Quasi-stationary-states in Hamiltonian mean-field dynamics

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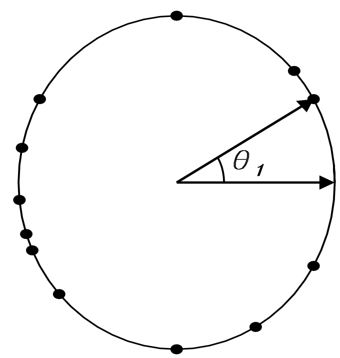
Statistical Mechanics Day III, The Weizmann Institute of Science, June 16, 2010

Plan

- Hamiltonian Mean Field (HMF) model
- Quasi-stationary-states (QSS)
- Klimontovich and Vlasov equations
- Lynden-Bell theory
- Non equilibrium phase transition
- Mean-field equilibria
- Application to the free electron laser

HMF model

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} \left(1 - \cos(\theta_i - \theta_j)\right)$$



Magnetization $\mathbf{M} = \lim_{N\to\infty} \left(\frac{\sum_{i=1}^{N} \cos\theta_i}{N}, \frac{\sum_{i=1}^{N} \sin\theta_i}{N} \right) = (M_x, M_y)$

Energy $U = \lim_{N\to\infty} \frac{H}{N}$

Equations of motion

$$\begin{aligned}\dot{\theta}_i &= \frac{\partial H}{\partial p_i} = p_i \\ \dot{p}_i &= -\frac{\partial H}{\partial \theta_i} = -\frac{1}{N}\sum_{j=1}^{N}\sin(\theta_i - \theta_j)\end{aligned}$$

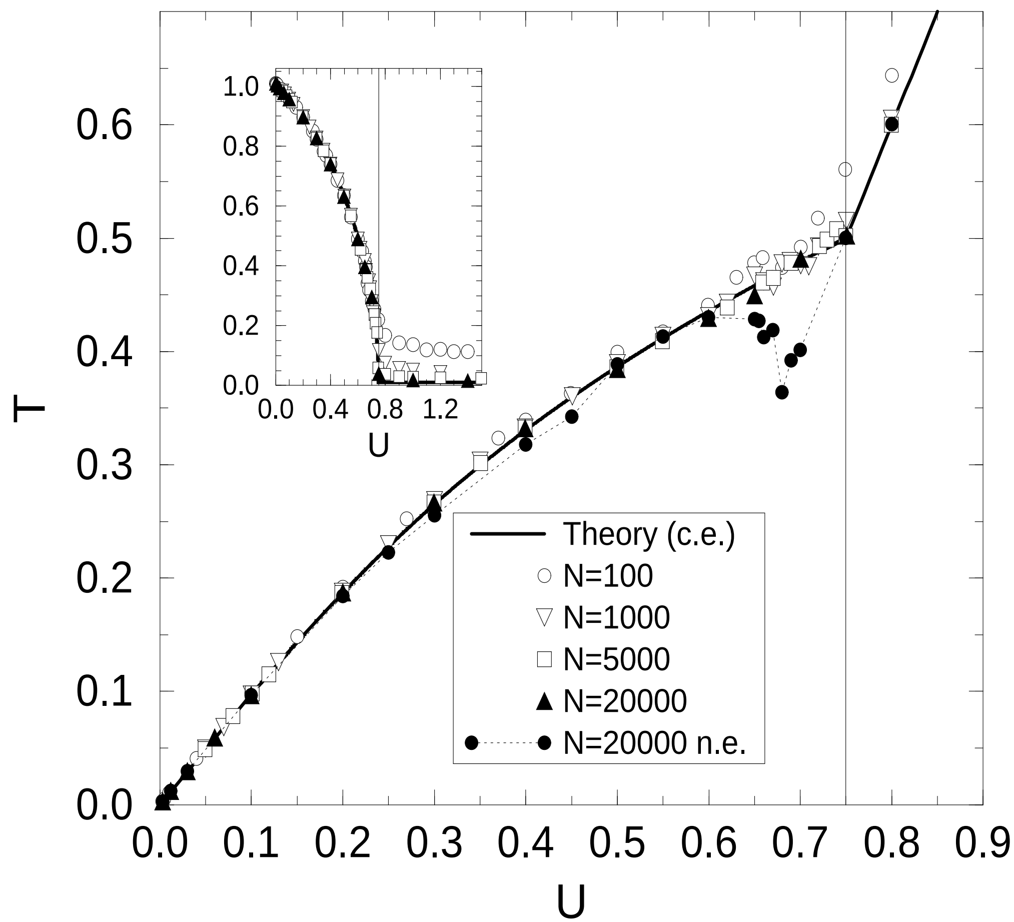
$$\begin{aligned}\dot{\theta}_i &= p_i \\ \dot{p}_i &= -M\sin(\theta_i - \phi)\end{aligned}$$

$\tan(\phi) = \frac{M_y}{M_x}$

Equilibrium phase transition

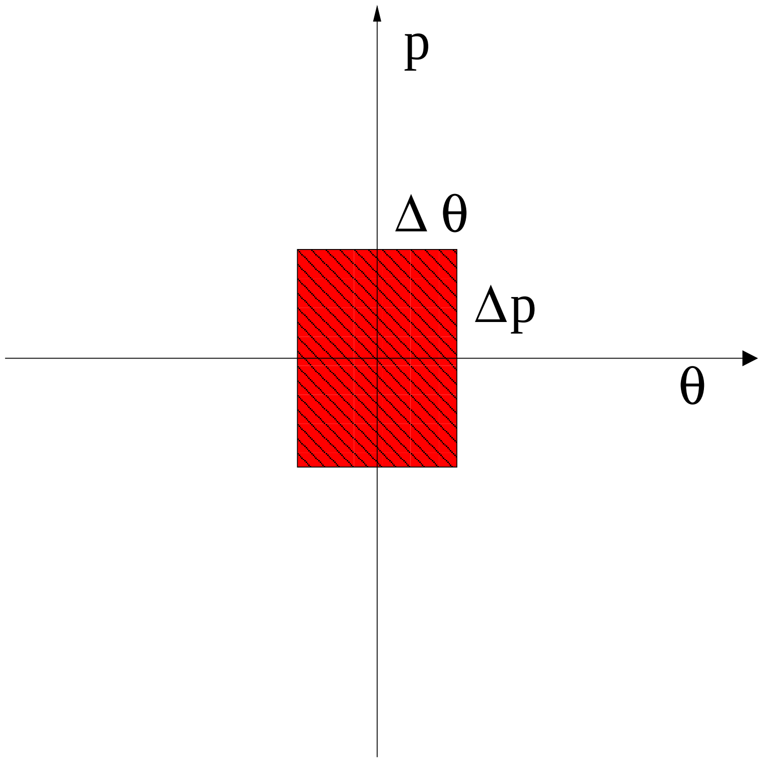
Fig. 1 revised

Latora, Rapisarda & Ruffo − Lyapunov instability and finite size...



$$U_c = 3/4 = 0.75,\, T = (\partial S/\partial U)^{-1} = \lim_{N\to\infty} \sum_i p_i^2/N$$

