Previously:

Tensor products of Hilbert spaces $V \otimes W$

$$|\psi\rangle = \sum_{ij} g_{ij} |v_i\rangle \otimes |w_j\rangle$$

Entangled state: any state that cannot be written as $|\psi\rangle = |v\rangle \otimes |w\rangle$ ($v \Downarrow$, $W \Downarrow$)

Operators: $A \otimes B |\psi\rangle = \sum_{ij} g_{ij} A|v_i\rangle \otimes B|w_j\rangle$

tensor prod of density matrices $\rho_A \otimes \rho_B = (\sum_i p_i |i\rangle\langle i|) \otimes (\sum_j p_j' |j\rangle\langle j|) = \sum_{ij} p_i p_j' |i\rangle\langle i| \otimes |j\rangle\langle j|$

- Entangled / Local

Commutation Relations:

$\sum_{ij} p_i p_j' = (\sum_i p_i)(\sum_j p_j') = 1 \cdot 1 = 1$

$$[A \otimes B, C \otimes D] = (A \otimes B)(C \otimes D) - (C \otimes D)(A \otimes B)$$

$$= AC \otimes BD - CA \otimes DB$$

if $[A,C] = 0$ $\quad AC = CA \Rightarrow [\ ] = AC \otimes [B,D]$

if both commute $[\ ] = 0$

if both don't $\Rightarrow$ stuck with

for the special case of σ's.

$$\sigma_i \sigma_j = i \varepsilon_{ijk} \sigma_k \Rightarrow [\sigma_i, \sigma_j] = 0$$

or

$$\{\sigma_i, \sigma_j\} = 0$$

$0 = [\sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y]$

$\Rightarrow |\phi^{\pm}\rangle, |\psi^{\pm}\rangle$ sim diag.

if Both (A,C) & $(B,D) = 0 \Rightarrow [\ ] = 0$ then

$\Rightarrow$ if Both $\{\ \}, \{\ \} = 0 \Rightarrow [\ ] = 0$

if $[\ ] = 0$ & $\{\ \} = 0 \Rightarrow \{\ \} = 0$

Partial trace / or Reduced density matrix

the bi-partite.

Assume ~~entangled~~ state:

$$|\psi\rangle_{AB} = \alpha|0\rangle_A \otimes |0\rangle_B + b\,|1\rangle_A|1\rangle_B$$

let's say system B is inaccesible to me. (locked in a vault, escaped to a caribbean beach)

and I want to know the expectation value

of $\hat{M}_A$. $\Rightarrow$ calculate $\langle M_A \otimes I_B\rangle$

$$\langle M_A \otimes I_B\rangle = |\alpha|^2 \, {}_A\langle 0|M_A|0\rangle_A \, {}_B\langle 0|0\rangle_B +$$

$$+ |b|^2 \, {}_A\langle 1|M_A|1_A\rangle \, {}_B\langle 1|1\rangle_B + \alpha^* \beta \, {}_A\langle 1|M_A|0\rangle_A \, {}_B\langle 0|1\rangle_B$$

$$+ \alpha\beta^* \, {}_A\langle 0|M_A|1\rangle_A \, \langle 1|0\rangle_B = |\alpha|^2 \langle 0|M_A|0\rangle$$

$$+ |\beta|^2 \langle 1|M_A|1\rangle =$$

$\langle 0|M|0\rangle$ m

$$= \mathrm{tr}\left[\rho_A M_A\right]$$ spectrally decomposed.

where $\rho_A = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}$

the state of A reduced to a statistical mixture.

