Long-range steady state density profiles induced by localized drives

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Joint work with
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The problem

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- Most non-equilibrium stationary states, have long-range correlations.
  Example: Power-law profiles in boundary driven lattice gas, heat conduction models *etc.*
- What happens when detailed balance is broken locally, inside bulk, in an otherwise equilibrium system?

Main result

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2. Decay exponent depends on the geometry of the drive.

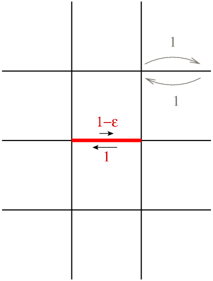
3. A correspondence with electrostatic is established where we can show that the density profile is related to the potential of different arrangement of electric dipoles.

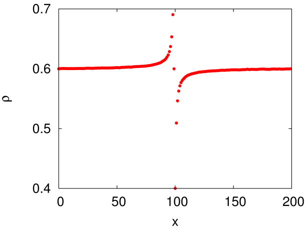
Outline

- Locally driven non-interacting particles
  - Analogy to electrostatic potential due to charges
  - Exact solution
- Local drive with exclusion interaction
- Summary

Non-interacting particles

- N non-interacting particles on square lattice.
- Drive across a single bond.
- When $\epsilon = 0$, detailed balance is satisfied w.r.t a flat density profile.
- For non-zero ϵ, detailed balance is broken, and change in density profile decays as $1/r$ for large r.





Master equation

The equation for the density profile $\phi(\vec{r}, t)$:

$$\partial_t \phi(\vec{r}, t) = \nabla^2 \phi(\vec{r}, t) + \epsilon \phi(\vec{0}) \left[\delta_{\vec{r},\vec{0}} - \delta_{\vec{r},\vec{e}_1} \right],$$

where discrete Laplacian

$$\nabla^2 \phi(m, n) = \phi(m+1, n) + \phi(m-1, n) + \phi(m, n+1) + \phi(m, n-1) - 4\phi(m, n)$$

and $\vec{0} \equiv (0, 0)$, $\vec{e_1} \equiv (1, 0)$

- Steady state equation

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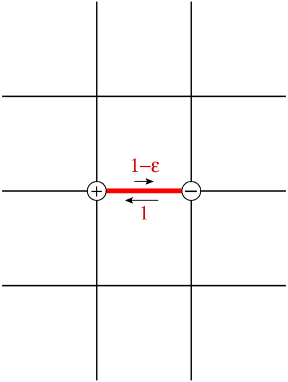

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- Solution:

$$\phi(\vec{r}) = \rho + \epsilon\phi(\vec{0})\left[G(\vec{r},\vec{0}) - G(\vec{r},\vec{e}_1)\right],$$

where G is the lattice greens function $\nabla^2 G(\vec{r},\vec{r}_o) = -\delta_{\vec{r},\vec{r}_o}$,
ρ is the global average density, and

$$\phi(\vec{0}) = \rho/(1-\epsilon/4)$$

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- At large $\vec{r}$,

$$\phi(\vec{r}) = \rho - \frac{\epsilon\phi(\vec{0})}{2\pi}\frac{\vec{e}_1\cdot\vec{r}}{r^2} + \mathcal{O}(\frac{1}{r^2})$$

and current

$$\vec{j}(\vec{r}) = -\nabla\phi(\vec{r}) = \frac{\epsilon\phi(\vec{0})}{2\pi}\frac{1}{r^2}\left[\vec{e}_1 - \frac{2(\vec{e}_1\cdot\vec{r})\vec{r}}{r^2}\right] + \mathcal{O}(\frac{1}{r^3}).$$

In d-dimensions

The analogy to electrostatics holds in higher dimensions.

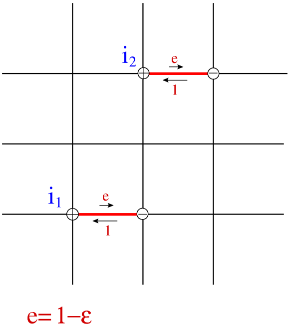
- Then, in $d \geq 2$

$$\phi(\vec{r}) \sim 1/r^{d-1}$$

- In $d = 1$, Green's function $G(x, x_o) = -|x - x_o|/2$, then

$$\phi(x) = \rho - (\epsilon/2)\,\phi(0)\,\mathrm{sgn}(x),$$

Arbitrary driving configuration



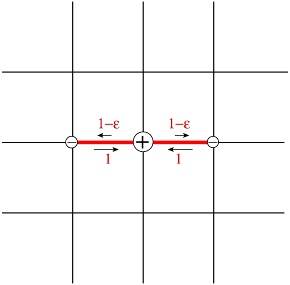
$$\phi(\vec r) = \rho + \epsilon\phi(\vec i_1)\left[G(\vec r, \vec i_1) - G(\vec r, \vec i_1 + \vec 1)\right] + \epsilon\phi(\vec i_2)\left[G(\vec r, \vec i_2) - G(\vec r, \vec i_2 + \vec 1)\right] + \cdots$$

n self-consistency equations obtained by putting $\vec r = \vec i_1, \vec i_2 \cdots$.