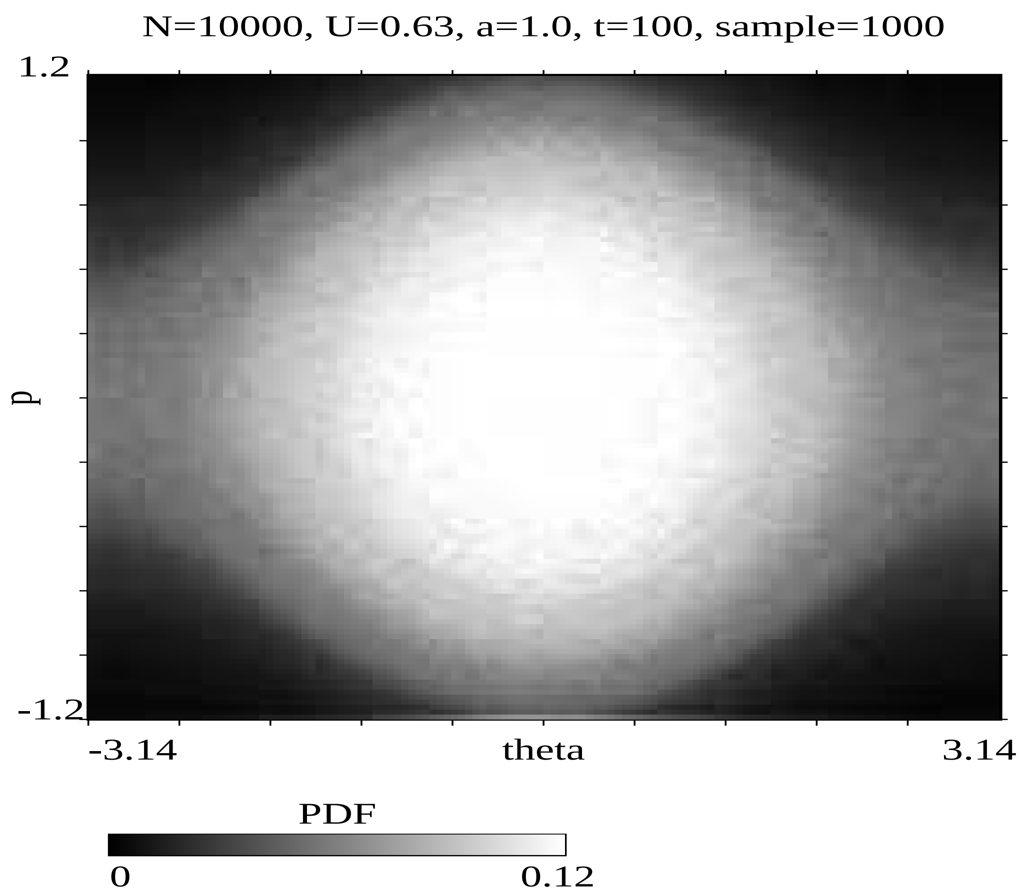
Waterbag



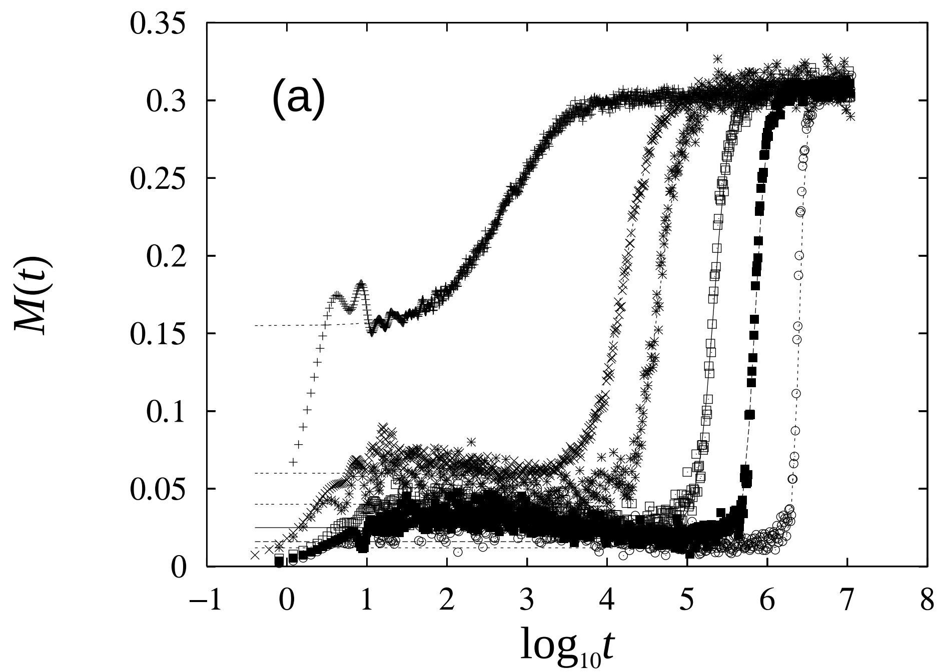
$$M_0 = \frac{\sin \Delta\theta}{\Delta\theta}$$

$$U = \lim_{N\to\infty} \frac{H}{N} = \frac{(\Delta p)^2}{6} + \frac{1-(M_0)^2}{2}$$

Equilibrium

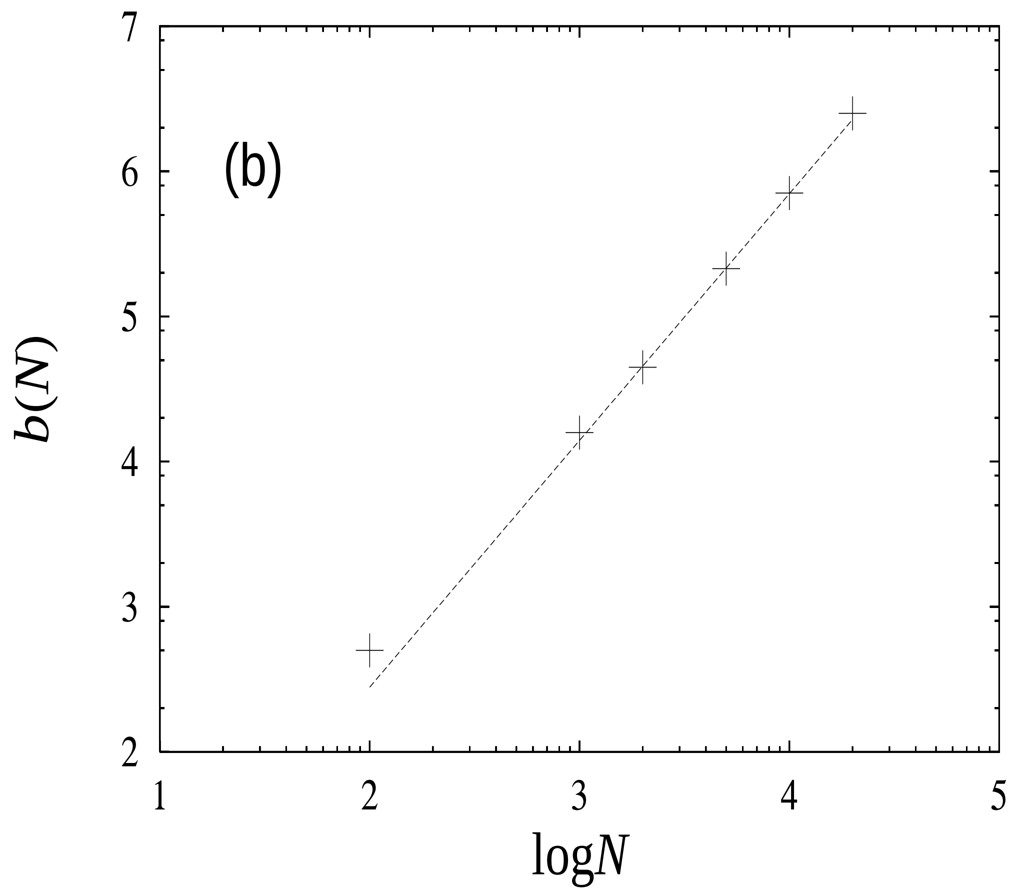


Quasi-stationary states



$U = 0.69$, from left to right
$N = 10^2, 10^3, 2\times10^3, 5\times10^3, 10^4, 2\times10^4$.
Initially $\Delta\theta = \pi$, hence $M_0 = 0$.

Power law



Power law increase of the lifetime, exponent 1.7

Separation of time scales

Initial Condition

Violent relaxation $\quad \tau_v = O(1)$

$\downarrow$

Vlasov's Equilibrium

Collisional relaxation $\quad \tau_c = N^{\delta}$

$\downarrow$

Boltzmann's Equilibrium

Klimontovich equation

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + U(\theta_i)$$

$$U(\theta_1, .., \theta_N) = \frac{1}{N} \sum_{i<j}^{N} V(\theta_i - \theta_j)$$

Discrete, one-particle, time dependent density function

$$f_d(\theta, p, t) = \frac{1}{N} \sum_{i=1}^{N} \delta(\theta - \theta_i(t))\delta(p - p_i(t))$$

Klimontovich equation

$$\frac{\partial f_d}{\partial t} + p\frac{\partial f_d}{\partial \theta} - \frac{\partial v}{\partial \theta}\frac{\partial f_d}{\partial p} = 0$$

where

$$v(\theta, t) = \int d\theta' dp' V(\theta - \theta') f_d(\theta', p', t)$$

From Klimontovich to Vlasov

$$f_d = \langle f_d(\theta, p, t) \rangle + \frac{1}{\sqrt{N}} \delta f(\theta, p, t) = f(\theta, p, t) + \frac{1}{\sqrt{N}} \delta f(\theta, p, t)$$

where the average $\langle \cdot \rangle$ is taken over initial conditions.

$$v(\theta, t) = \langle v \rangle(\theta, t) + \frac{1}{\sqrt{N}} \delta v(\theta, t)$$

where

$$\langle v \rangle(\theta, t) = \int d\theta' dp' V(\theta - \theta') f(\theta', p', t)$$

