# Quantum Error Correction

Digital error correction - classical

use redundancy to protect data.

(B - Bravo, C - Charlie ...)

Example (simplest): repetition code

$$0 \to 000 = \tilde{0}$$
$$1 \to 111 = \tilde{1}$$

protects against: independent bit flips.
assumption

decoding: majority ruling:

$$\begin{matrix} 000 \\ 100 \\ 010 \\ 001 \end{matrix} \to \tilde{0} \qquad \begin{matrix} 111 \\ 110 \\ 101 \\ 011 \end{matrix} \to \tilde{1}$$

$p$ = prob. for bit flip.

$$\Rightarrow P_e = 3p^2(1-p) + p^3 < p$$

$$\Rightarrow p < \frac{1}{2}$$

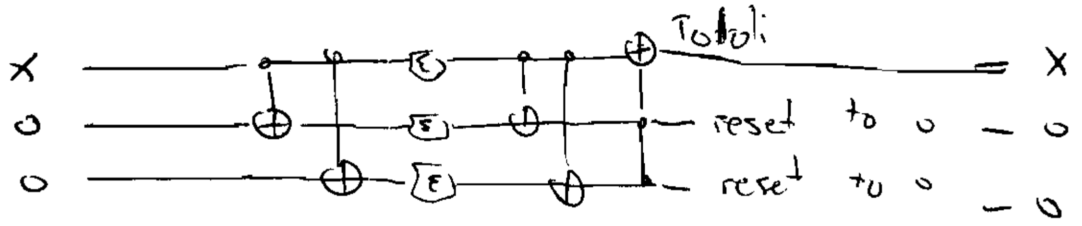
- Can we use the repetition code to protect Q information against bit flips? $\sigma_x \equiv \tilde{X}$ ?

Difficulties: - No cloning.

- Meas. projects onto meas. basis.

- errors are continuous.

find a reversible circuit that performs the above EC.




ancilla bits. Decode?

notice that the circuit above

| | | |
|---|---|---|
| $\tilde{0} \to$ | $000 \to 000$ | $111 \to 100$ |
| | $100 \to 111$ | $\tilde{1} \to 011 \to 011$ |
| | $010 \to 010$ | $101 \to 110$ |
| | $001 \to 001$ | $110 \to 101$ |

← only error in 1st qubit.

<u>QM</u>: $x \to |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

After encoding: $|\psi\rangle \otimes |0\rangle \otimes |0\rangle \to \alpha|000\rangle + \beta|111\rangle$

<u>Errors</u>: bit-flip channel. $E_1 = \sqrt{1-p}\,\hat{I}$ $E_2 = \sqrt{p}\,\hat{X}$

$I \otimes I \otimes I$ : $(\alpha|0\rangle + \beta|1\rangle)|0\rangle|0\rangle \to \alpha|000\rangle + \beta|111\rangle \to$

$\to (\alpha|0\rangle + \beta|1\rangle)|00\rangle$ ✓

$I \times I$: $\alpha|0\rangle + \beta|1\rangle \to \alpha|0\,0\,0\rangle + \beta|1\,1\,1\rangle$

$\to \alpha|0\,1\,0\rangle + \beta|1\,1\,0\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes |1\,0\rangle$

$X\,11 \to \alpha|1\,1\,0\,0\rangle + \beta|0\,1\,1\rangle \to \alpha|1\,1\,1\rangle + \beta|0\,1\,1\rangle =$

$= (\alpha|1\rangle + \beta|0\rangle)\;|1\,1\rangle$

↑ bit flip.

Either run a Toffoli or meas. ancillas and correct.

<u>What about continuous rotations?</u>

$E \to e^{i\frac{\theta}{2}X} \otimes I \otimes I\;(\alpha|000\rangle + \beta|111\rangle)$

$= \cos(\frac{\theta}{2})\left[\alpha|000\rangle + \beta|111\rangle\right] + i\sin\frac{\theta}{2}\left(\alpha|100\rangle + \beta|011\rangle\right)$

$\{\alpha|0\rangle + \beta|1\rangle\} \otimes |00\rangle$

Decoding $\to \cos\left(\frac{\theta}{2}\right)\left[\alpha|000\rangle + \beta|100\rangle\right] +$

$+ i\sin\left(\frac{\theta}{2}\right)\left\{\alpha|111\rangle + \beta|011\rangle\right]$

$\{\alpha|1\rangle + \beta|0\rangle\} \otimes |11\rangle$

meas. ancillas or Toffoli and apply either $III$ (if meas. Syndrome $00$) or $X11$ (if meas. $11$)