These are linear set of equations, and can be solved using known solutions of $G(\vec r, \vec 0) - G(\vec 0, \vec 0)$

(i,j)	0	1	2
0	0	$-\frac{1}{4}$	$\frac{2}{\pi} - 2$
1	$-\frac{1}{4}$	$-\frac{1}{\pi}$	$\frac{2}{\pi} - 2$
2	$\frac{2}{\pi} - 2$	$\frac{1}{4} - \frac{2}{\pi}$	$\frac{4}{3\pi}$

Quadrupolar charge configuration



The steady state equation

$$\nabla^2\phi(\vec{r}) = -\epsilon\phi(\vec{0})\left[2\delta_{\vec{r},\vec{0}} - \delta_{\vec{r},\vec{e}_1} - \delta_{\vec{r},-\vec{e}_1}\right].$$

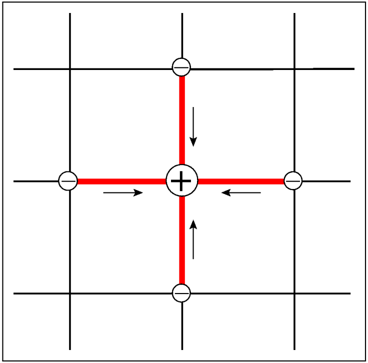
Solution

$$\phi(\vec{r}) = \rho - \frac{\epsilon\phi(\vec{0})}{2\pi}\left[\frac{1}{r^2} - 2\left(\frac{\vec{e}_1\cdot\vec{r}}{r^2}\right)^2\right] + \mathcal{O}(\frac{1}{r^4}),$$

with $\phi(\vec{0}) = \rho/(1-\epsilon/2)$.

A side note

- Collection of biased bonds do not necessarily imply breakdown of detailed balance.
- Detailed balance with respect to potential $V(\vec{r}) = -\ln(1-\epsilon)\,\delta_{\vec{r},\vec{0}}$
- Consequently, the density profile $\phi(\vec{r}) \propto \exp[-V(\vec{r})]$



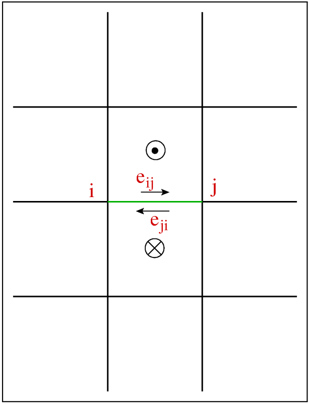
Analogy to magnetic fields

- In 2-d, magnetic field by $(i \to j)$ link

$$H = \ln[e_{ij}]$$

- Then for a bond

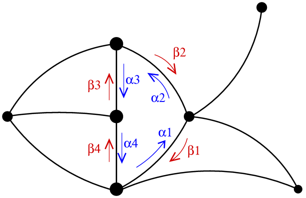
$$\begin{aligned} H &= \ln[e_{ij}] - \ln[e_{ji}] \\ &= \ln[\frac{e_{ij}}{e_{ji}}] \end{aligned}$$



- Kolmogorov criterion: Detailed balance if and only if

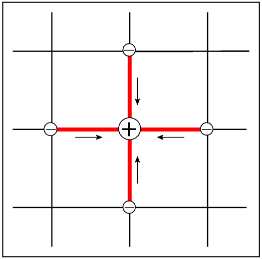
$$\alpha_1\alpha_2\alpha_3\alpha_4 = \beta_4\beta_3\beta_2\beta_1$$

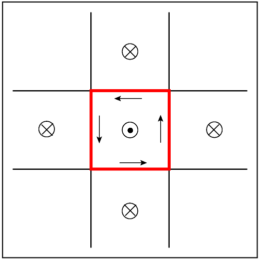
on all loops



- In terms of magnetic field:

$$H = \begin{cases} \text{zero} \iff & \text{Detailed balance} \\ \text{non-zero} \iff & \text{No detailed balance} \end{cases}$$





Exclusion interaction

- The steady state equation for density

$$\nabla^2\phi(\vec{r}) = -\epsilon\langle\tau(\vec{0})(1-\tau(\vec{e}_1))\rangle\left[\delta_{\vec{r},\vec{0}} - \delta_{\vec{r},\vec{e}_1}\right],$$

where

$$\tau(\vec{r}) = \begin{cases} 1 & \text{If there is a particle} \\ 0 & \text{No particle} \end{cases} \quad \Bigg| \quad \text{and } \phi(\vec{r}) = \langle\tau(\vec{r})\rangle$$

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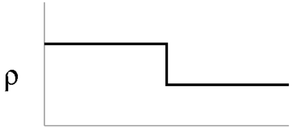
- Unlike the non-interacting case, the pre-factor has to be determined separately.
  However, the profile remains the same, at large r.