Using Klimontovic equation, one gets

$$\frac{\partial f}{\partial t} + p \frac{\partial f}{\partial \theta} - \frac{\partial \langle v \rangle}{\partial \theta} \frac{\partial f}{\partial p} = \frac{1}{N} \langle \frac{\partial \delta v}{\partial \theta} \frac{\partial \delta f}{\partial p} \rangle$$

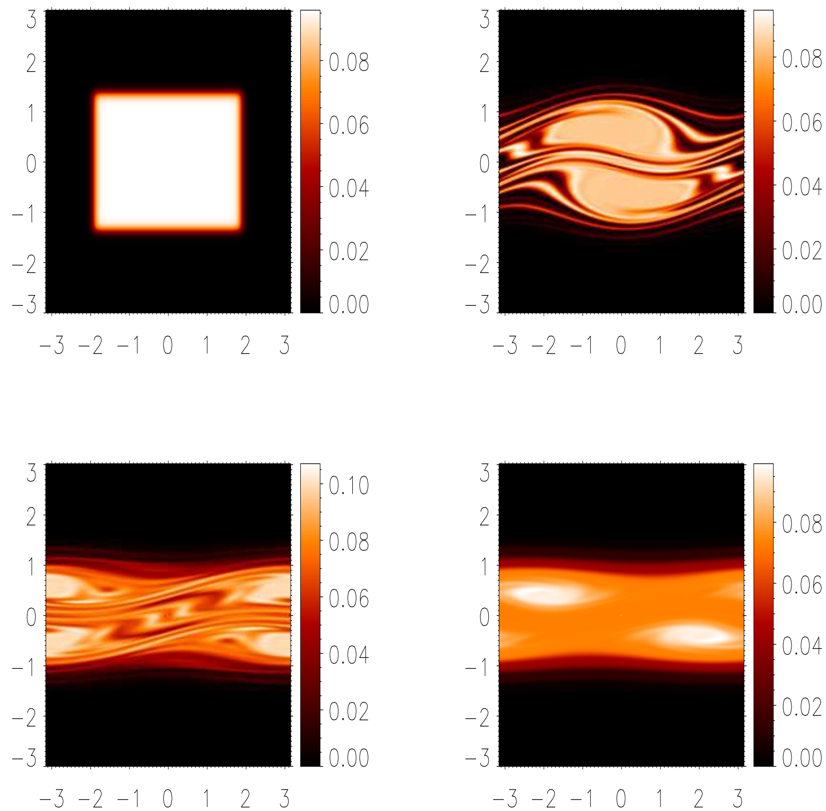
HMF Vlasov equation

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial \theta} - \frac{dV}{d\theta}\frac{\partial f}{\partial p} = 0 \quad ,$$

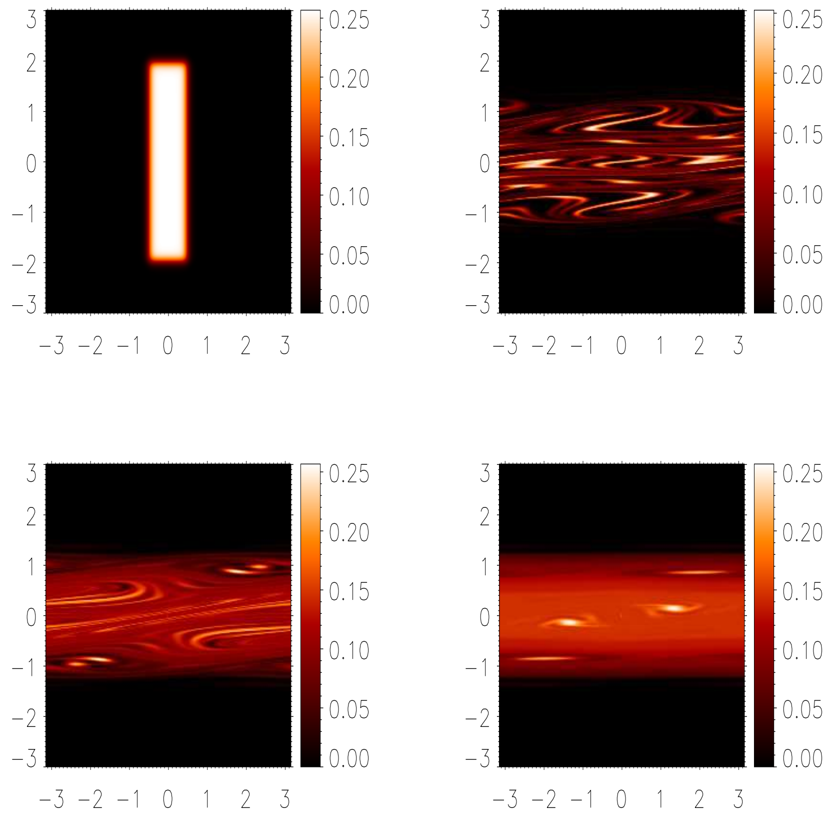
$$\begin{aligned}
V(\theta)[f] &= 1 - M_x[f]\cos(\theta) - M_y[f]\sin(\theta)\ , \\
M_x[f] &= \int f(\theta,p,t)\ \cos\theta \mathrm{d}\theta\mathrm{d}p \quad , \\
M_y[f] &= \int f(\theta,p,t)\ \sin\theta \mathrm{d}\theta\mathrm{d}p \quad .
\end{aligned}$$

Specific energy
$e[f] = \int (p^2/2) f(\theta,p,t)\mathrm{d}\theta\mathrm{d}p + 1/2 - (M_x^2 + M_y^2)/2$ and momentum $P[f] = \int p f(\theta,p,t)\mathrm{d}\theta\mathrm{d}p$ are conserved.

