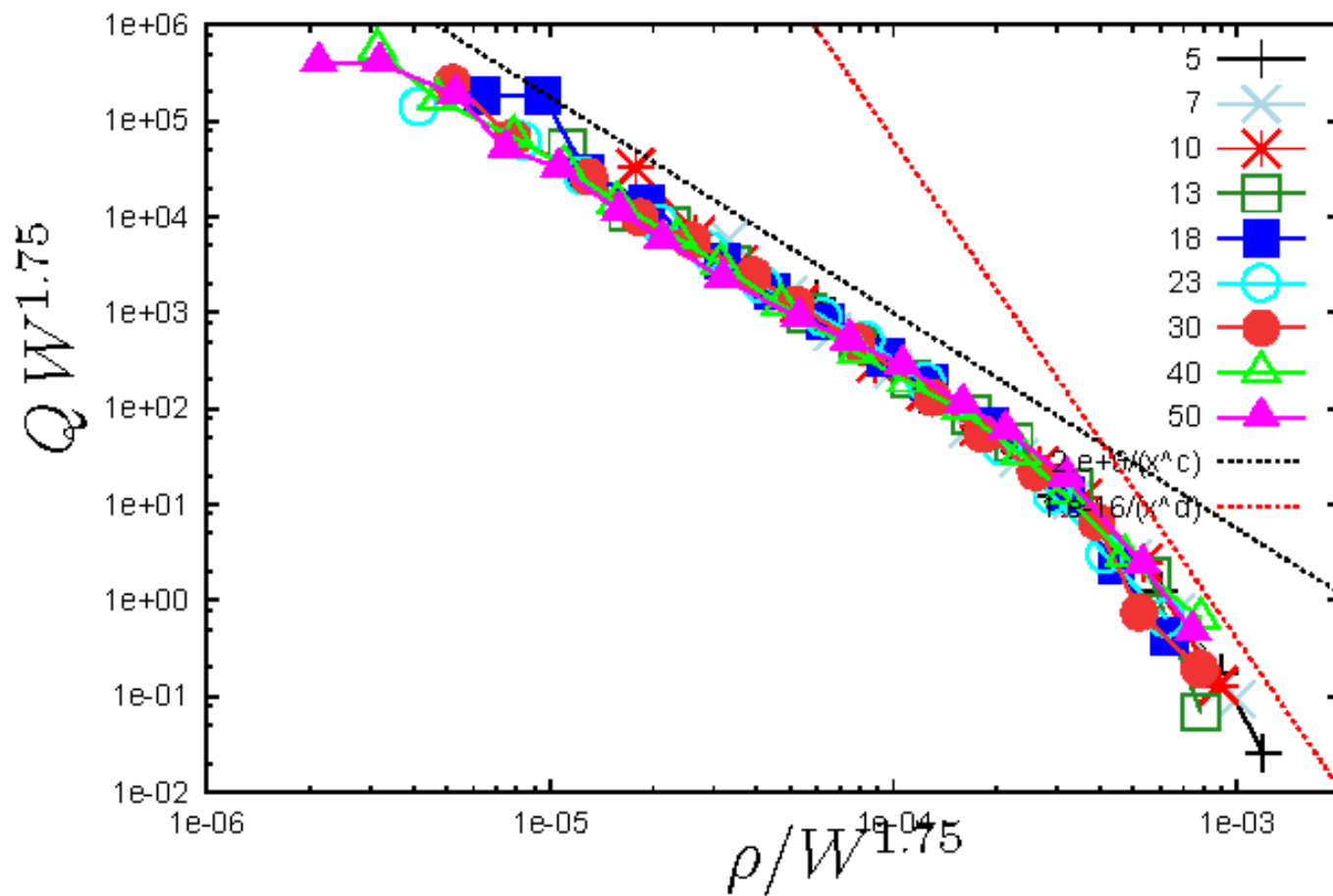
$$\sum_{x=1}^L |\psi(x)|^2 \quad \text{conserved}$$

$$p_{reg} = \bar{p}_{reg}^L \rightarrow \ln p_{reg} = L \ln \bar{p}_{reg} \square L \rho^{2.25} \square L^{-1.25}$$


In the limit $L \rightarrow \infty \quad \ln p_{reg} \rightarrow 0 \rightarrow p_{reg} = 1$

Crossover point $\rho \approx 0.005$

length 16 scaled a=1.75 b=1.75 c=2.25 d=5.2



Spreading



No chaos



Localization ?