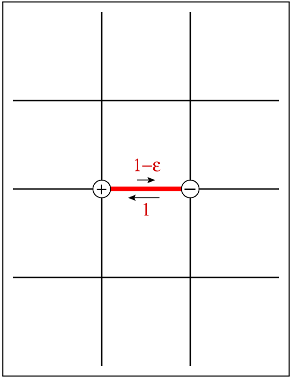
Exclusion interaction

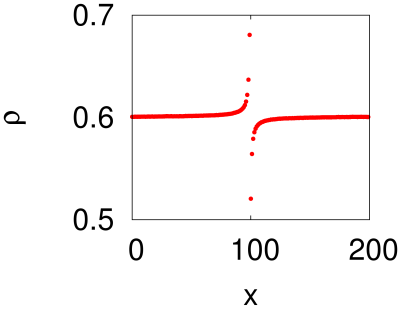
- The $d = 1$ result is very similar to the profile obtained in SSEP with a battery by [Bodineau, Derrida and Lebowitz].



Exclusion interaction

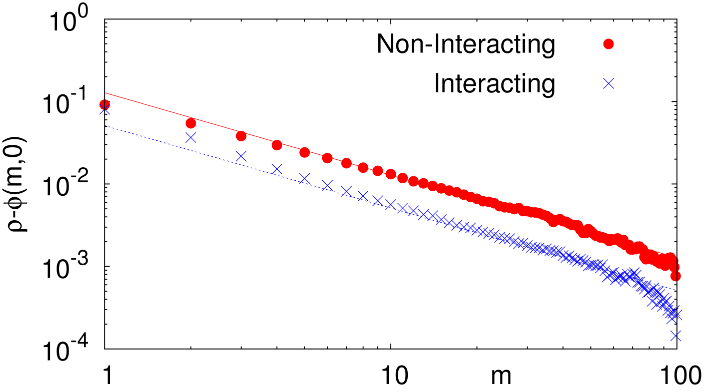
- In $d = 2$





Numerical results

On a 200×200 lattice with $\rho = N/L^2 = 0.6$

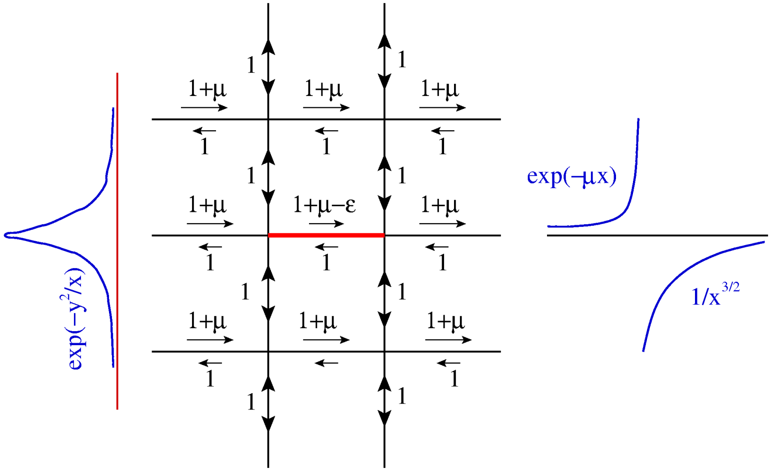


Non-Interacting: $\phi(\vec{0}) = \frac{\rho}{1-\epsilon/4}$

Exclusion interaction: $\langle \tau(\vec{0})(1-\tau(\vec{e}_1)) \rangle = 0.3209$ measured separately

Steady state equation for Non-interacting case

$$\nabla^2 \phi(\vec{r}) = -\epsilon \phi(\vec{0}) \left[\delta_{\vec{r},\vec{0}} - \delta_{\vec{r},\vec{e}_1} \right] + \mu \left[\phi(\vec{r}) - \phi(\vec{r} - \vec{e}_1) \right]$$



Summary

- In diffusive systems, both with and without exclusion interaction, localized drive can give rise to algebraically decaying density profiles at large distances.
- An electrostatic analogy is established.
- What happens when other kinds of local interactions are switched on?

Thank you