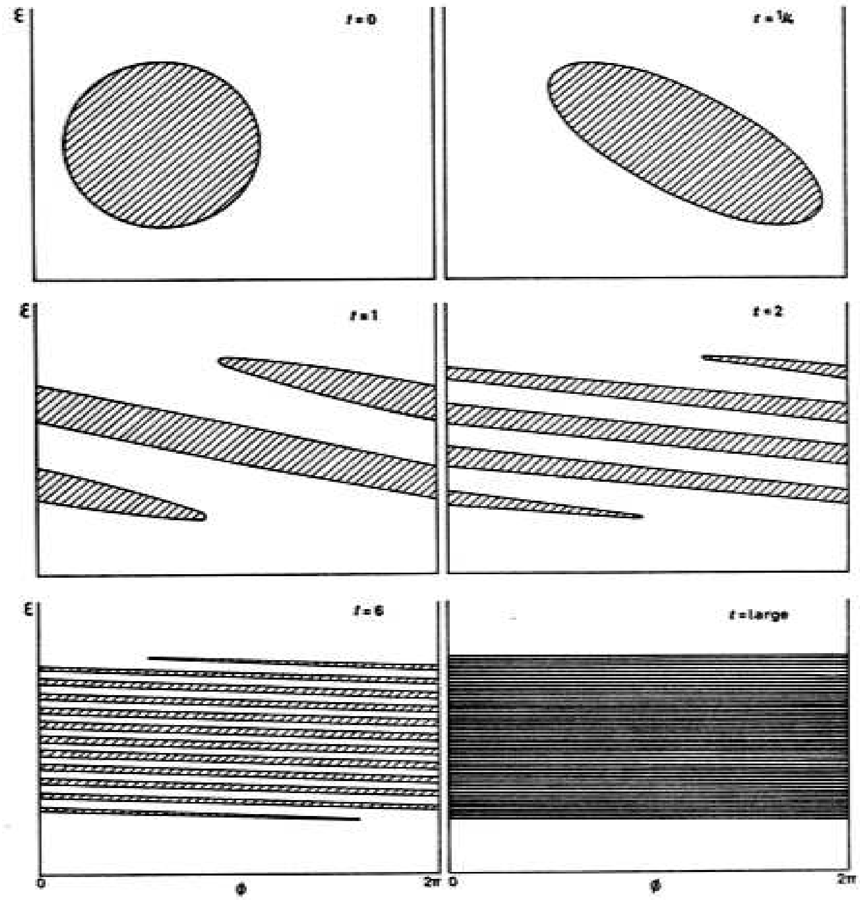
Vlasov fluid-I



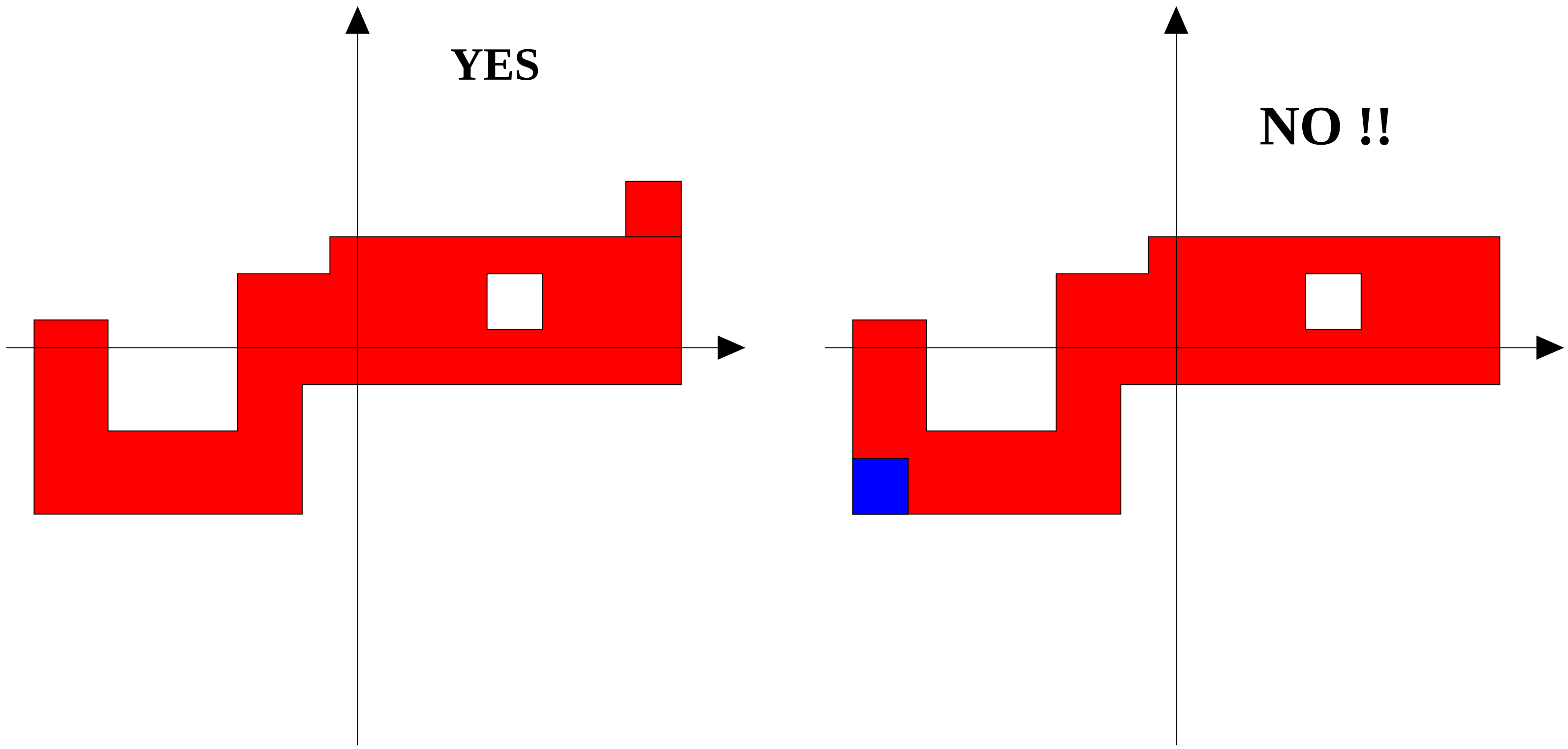
Vlasov fluid-II



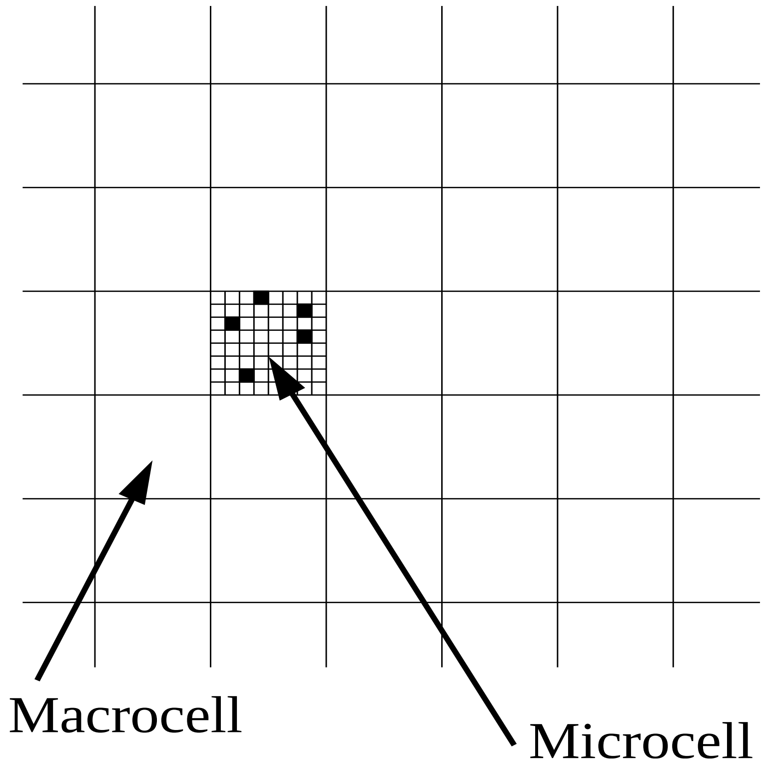
Stirring



Allowed moves



Coarse graining for two-level distr.



ν microcells of volume ω in a macrocell. A macroscopic configuration has n_i microcells occupied with level f_0 in the i-th macrocell (the remaining are occupied with level 0). The distribution is coarse-grained on a macrocell, $\bar{f}(\theta, p)$. The total number of occupied microcells is $\mathcal{N}$, such that the conserved total mass is $mass = \int f(\theta, p) d\theta dp = \mathcal{N} \omega f_0$

Lynden-Bell entropy

$$S_{LB}(\bar{f}) = -\frac{1}{\omega}\int \mathrm{d}p\mathrm{d}\theta \left[\frac{\bar{f}}{f_0}\ln\frac{\bar{f}}{f_0} + \left(1-\frac{\bar{f}}{f_0}\right)\ln\left(1-\frac{\bar{f}}{f_0}\right)\right].$$

$$\bar{f}(\theta,p) = \sum_{i=1}^{2} \rho(\theta,p,\eta_i)\,\eta_i \ ,$$

$\eta_1 = 0$, $\eta_2 = f_0$

$$W(\{n_i\}) = \frac{\mathcal{N}!}{\prod_i n_i!} \times \prod_i \frac{\nu!}{(\nu-n_i)!} \ .$$