Perturbation Theory

The nonlinear Schroedinger Equation on a Lattice in 1D

$$i \frac{\partial}{\partial t} \psi = \mathcal{H}_0 \psi + \beta |\psi|^2 \psi$$

$$\mathcal{H}_0 \psi(x) = -(\psi(x+1) + \psi(x-1)) + \varepsilon(x) \psi(x)$$

$$\varepsilon(x) \text{ random} \longrightarrow \mathcal{H}_0 \quad \text{Anderson Model}$$

Eigenstates

$$\mathcal{H}_0 u_m(x) = E_m u_m(x)$$

$$\psi(x, t) = \sum_m c_m(t) e^{-iE_m t} u_m(x)$$

Perturbation theory steps

- Expansion in nonlinearity
- Removal of secular terms
- Control of denominators
- Probabilistic bound on general term
- Control of remainder
- **Use perturbation theory to obtain a numerical solution that is controlled a posteriori**

$$i \frac{\partial}{\partial t} c_n = \beta \sum_{m_1, m_2, m_3} V_n^{m_1, m_2, m_3} c_{m_1}^* c_{m_2} c_{m_3} e^{i(E_n + E_{m_1} - E_{m_2} - E_{m_3})t}$$

Overlap $V_n^{m_1, m_2, m_3} = \sum_x u_n(x) u_{m_1}(x) u_{m_2}(x) u_{m_3}(x)$

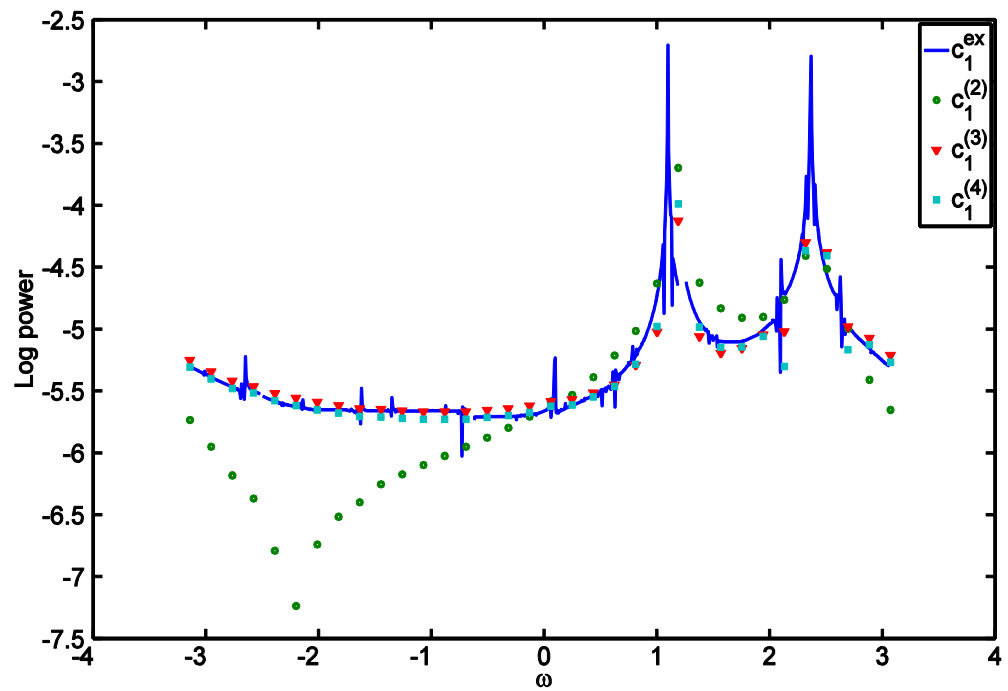
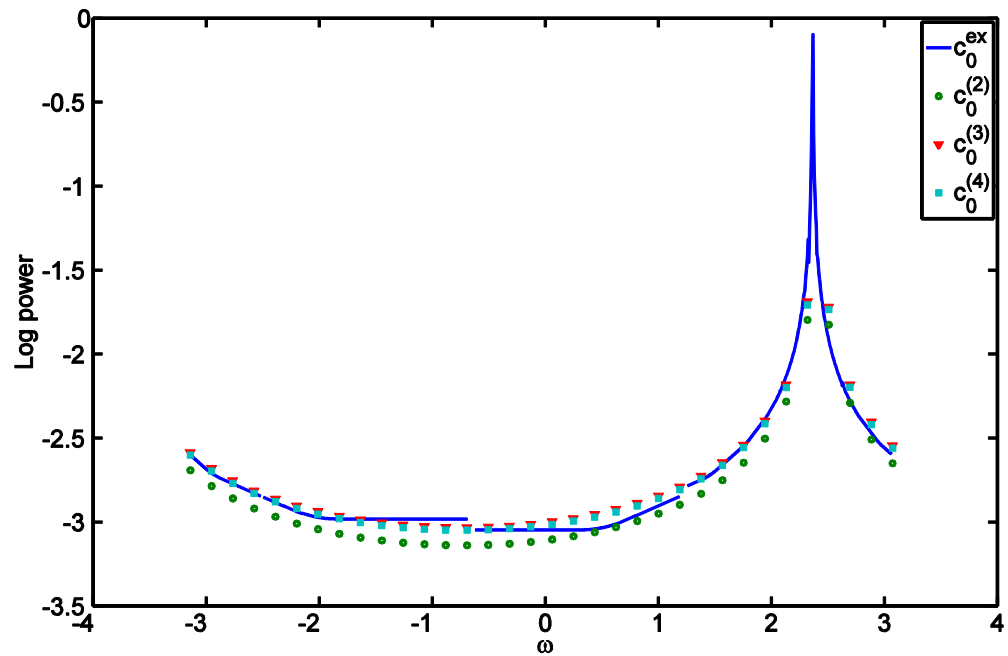
of the range of the localization length ξ

