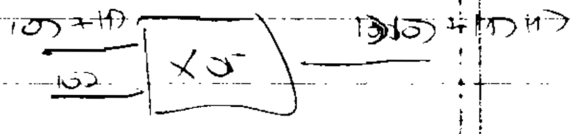
more Qubits:

How do you combine Hilbert spaces:

tensor products.
(Direct)



V & W are Hilbert spaces of dimension m & n

$V \otimes W$ is a Hilbert space of dimension mn
↳ all possible linear combinations of $|v\rangle$'s $\in V$ $|w\rangle \in W$: $|\psi\rangle = \sum_{ij} g_{ij} |v\rangle \otimes |w\rangle$
if $|i\rangle$ and $|j\rangle$ are basis states for V & W

then $|i\rangle \otimes |j\rangle$ are the basis states for $V \otimes W$

1. for any $|v\rangle \in V$ $|w\rangle \in W$

$$a\left(|v\rangle \otimes |w\rangle\right) = a|v\rangle \otimes |w\rangle = |v\rangle \otimes a|w\rangle$$

2. for any $|v\rangle, |v'\rangle \in V$ $|w\rangle \in W$

$$\left(|v\rangle + |v'\rangle\right) \otimes |w\rangle = |v\rangle \otimes |w\rangle + |v'\rangle \otimes |w\rangle$$

3. and vice versa.

$$|v\rangle \otimes \left[|w\rangle + |w'\rangle\right] = |v\rangle \otimes |w\rangle + |v\rangle \otimes |w'\rangle$$

operators acting on $V \otimes W$

A operator acting on V

commutation relations

B - " - " - W

5

$$(A \otimes B)(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle \Rightarrow (A \otimes B)(C \otimes D) = AC \otimes BD$$

$$\Rightarrow A \otimes B \sum_i a_i |v_i\rangle \otimes |w_i\rangle = \sum_i a_i A|v_i\rangle \otimes B|w_i\rangle$$

(inner scalar) product.

$$(|v\rangle \otimes |w\rangle, |v'\rangle \otimes |w'\rangle) = \langle v|v'\rangle\langle w|w'\rangle$$

$$\left(\sum_i a_i |v_i\rangle \otimes |w_i\rangle, \sum_j b_j |v_j'\rangle |w_j'\rangle\right) =$$

$$= \sum_{ij} a_i^* b_j \langle v_i|v_j'\rangle\langle w_i|w_j'\rangle$$

Matrix representation: Kronecker product.

$$A \otimes B = \begin{bmatrix} A_{11} B & A_{12} B & \cdots \\ A_{21} B & & \\ \vdots & & \end{bmatrix}$$

Holds both for state vectors as well as operators

example states:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1\,1 \\ \phi\,\phi \\ \phi\,1 \\ 0\,0 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

← Lexicographically ordered.

$(a,b) \leq (a',b')$ iff $a < a'$ or $a = a'$ and $b \leq b'$

like words in dictionary

Example operators:

$$\sigma_{x_1} \otimes \sigma_{x_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ \phi & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$|\uparrow\uparrow\rangle \equiv |\uparrow\rangle \otimes |\uparrow\rangle$

$\Rightarrow |\downarrow\rangle \otimes |\downarrow\rangle$

$$e^{i\left[\frac{\theta}{2}\right]\hat{\sigma}_{x_1}\otimes\hat{\sigma}_{x_2}} = \cos\left(\frac{\theta}{2}\right)\underbrace{\hat{\mathbb{I}}}_{4\times4} + i\left(\hat{\sigma}_{x_1}\otimes\hat{\sigma}_{x_2}\right)\sin\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \theta = \frac{\pi}{2} \Rightarrow \cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{2}}\hat{\mathbb{I}} + i\frac{1}{\sqrt{2}}\hat{\sigma}_x\otimes\hat{\sigma}_x$$

$$\Rightarrow e^{i\frac{\pi}{4}\hat{\sigma}_{x_1}\otimes\hat{\sigma}_{x_2}}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \frac{1}{\sqrt{2}}|\uparrow\uparrow\rangle + \frac{i}{\sqrt{2}}|\downarrow\downarrow\rangle$$

$$e^{i\left[\frac{\theta}{2}\right]\left(\hat{\sigma}_{x_1}\otimes\hat{I}_2 + \hat{I}_1\otimes\hat{\sigma}_{x_2}\right)} = e^{i\left(\frac{\theta}{2}\right)\hat{\sigma}_x\otimes\hat{I}}\; e^{i\left[\frac{\theta}{2}\right]\hat{I}\otimes\hat{\sigma}_{x_2}} =$$

commute

always separable

$$= \left[\cos\left(\frac{\theta}{2}\right)\hat{\mathbb{I}} + i\sin\left(\frac{\theta}{2}\right)\hat{\sigma}_{x_1}\otimes\hat{I}\right]$$

$$\left[\cos\left(\frac{\theta}{2}\right)\hat{\mathbb{I}} + i\sin\left(\frac{\theta}{2}\right)\hat{I}_1\otimes\hat{\sigma}_{x_2}\right] =$$

$$= \cos^2\left(\frac{\theta}{2}\right)\hat{\mathbb{I}} + i\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\left[\hat{\sigma}_{x_1}\otimes I + I\otimes\hat{\sigma}_x\right]$$

$$- \sin^2\left(\frac{\theta}{2}\right)\hat{\sigma}_{x_1}\otimes\hat{\sigma}_{x_2}$$

eigen states:

$Z \otimes Z\, |1\rangle|1\rangle = Z|1\rangle \otimes Z|1\rangle = |1\rangle|1\rangle$ degenerate.

$Z \otimes Z\, |0\rangle \otimes |0\rangle = Z|0\rangle \otimes Z|0\rangle = (-1)^2 |0\rangle|0\rangle = |0\rangle|0\rangle$

$Z \otimes Z\, |0\rangle|1\rangle = Z \otimes Z |1\rangle|0\rangle = -|01\rangle$

$Z \otimes Z\, |1\rangle|0\rangle = -|10\rangle$

$X|1\rangle = |0\rangle$; $X|0\rangle = |1\rangle$

$X \otimes X\, |1\rangle|1\rangle = |0\rangle|0\rangle$

$X \otimes X\, |1\rangle|0\rangle = |0\rangle|1\rangle$

$X \otimes X \left\{ \frac{1}{\sqrt{2}} \left[|0\rangle|1\rangle \pm |1\rangle|0\rangle \right] \right\} = \pm\frac{1}{\sqrt{2}} \left[|1\rangle|0\rangle \pm |0\rangle|1\rangle \right]$

$X \otimes X \left\{ \frac{1}{\sqrt{2}} \left[|0\rangle|0\rangle \pm |1\rangle|1\rangle \right] \right.$

4 Bell states.