$\rho_i(f_0) = n_i/\nu = \bar{f}_i/f_0$

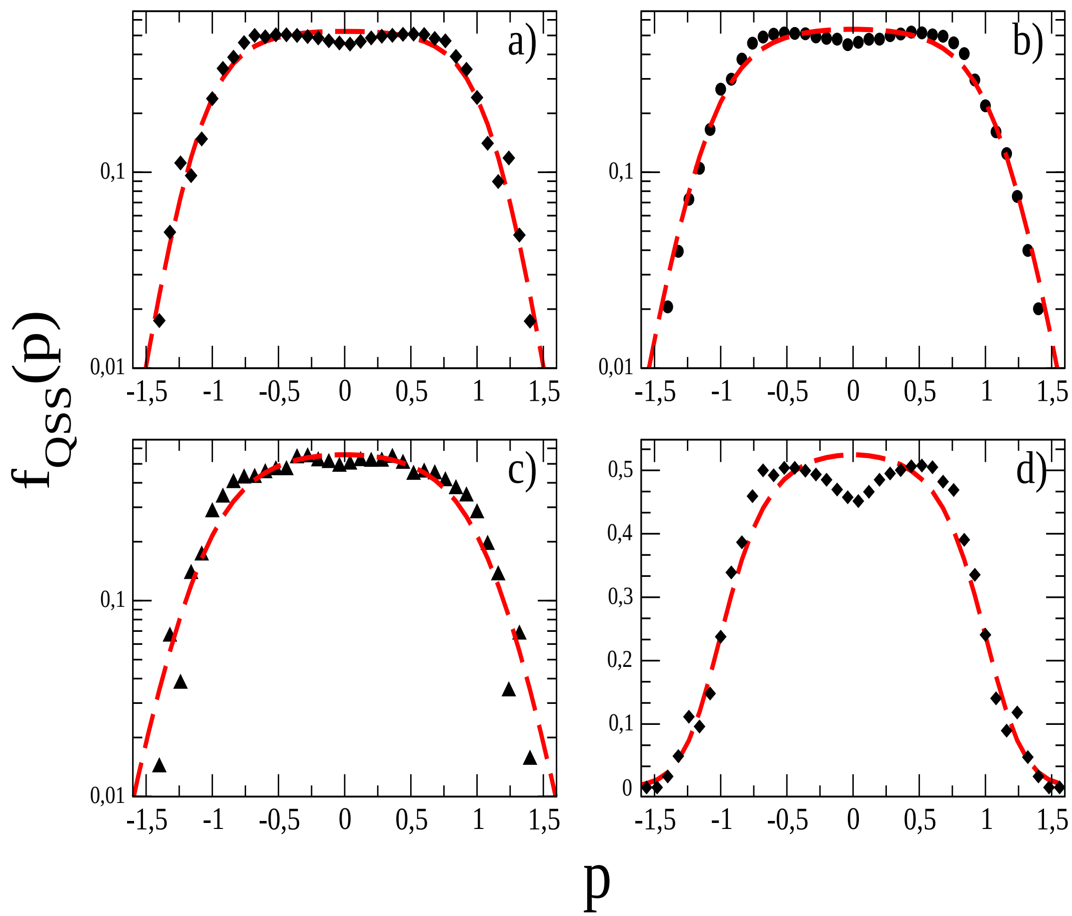
Maximal Lynden-Bell entropy states

$$\bar{f}(\theta,p) = \frac{f_0}{e^{\beta(p^2/2 - M_y[\bar{f}]\sin\theta - M_x[\bar{f}]\cos\theta) + \lambda p + \mu} + 1}.$$

$$\begin{aligned}
f_0 \quad & \frac{x}{\sqrt{\beta}} \int \mathrm{d}\theta e^{\beta \mathbf{M}\cdot\mathbf{m}} F_0\left(x e^{\beta \mathbf{M}\cdot\mathbf{m}}\right) = 1 \\
f_0 \quad & \frac{x}{2\beta^{3/2}} \int \mathrm{d}\theta e^{\beta \mathbf{M}\cdot\mathbf{m}} F_2\left(x e^{\beta \mathbf{M}\cdot\mathbf{m}}\right) = U + \frac{M^2-1}{2} \\
f_0 \quad & \frac{x}{\sqrt{\beta}} \int \mathrm{d}\theta \cos\theta e^{\beta \mathbf{M}\cdot\mathbf{m}} F_0\left(x e^{\beta \mathbf{M}\cdot\mathbf{m}}\right) = M_x \\
f_0 \quad & \frac{x}{\sqrt{\beta}} \int \mathrm{d}\theta \sin\theta e^{\beta \mathbf{M}\cdot\mathbf{m}} F_0\left(x e^{\beta \mathbf{M}\cdot\mathbf{m}}\right) = M_y
\end{aligned}$$

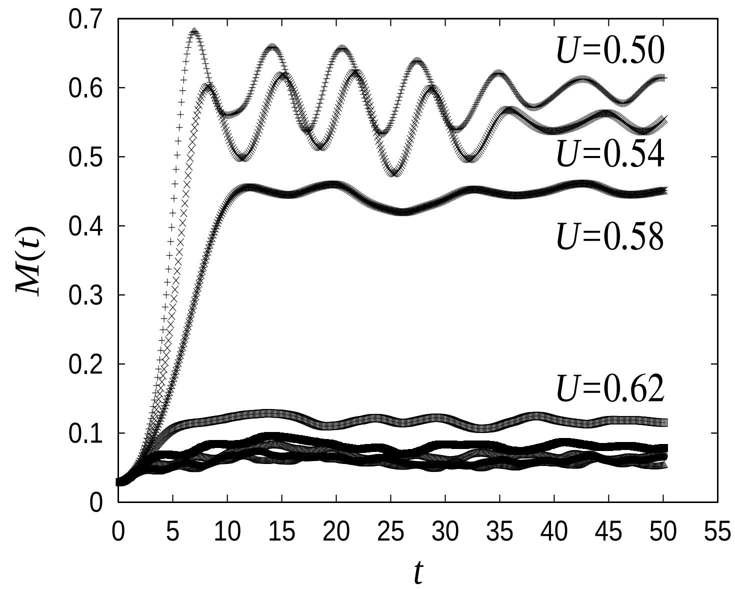
$\mathbf{M} = (M_x, M_y)$, $\mathbf{m} = (\cos\theta, \sin\theta)$.
$F_0(y) = \int \exp(-v^2/2)/(1 + y\exp(-v^2/2))\mathrm{d}v$,
$F_2(y) = \int v^2 \exp(-v^2/2)/(1 + y\exp(-v^2/2))\mathrm{d}v$.
$f_0 = 1/(4\Delta\theta_0\Delta p_0)$

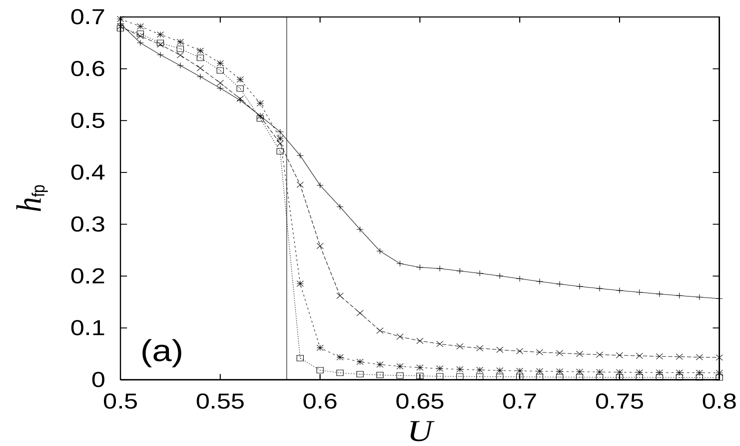
Velocity PDF



$|M_0| = \sin(\Delta\theta_0)/\Delta\theta_0 = 0.3, 0.5, 0.7$. $U = 0.69$. All QSS are homogeneous ($M = 0$).