For non entangled states
~~In general~~:

$$\rho_{AB} = \sum_{\mu,\nu} g_{\mu\nu}\, \rho_\mu^A \otimes \sigma_\nu^B \Rightarrow \rho_A = \sum_\mu \sum_\nu g_{\mu\nu} \underbrace{\sum_i \langle i | \sigma_\nu^B | i \rangle_B}_{\mathrm{tr}=1}\, \rho_\mu$$

prob. to be in ρ_μ — orthonormal basis.

$$= \sum_\mu \Big(\sum_\nu g_{\mu\nu}\Big) \rho_\mu$$

for a

any measurement would project on ρ_μ

in an entry-by-entry form:

$$\rho_{AB} = \sum_{\mu,\nu,\gamma,\delta} g_{\mu\nu\gamma\delta}\, |\mu\rangle\langle\nu|_{AA} \otimes |\gamma\rangle\langle\delta|_{BB}$$

$$\rho_A = \sum_{\mu\nu} \Big(\sum_{\gamma,\delta} \langle\gamma|\delta\rangle\, g_{\mu,\nu,\gamma,\delta} \Big) |\mu\rangle\langle\nu|_{AA}$$

?

$$\mathrm{tr}\left[|\gamma\rangle\langle\delta|\right] = \sum_i \langle\mu_i|\gamma\rangle\langle\delta|\mu_i\rangle = \sum_i \langle\delta|\mu_i\rangle\langle\mu_i|\gamma\rangle$$

orthonormal!

$$= \langle\delta|\gamma\rangle$$

Back to prev. example:

$$\rho_{AB} = |\alpha|^2\, |0\rangle\langle 0| \otimes |0\rangle\langle 0| + \alpha^*\beta\, |0\rangle\langle 1| \otimes |0\rangle\langle 1|$$

$$+ \beta^*\alpha\, |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |\beta|^2\, |1\rangle\langle 1| \otimes |1\rangle\langle 1|$$

Prob. of finding

B in 0 $\Rightarrow P_{B_0} = \langle 0|\rho_{AB}|0\rangle$

& what A is projected on.

If $\hat{\rho}_{AB} = \rho_A \otimes \sigma_B$ ← i.e. ~~pure~~ non entangled state

$\Rightarrow$ $\hat{\rho}_B$ $tr_A [\hat{\rho}_{AB}] = \sigma_B$

ρ_A $tr_B [\rho_{AB}] = \sigma_A$

(linear map from $\mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_{A/B}$

– without proving: Unique ~~fraction~~ map such that.

$$\mathrm{Tr}[\hat{M}_A \rho_A] = \mathrm{Tr}[(M_A \otimes I_B)\, \rho_{AB}]$$

Schmidt decomposition:

A bi-partite pure state can be decomposed into

$$|\psi\rangle = \sum_i \overset{\text{positive}}{\lambda_i} |u_i\rangle |v_i\rangle$$

where u_i & v_i are orthonormal bases

$$\Rightarrow \langle v_i | v_j \rangle = \delta_{ij} \quad \text{etc}$$

Proof: ~~By construction~~

$$|\psi\rangle_{AB} = \sum_{i,\mu} g_{i\mu} |i\rangle_A \otimes |\mu\rangle_B = \sum_i |i\rangle \otimes \overline{|i\rangle}_B$$

where $\overline{|i\rangle}_B = \sum_\mu g_{i\mu} |\mu\rangle_B$

need not be orthogonal nor normalized.

$\Rightarrow$ choose $|i\rangle$ to be the set that diagonalized

$$\rho_A \Rightarrow \rho_A = \sum_i p_i |i\rangle\langle i|$$

$$\rho_A = \mathrm{Tr}_B \left[|i\rangle\langle j|_{AA} \otimes \overline{|i\rangle}\overline{\langle j|}_{BB} \right] = \sum_{i,j} \langle \bar{i} | \bar{j} \rangle |i\rangle\langle j|$$

$$\Rightarrow \langle \bar{i} | \bar{j} \rangle = p_i \delta_{ij}$$

$\Rightarrow$ $\overline{|i\rangle}$ orthogonal!

$|i\rangle_B = \sqrt{p_i}\,\overline{|i\rangle}$ $\Rightarrow$ an orthonormal basis

$\Rightarrow$ $|\psi\rangle_{AB} = \sum_i \sqrt{p_i}\, |i\rangle_A \otimes |i\rangle_B$

1. freedom of phase $\Rightarrow$ absorbed in states.