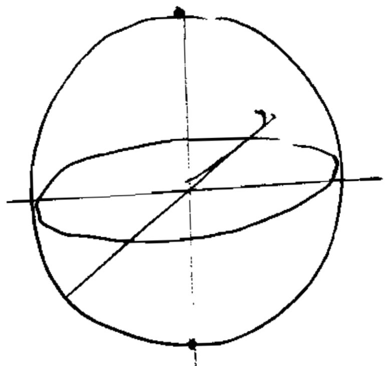
<u>Morals:</u>

Morals: Mitigated difficulties:

- No cloning $\Leftrightarrow$ entanglement.
- Meas. projects $\Leftrightarrow$ ancilla meas. tells us nothing about $\alpha$ & $\beta$.
- continuity of error: meas. projects on a discrete set of options.

More generally: we encode our data in the subspace.

$|111\rangle = |\tilde{\psi}\rangle$



$|000\rangle = |\psi\rangle$

Error: $X_1 11$, $1 X_2 1$, $1 1 X_3$ move superpositions $\alpha|\psi\rangle + \beta|\tilde{\psi}\rangle \to$ to an orthogonal subspace

but do not distort the superposition

i.e. $X11|\psi\rangle$ orthogonal to $X11|\tilde{\psi}\rangle$ etc.

decoding: meas. in which subspace our superposition is:

<u>operators</u>: $S_1 = Z_1 Z_2 I$  $S_2 = Z_1 \cancel{Z_2} Z_3$ ← parity checks.

$|\psi\rangle$ and $|\tilde{\psi}\rangle$ are degenerate (+1) eigenstate of $S_1, S_2$.

⇒ meas. these tells nothing about $\alpha$ & $\beta$.

subspaces after error:

| | | | |
|---|---|---|---|
| $\|100\rangle$ | $\|011\rangle$ | e.g. | $-1, -1$ |
| $\|010\rangle$ | $\|101\rangle$ | -"- | $-1, 1$ |
| $\|001\rangle$ | $\|110\rangle$ | -"- | $+1, -1$ |

<u>Identity subspace.</u>

<u>Gates in the protected subspace?</u>

$\tilde{X} = X_1 X_2 X_3$    $|000\rangle \to |111\rangle$

$\tilde{Z} = Z_1 I I$    $\frac{1}{\sqrt{2}}[|000\rangle + |111\rangle]$

$\to \frac{1}{\sqrt{2}}[|000\rangle - |111\rangle]$

<u>redundancy</u>: since deg. eig. of $S_1 = Z_1 Z_2 I$, $S_2 = Z_1 I Z_3$

(also of $I Z_2 Z_3 = S_1 \cdot S_2$)

$\tilde{X} = (X_1 X_2 X_3)(Z_1 I Z_3) \propto Y_1 X_2 Y_3$

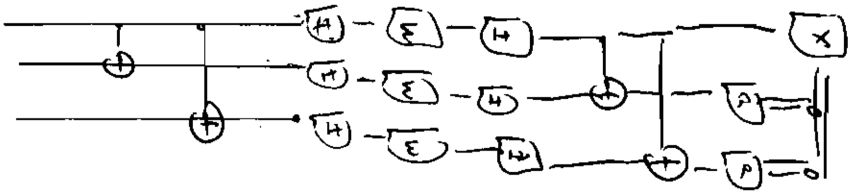
$\tilde{Z} = (Z_1 I I)(Z_1 Z_2 I) \propto I Z_2 I$

## what about phase errors? flip.

$Z11$, $1Z1$, $11Z$ ?

encode in: $|{+}{+}{+}\rangle$ $\quad |{-}{-}{-}\rangle$

where $|\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$




$$S_1, S_2 \Rightarrow X_1 X_2 I \;,\; X_1 I X_3$$

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## Shor code: protect against both bit & phase flip.

$$|\psi\rangle = \left[\tfrac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\right]^{\otimes 3} = \frac{1}{2\sqrt{2}}\left(|000\rangle + |111\rangle\right)\otimes\left[|000\rangle + |111\rangle\right]\otimes\left(|000\rangle + |111\rangle\right)$$

$$|\tilde{\psi}\rangle = \left(\tfrac{1}{\sqrt{2}}(|000\rangle - |111\rangle)\right)^{\otimes 3} = \frac{1}{2\sqrt{2}}\left(|000\rangle - |111\rangle\right)\otimes\left[|000\rangle - |111\rangle\right]\otimes\left(|000\rangle - |111\rangle\right)$$

$S_1 = Z_1 Z_2$ $\qquad S_7 = (X_1 X_2 X_3)(X_4 X_5 X_6)$

$S_2 = Z_1 Z_3$

$S_8 = (X_1 X_2 X_3)(X_7 X_8 X_9)$

$S_3 = Z_4 Z_5$

$S_4 = Z_5 Z_6$ $\qquad$ 8 bit sindrome

$S_5 = Z_7 Z_8$ $\qquad$ that tells us which error occured.