Non equilibrium phase transition

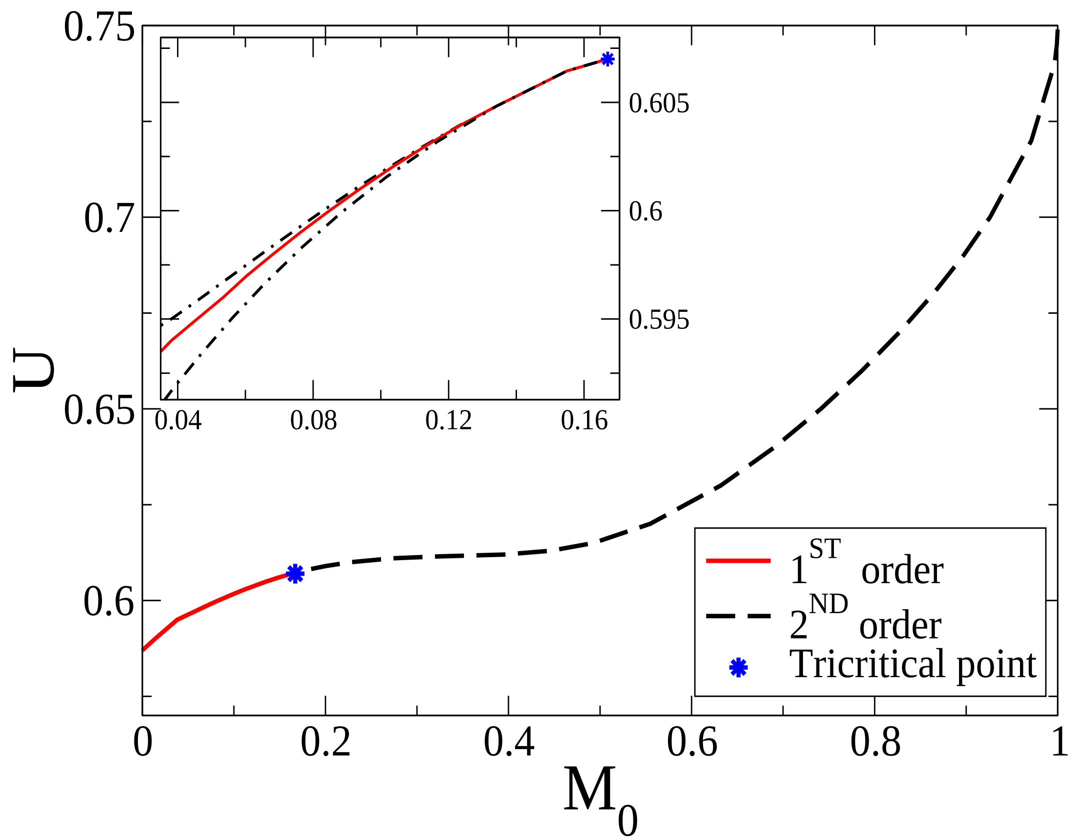




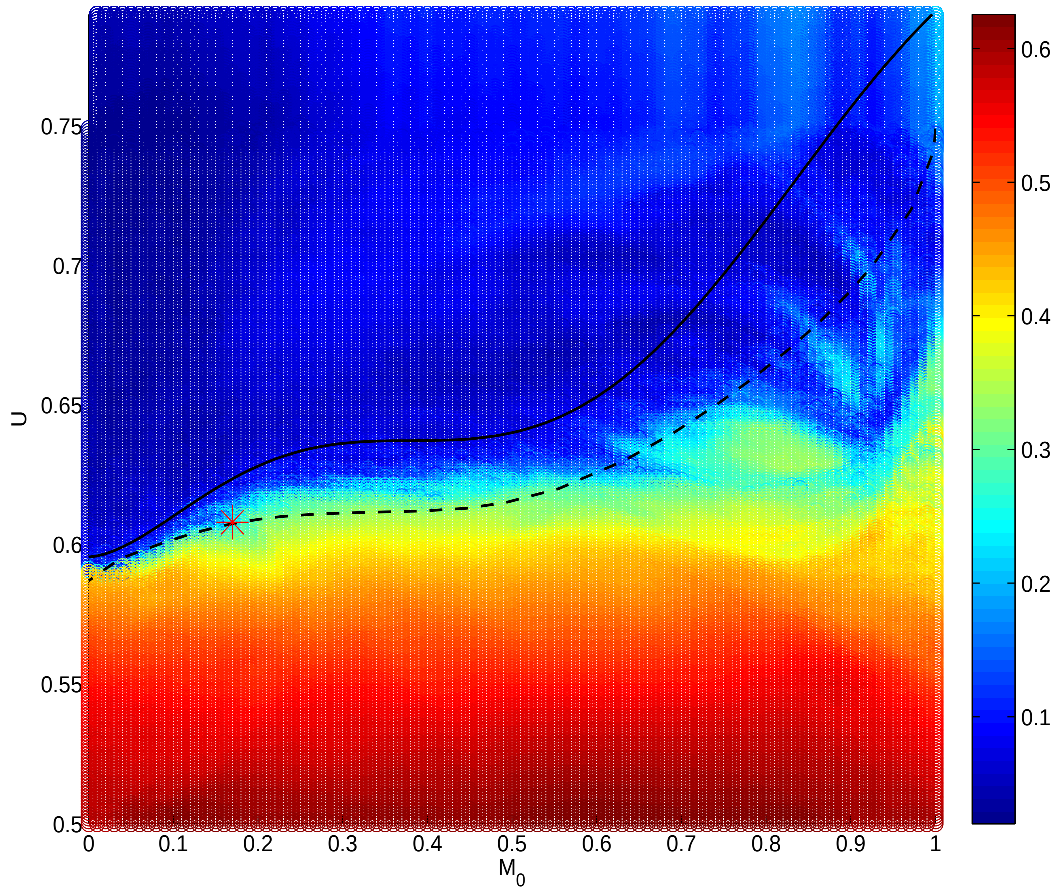
LEFT: $N = 10^3$

RIGHT: First peak height as a function of U for increasing values of N $(10^2, 10^3, 10^4, 10^5)$. The transition line is at $U = 7/12 = 0.583...$.