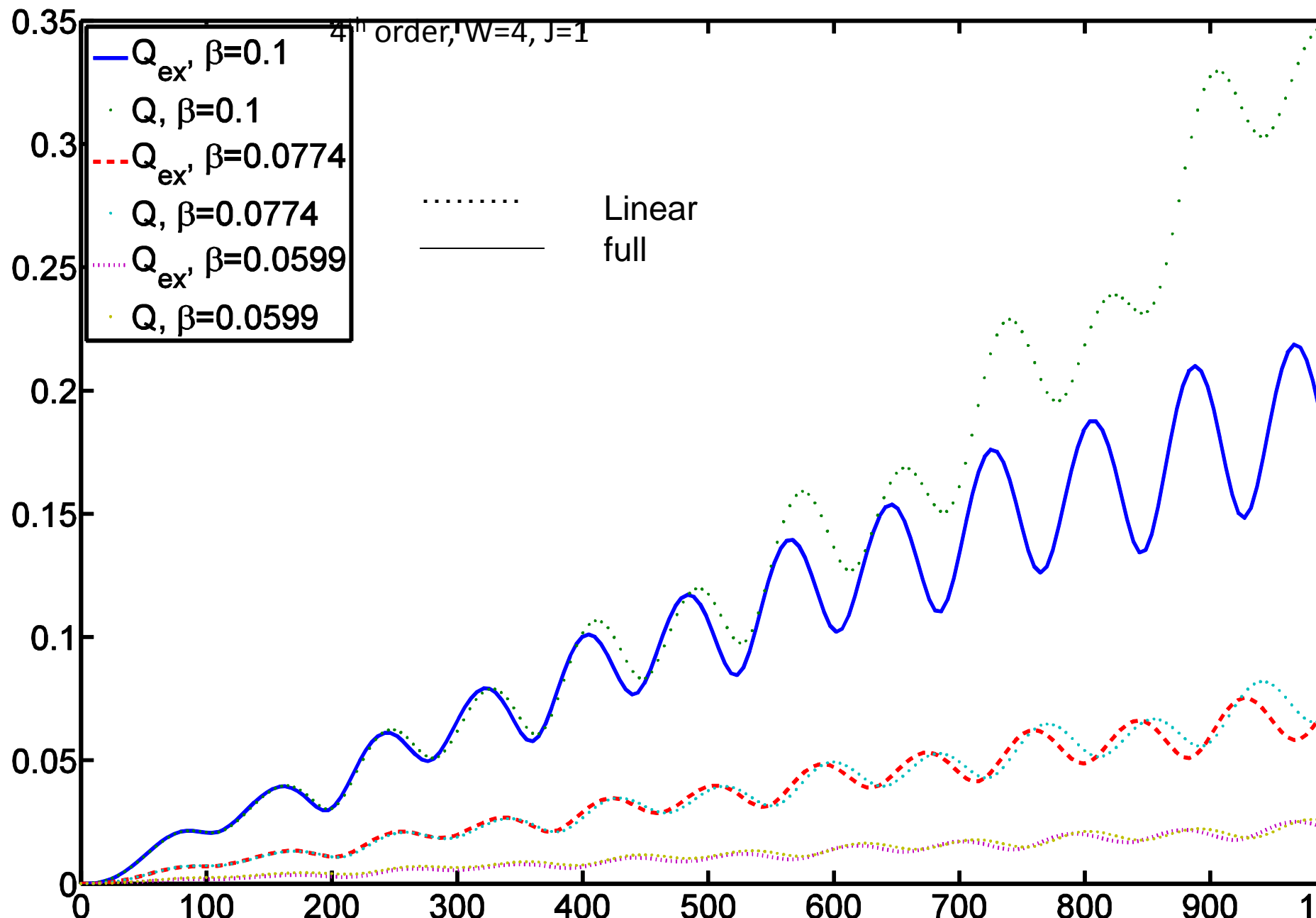
perturbation expansion

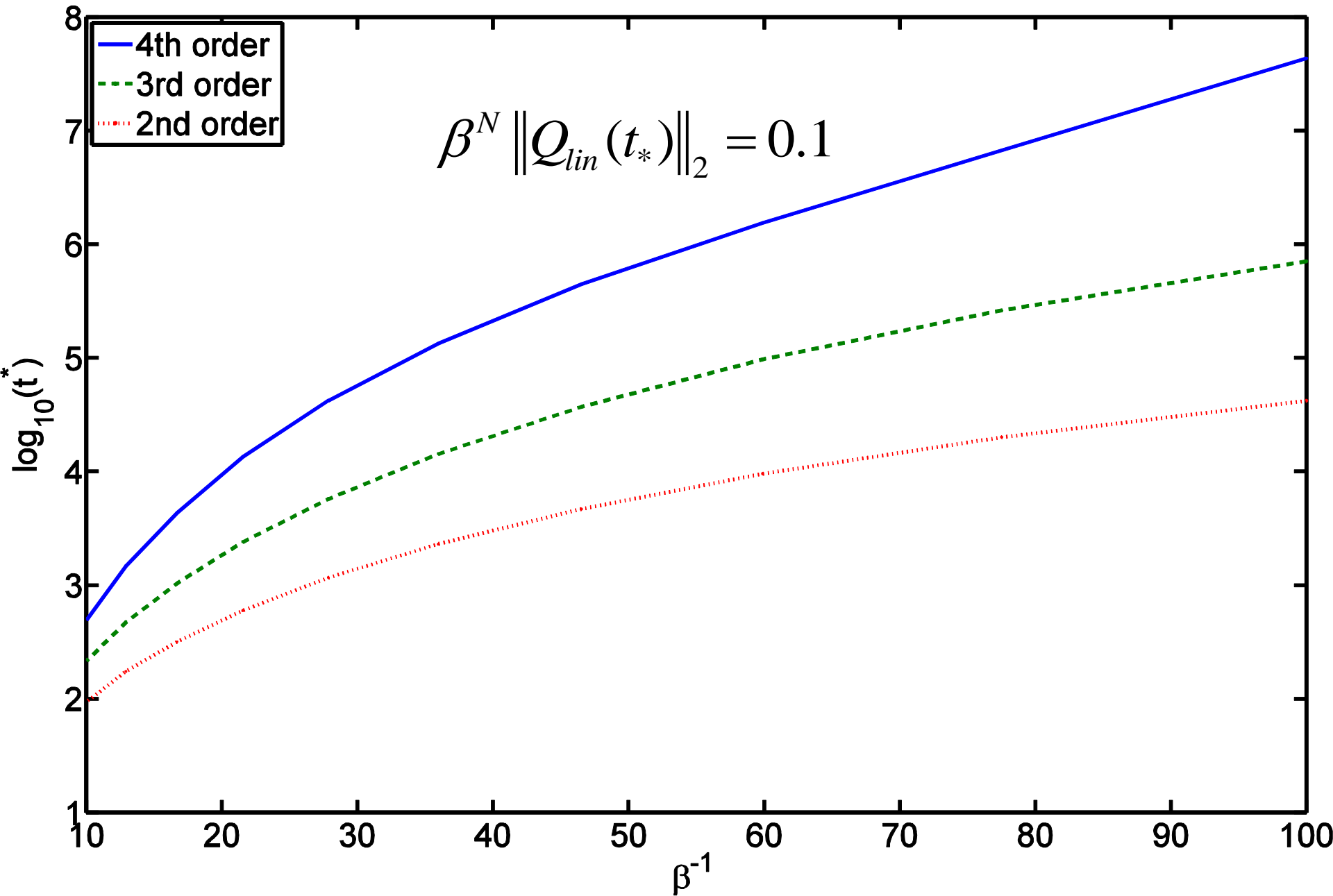
$$c_n(t) = c_n^{(0)} + \beta c_n^{(1)} + \beta^2 c_n^{(2)} + \dots + \beta^{N-1} c_n^{(N-1)} + \beta^N Q_N(n)$$