$S_6 = Z_7 Z_9$ $\qquad$ 12 non-trivial sindrome outcomes.

vs. $9 \times 2$ possible single qubit errors.
$\Rightarrow$ degenerate code

Gates: ? $\tilde{X}$ $\quad |\psi\rangle \Leftrightarrow |\tilde{\psi}\rangle$

e.g. $\check{X}_1 : Z_1 Z_4 Z_7 \qquad \tilde{X}_i = \check{X}_1 S_i$

$\check{X}_2 : Z_2 Z_5 Z_8$

etc.

$\tilde{Z}$

$$\frac{1}{\sqrt{2}}\left[ |\psi\rangle + |\tilde{\psi}\rangle \right] \Rightarrow \frac{1}{\sqrt{2}}\left[ |\psi\rangle - |\tilde{\psi}\rangle \right]$$

e.g. $\check{Z}_1 : X_1 X_2 X_3$

etc..

Shor code protects also against $Y\mathbb{1}\mathbb{1}$, $\mathbb{I}Y\mathbb{I}$, $\mathbb{I}\mathbb{I}Y$.

Why?

$$Y = iZX$$

$\Rightarrow$ corrects against any single qubit error.

$\therefore$ also corrects for any linear combination single qubit errors (syndrom projects or are optim) and any superoperator (Quantum operation) containing only single qubit errors

<u>Remark</u>: single qubit $\mathbb{I}$ in Hamiltonian $\Rightarrow$ many correlated flips in time evolution.

$$U = e^{i\mathcal{H}t/\hbar}$$

<u>Solution</u>: correct repetitively after short times: $\bigotimes_9 (\mathbb{I} + \varepsilon X) = \mathbb{I} + \varepsilon(x \ldots\ )$
$+ \varepsilon^2(xx \ldots)) : ?$

## General properties of QECC

n- physical qubits $\Rightarrow \mathcal{H}$ Hil. of d. $2^n$.

k- logical qubits $\Rightarrow$ code subspace d. $2^k$.

$C$ $\qquad C \in \mathcal{H}$

C. protects against a set of errors $\{E_i\} \in G$

All possible generators in $\mathcal{H} \Rightarrow$ pauli Group. $G$

multiplication of $X, Y, Z, I$ for the different qubits

up to a phase $4^n$ elements in $G$.

- Weight $t$ of an element in $G$ = # of non-trivial operators in product.

$\Rightarrow$ Error occurs at $t$ qubits. typ. $\{E_i\}$ to be all elements in $G$ with weight up to and inc. $t$.

what are the conditions for a subspace $C$ to be a good QECC for $\{E_i\}$?

<u>necessary condition:</u> $\forall\ |\phi\rangle, |\psi\rangle \in C$ s.t. $\langle\phi|\psi\rangle = 0$

$$\langle\psi| E_b^{+} E_a |\phi\rangle = 0$$

i.e. Error cannot "distort" code superpositions

only move to outside the code

Remarks: $d$ = code distance for any orthonormal basis $\{|i\rangle\}$ of code $C_i$

$\Rightarrow d$ = the minimal weight operators $A_d$ such that:

$$\langle j | A_d | i \rangle \neq 0$$

in the logical basis: Hamming distance.

QEC code is denoted by $[n, k, d]$

what is the distance of Shor's rep. code?

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$\Rightarrow$ a code of distance $d$, can correct at most $t$ error satisfying. $d > 2t$

$$\Rightarrow d \geq 2t + 1$$

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Is this condition sufficient?

clearly not because if $\langle \phi | E_b^\dagger E_a | \phi \rangle = \delta_{(a,b)}$

our correction operation would depend on $|\phi\rangle$

Big no-no. $\Rightarrow$ we can't know anything about $|\phi\rangle$ to correct.

one sufficient condition is: $|i\rangle, |j\rangle \in$ orthonormal basis

$$\langle i | E_b^\dagger E_a | j \rangle = \delta_{ab} \delta_{ij}$$

$\Rightarrow$ different errors move states from $C_i$ to orthogonal sub-spaces.

$\Rightarrow$ Non-degenerate codes.

Quantum Hamming bound:

$$\sum_{j=0}^{t} 3^j \binom{n}{j} 2^k \leq 2^n$$

$\underbrace{\phantom{2^n}}$ # of linearly ind. states.