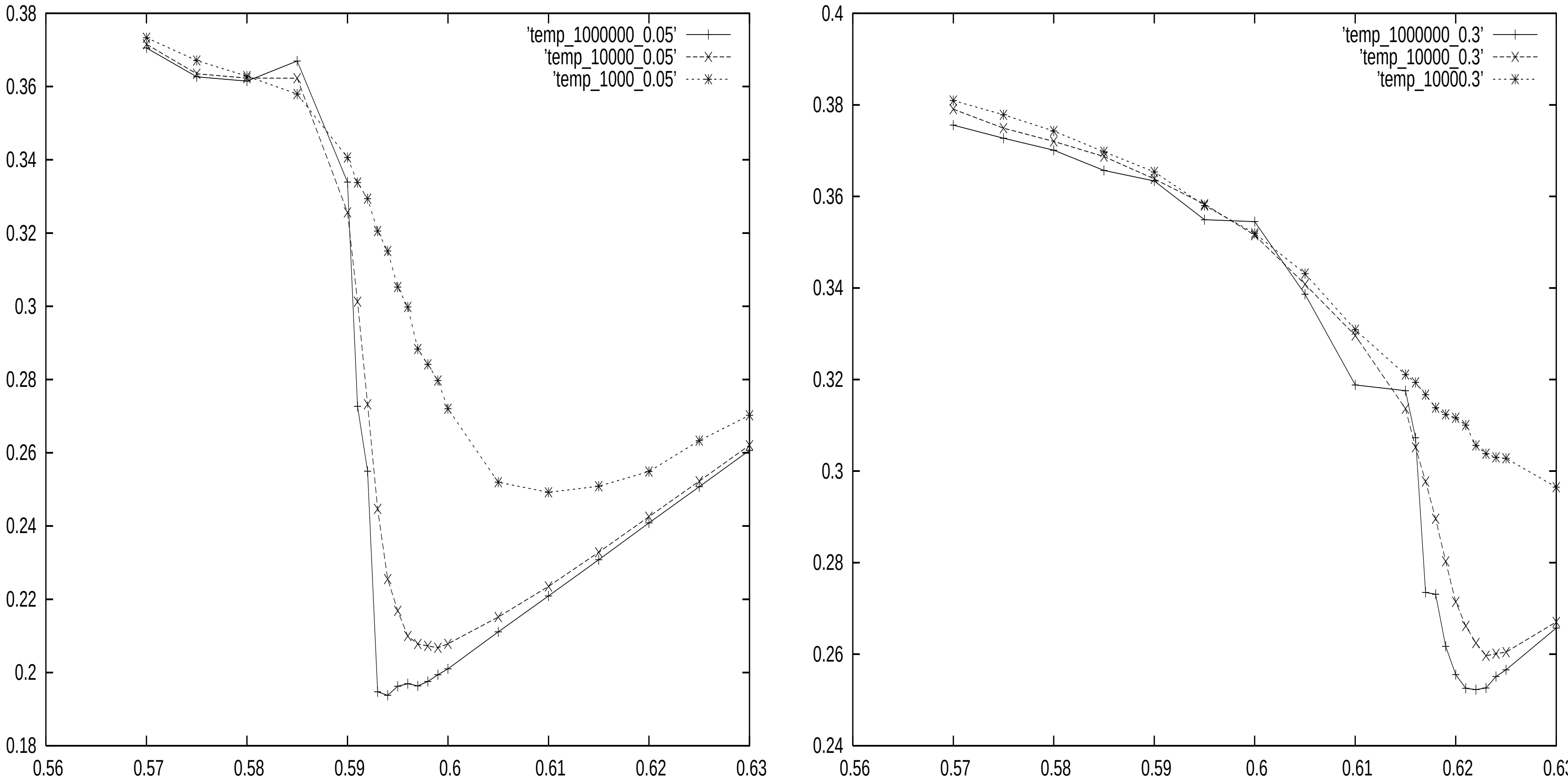
Tricritical point



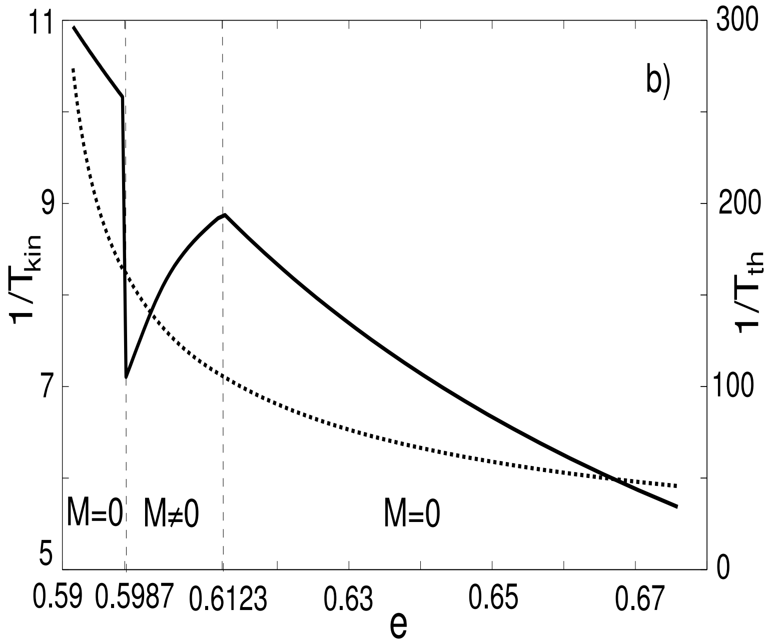
Magnetization plot

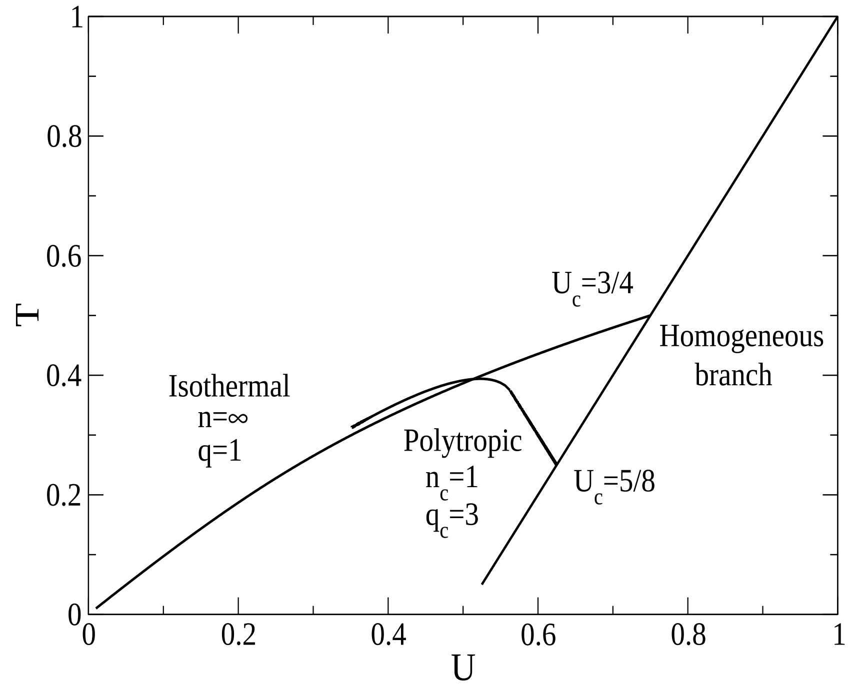


Negative heat capacity



Analytical results





F. Staniscia et al. arXiv:1003.0631, P. H. Chavanis and A. Campa arXiv:1001.2109v1

Mean-field equilibria

$$\epsilon = \frac{p^2}{2} - H\cos\theta$$

H constant.

$$P(\epsilon; \Delta\theta, \Delta p) = \frac{1}{4\Delta\theta\Delta p}\int_{-\Delta\theta}^{+\Delta\theta} d\theta \int_{-\Delta p}^{+\Delta p} dp\, \delta(\frac{p^2}{2} - H\cos\theta - \epsilon)$$

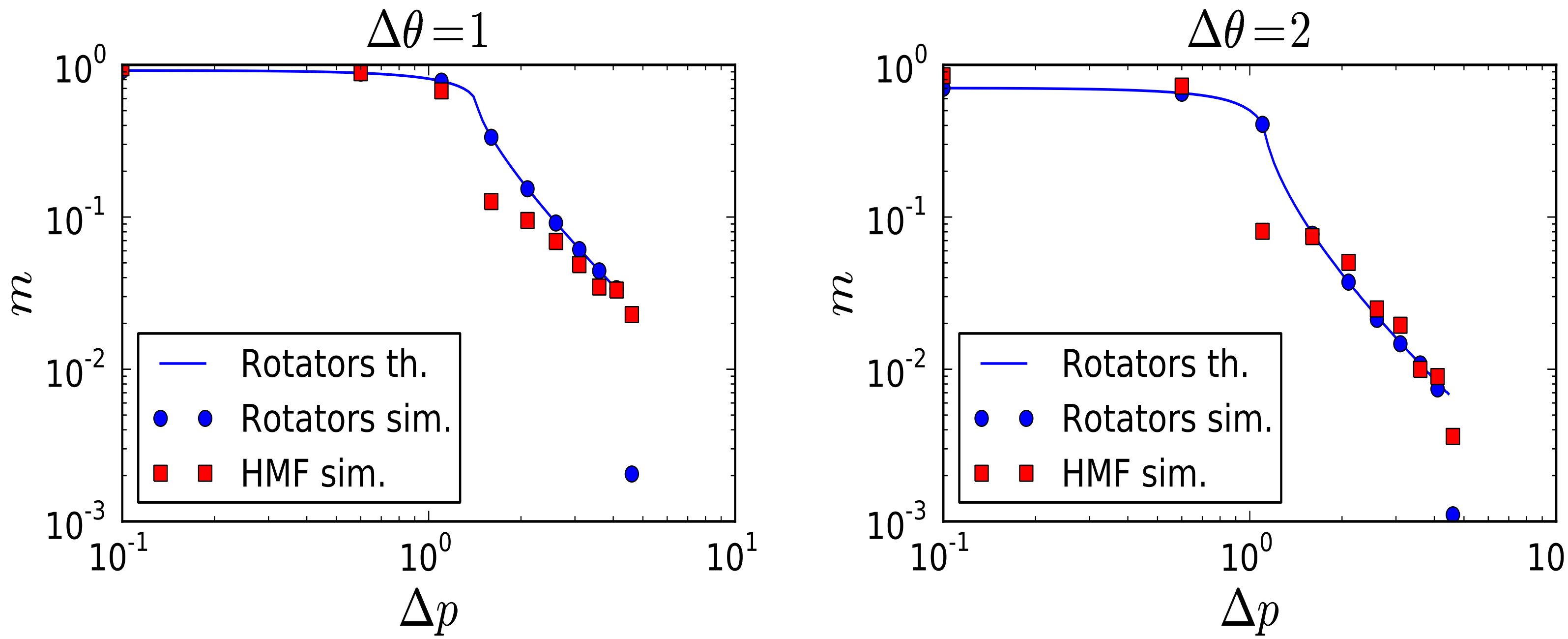
Then

$$P_H(\theta, p; \Delta\theta, \Delta p) = \frac{1}{4\Delta\theta\Delta p}\frac{P(\epsilon, \Delta p)}{Q(\epsilon, \Delta p)}$$

with $Q(\epsilon, \Delta p)$ a normalization.
Self consistency

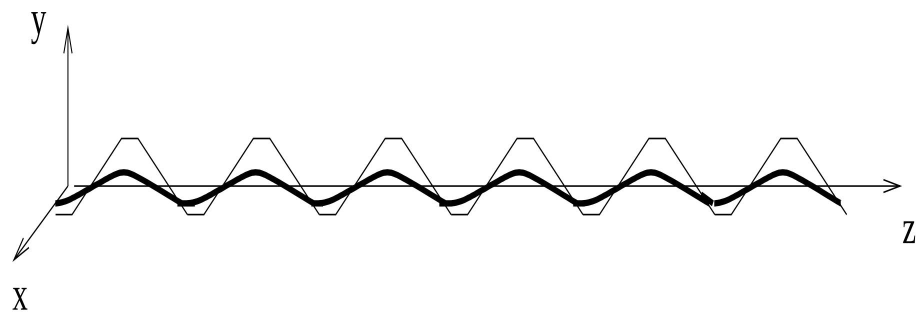
$$m(\Delta\theta, \Delta p) = \int_{-\pi}^{+\pi} d\theta dp \cos\theta P_m(\theta, p; \Delta\theta, \Delta p)$$

Numerical verification



P. de Buyl, D. Mukamel and SR, in progress

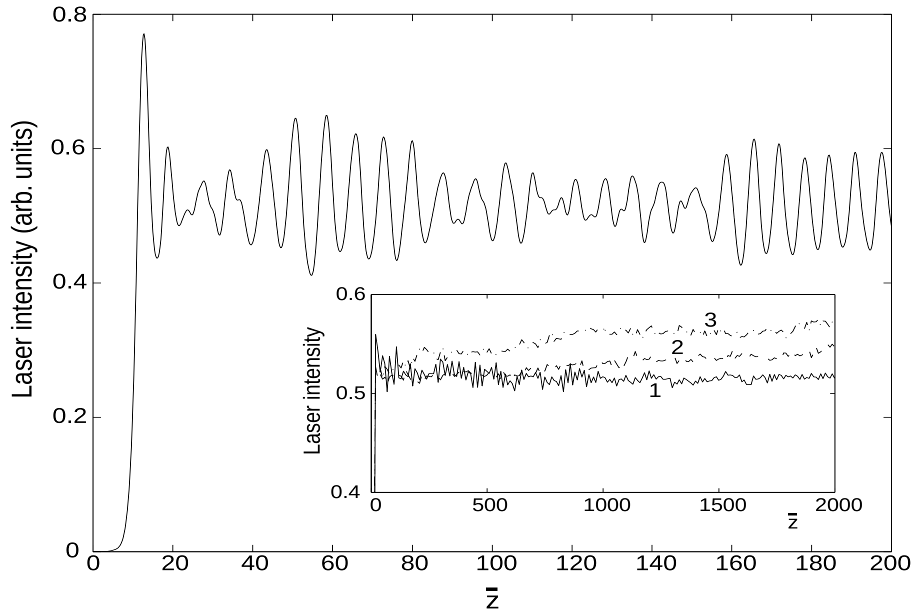
Free Electron Laser



Colson-Bonifacio model

$$
\begin{aligned}
\frac{d\theta_j}{dz} &= p_j \\
\frac{dp_j}{dz} &= -\mathbf{A}e^{i\theta_j} - \mathbf{A}^* e^{-i\theta_j} \\
\frac{d\mathbf{A}}{dz} &= i\delta\mathbf{A} + \frac{1}{N}\sum_j e^{-i\theta_j}
\end{aligned}
$$