Iterative calculation of $c_n^{(l)}$

start at $c_n^{(0)} = c_n(t=0) = \delta_{n0}$







The Bound on the remainder

$$\left| \beta^N Q_N(n) \right| \leq A(N, \gamma) \beta^N t e^{-\gamma |x_n|} = A e^{(\ln t - N |\ln \beta| - \gamma |x_n|)}$$

x_n Localization center of state n

For fixed order and time

$$\lim_{\beta \rightarrow 0} \frac{\beta^N Q_N(n)}{\beta^{N-1}} = 0 \quad \text{Expansion Asymptotic}$$

One can show that for strong disorder $A(N, \gamma) \xrightarrow{\gamma \rightarrow \infty} 0$

Looks that $A \propto \exp(-\gamma)$

Difficulties in the calculation of A

Front logarithmic in time $\bar{x} \propto \frac{1}{\gamma} \ln t$ For limited time

Localization for $|x| > \bar{x}$

Bound on error

- Solve linear equation for the remainder of order N
- If bounded to time t_0 perturbation theory accurate to that time.
- Order of magnitude estimate $\beta^N t_0 \ll 1$ if asymptotic $\beta^N N! \ll 1$ hence $t_0 \ll N!$ for optimal order (up to constants).
- $t_0 = \beta^{-1/\beta}$ validity time of perturbation theory

Summary Perturbation Theory

1. A perturbation expansion in β was developed
2. Secular terms were removed
3. A bound on the general term was derived
4. Perturbation theory was used to obtain a controlled numerical solution
5. A bound on the remainder was obtained, indicating that the series is asymptotic.
6. For limited time tending to infinity for small nonlinearity, front logarithmic in time $\bar{x} \propto \ln t$
7. Improved for strong disorder

Emerging Picture

- For small nonlinearity initially no spreading
- For strong nonlinearity some part does not spread
- For some nonlinearity wide regime of sub-diffusion
- Asymptotic spreading at most logarithmic:
 - a. perturbation theory
 - b. rigorous results in the limit of strong disorder
- Unlikely that sub-diffusion continues forever:
 - a. scaling theory
 - b. Effective noise “theories”

Coherent picture for various regimes?