2. Schmidt coeffs. give the spectral decomposition of reduced density mat.

differs only by # of zero eig's $\Leftarrow$ the same if of the same dim

ρ_A and ρ_B have the same non-zero eigenvalues.

3. Schmidt # / rank $\Rightarrow$ # of Schmidt coeffs $\neq 0$.

4. connection between entanglement & decoherence.

5. If $\hat{\rho}_A / \hat{\rho}_B$ have no degenerate non-zero eigenvalues $\Rightarrow$ $\hat{\rho}_A$ & $\hat{\rho}_B$ uniquely determine the Schmidt decomposition.

if there are degenerate non-zero eigenvalues

$\Rightarrow$ $\hat{\rho}_A$ & $\hat{\rho}_B$ are not enough.

for a pure state Schmidt # (rank) $> 1 \Rightarrow$ entanglement criteria!

$\Rightarrow$ necessary + sufficient.

<u>Recepie:</u> Given $\hat{\rho}_{AB}$ ← Guaranteed pure.

$$tr_A \{ \hat{\rho}_{AB} \} = \hat{\rho}_B \Rightarrow \text{diag. } \rho_A \Rightarrow \# \text{ of } \neq 0 \text{ eigenvalues} > 1 \; ?$$

what if $\hat{\rho}_{AB}$ is non-pure?

what does mixed-state entanglement mean?

we say that $\hat{\rho}_{AB}$ is separable iff $\hat{\rho}_{AB}$ <u>can</u> be written as

$$\rho_{AB} = \sum_{i,j} P_{ij} \, \rho_{A,i} \otimes \rho_{B,j}$$

A separable mixed state can also be written as a stat. mixture of entangled states:

<u>example:</u>

$$\rho = \frac{1}{4} I \otimes I \qquad \begin{bmatrix} 0.25 & & & 0 \\ & 0.25 & & \\ & & 0.25 & \\ 0 & & & 0.25 \end{bmatrix}$$

$$\rho = \frac{1}{4} \{ \underset{|1\rangle\langle 1| \otimes |1\rangle\langle 1|}{|11\rangle\langle 11|} + \underset{|1\rangle\langle 1| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |0\rangle\langle 0| \otimes |0\rangle\langle 0|}{|10\rangle\langle 10| + |01\rangle\langle 01| + |00\rangle\langle 00|} \} = \frac{1}{4} \{ |\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-| \}$$

Peres Horodecki Criteria

- How do I know whether a density matrix

is entangled — sufficient

separable — necessary

Partial transpose

$\hat{T}$ operator: $\rho = \sum_{ij} g_{ij} |i\rangle\langle j| =$

$$\rho^T = \sum_{ij} g_{ij} |j\rangle\langle i|$$

T: preserves trace + it is a positive map
also hermitian

i.e. if ρ is Hermitic then $\rho^{T\dagger} = \rho^T$ ✓ and has the same

real eigenvalues.

$\Rightarrow$ ρ^T is positive $\Rightarrow$ T a positive map.

However T is not completely positive.

$PT = T \otimes I$ is not a positive map

$|\psi\rangle = \frac{1}{\sqrt{2}} \left[|00\rangle + |11\rangle \right]$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}}_{\rho} \quad \overset{\text{SWAP}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \quad \pm 1 \text{ eig's}$$

$\rho = \frac{1}{2} \left[|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11| \right]$

$\rho^{PT} = \frac{1}{2} \left[|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11| \right]$

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For a separable state:

$$\rho_{AB} = \sum_{i,j} p_{ij}\, \rho_i^A \otimes \rho_j^B \qquad \rho_{AB}^{PT} = \sum_{i,j} p_{ij}\, \rho_i^{AT} \otimes \rho_j^B$$

legit. density op.

Eig of ρ_{AB}^{PT} <u>always</u> positive.

For $2\otimes 2$ density matrices (i.e. 2 qubits)

necessary + sufficient.