Quasi-stationary states



$N = 5000$ (curve 1), $N = 400$ (curve 2), $N = 100$ (curve 3)

On a first stage the system converges to a quasi-stationary state. Later it relaxes to Boltzmann-Gibbs equilibrium on a time $O(N)$. The quasi-stationary state is a Vlasov equilibrium, sufficiently well described by Lynden-Bell's Fermi-like distributions.

Vlasov equation

In the $N \to \infty$ limit, the single particle distribution function $f(\theta, p, t)$ obeys a Vlasov equation.

$$\begin{aligned}
\frac{\partial f}{\partial z} &= -p\frac{\partial f}{\partial \theta} + 2(A_x \cos\theta - A_y \sin\theta)\frac{\partial f}{\partial p} \quad , \\
\frac{\partial A_x}{\partial z} &= -\delta A_y + \frac{1}{2\pi}\int f \cos\theta \, d\theta dp \quad , \\
\frac{\partial A_y}{\partial \bar{z}} &= \delta A_x - \frac{1}{2\pi}\int f \sin\theta \, d\theta dp \ .
\end{aligned}$$

with $\mathbf{A} = A_x + iA_y = \sqrt{I}\exp(-i\varphi)$

Vlasov equilibria

Lynden-Bell entropy maximization

$$S_{LB}(\bar{f}) = -\int dpd\theta \left(\frac{\bar{f}}{f_0} \ln \frac{\bar{f}}{f_0} + \left(1 - \frac{\bar{f}}{f_0}\right) \ln \left(1 - \frac{\bar{f}}{f_0}\right) \right).$$

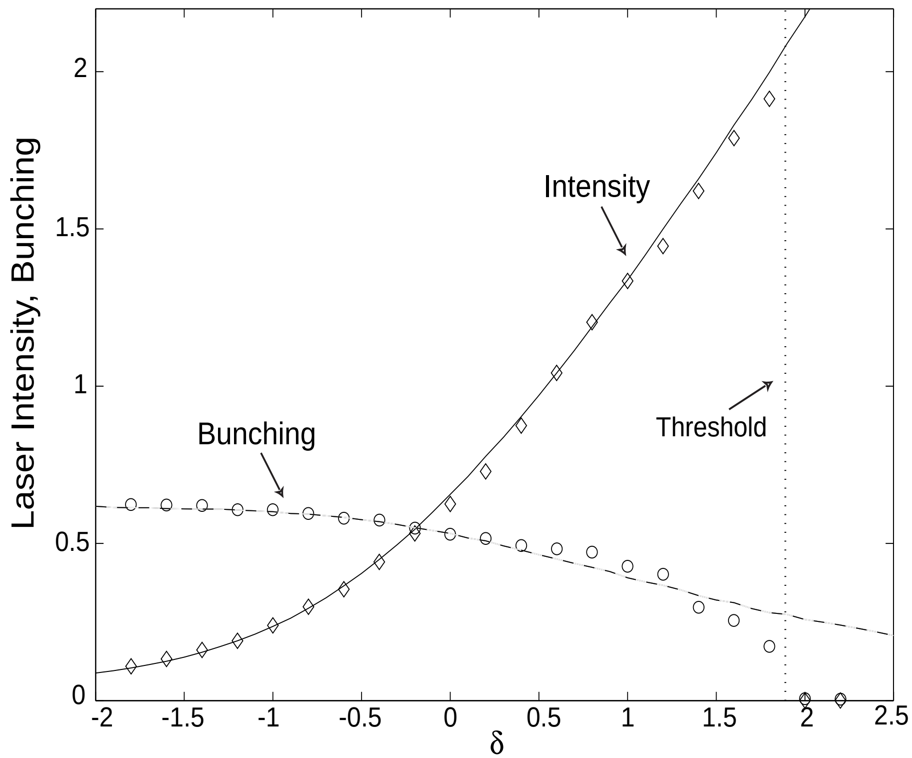
$$S_{LB}(\varepsilon, \sigma) = \max_{\bar{f}, A_x, A_y} [S_{LB}(\bar{f}) | H(\bar{f}, A_x, A_y) = N\varepsilon; \int d\theta dp \bar{f} = 1; P(\bar{f}, A_x, A_y) = \sigma].$$

$$\bar{f} = f_0 \frac{e^{-\beta(p^2/2 + 2A\sin\theta) - \lambda p - \mu}}{1 + e^{-\beta(p^2/2 + 2A\sin\theta) - \lambda p - \mu}} .$$

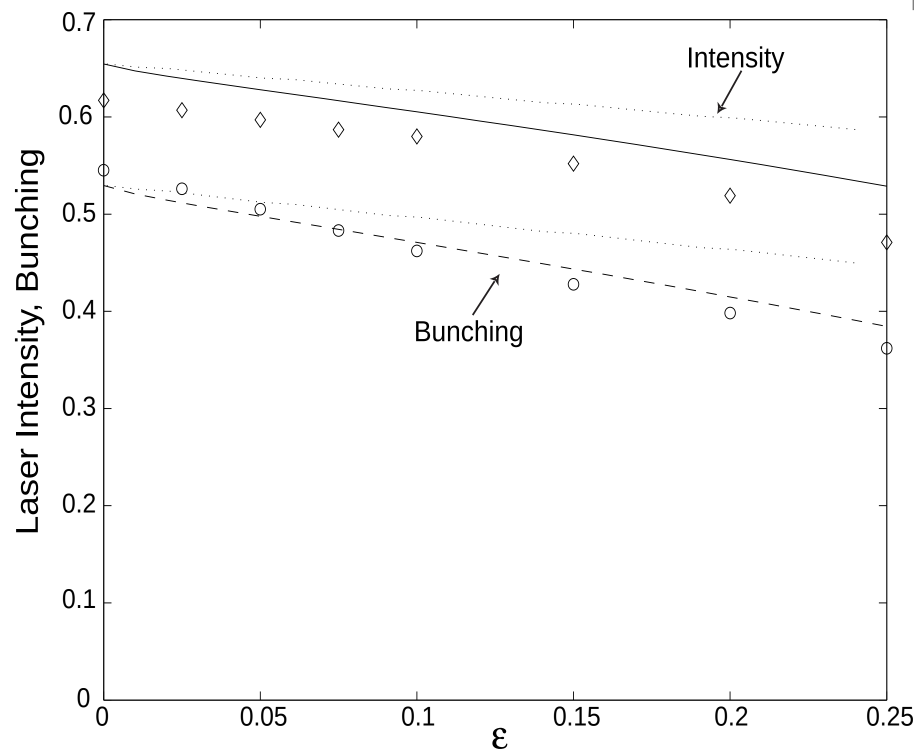
Non-equilibrium field amplitude

$$A = \sqrt{A_x^2 + A_y^2} = \frac{\beta}{\beta\delta - \lambda} \int dpd\theta \, \sin\theta \bar{f}(\theta, p).$$

Results



Laser Intensity, Bunching
Intensity
Bunching
Threshold
δ



Laser Intensity, Bunching
Intensity
Bunching
ε

Conclusions

- Mean-field Hamiltonian systems with many degrees of freedom display interesting statistical and dynamical properties.
- Non equilibrium quasi-stationary states arise “naturally" from water-bag initial conditions. Their life-time increases with a power of system size.
- Vlasov (non collisional) equation correctly describes the “initial" dynamics.
- Lynden-Bell maximum entropy principle provides a theoretical approach to quasi-stationary states.
- The results are derived for mean-field systems, but some of them are expected to be valid also for decaying interactions.

References

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