Stabilizer codes

Quantum Error correction: reminder

An example: Shor's repetition code.

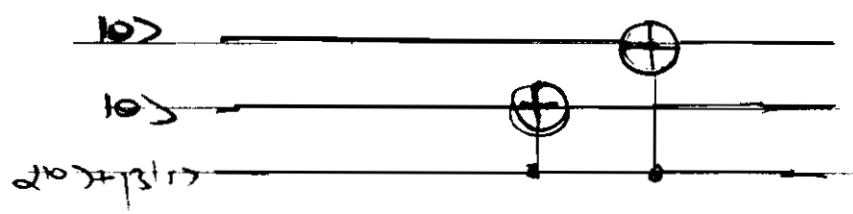
$\alpha|0\rangle_L + \beta|1\rangle_L \Rightarrow \alpha|000\rangle + \beta|111\rangle$

protects against $\hat{\sigma}_x$ noise on one qubit

Errors: $\hat{\sigma}_x^1 \otimes \hat{I}^2 \otimes \hat{I}^3$; $\hat{I}^1 \otimes \hat{\sigma}_x^2 \otimes \hat{I}^3$; $\hat{I}^1 \otimes \hat{I}^2 \otimes \hat{\sigma}_x^3$

attribute: X I I ; I X I ; I I X

encode 2 C-NOTs



$$C_{\hat{\sigma}_x}\hat{I}\otimes\hat{I}\left(\alpha|0\rangle_L + \beta|1\rangle_L\right) = \alpha\left(\cos\tfrac{\theta}{2}|000\rangle - i\sin\tfrac{\theta}{2}|100\rangle\right)$$

$$+\beta\left(\cos\tfrac{\theta}{2}|111\rangle - i\sin\tfrac{\theta}{2}|011\rangle\right) =$$

$$= \cos\tfrac{\theta}{2}\left(\alpha|000\rangle + \beta|111\rangle\right) - i\sin\tfrac{\theta}{2}\left(\alpha|100\rangle + \beta|011\rangle\right)$$

How can we tell which error occurred?

Decode again 2 Qubits $00 \Rightarrow$ No error $I \otimes I \otimes I$

$11 \Rightarrow \sigma_x$ on qubit L. $X \otimes I \otimes I$

Same protocole:	Syndrom	Error
	00	$I \otimes I \otimes I$
	11	$X \otimes I \otimes I$
	10	$I \otimes X \otimes I$
	01	$I \otimes I \otimes X$

To correct against z errors $\Rightarrow$ Looks like X error in the $|0\rangle + |1\rangle$ $|0\rangle - |1\rangle$ basis

to correct both x and z errors: concatenate

$\Rightarrow |0\rangle_L = (|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$

$|1\rangle_L = (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$

inner layer corrects for x errors, outer layer: z errors.

Since $XZ = -iY$, this code also corrects for Y errors $\Rightarrow$ any single qubit errors.

called Shor's 9 qubit code.

Repetition code: stabilizers

what meas. are we performing?

parity check matrix $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ =

3 qubits code. parity of 1 & 2 and 2 & 3: $= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

$\hat{S}_1 = Z\,Z\,I$ and $I\,Z\,Z = \hat{S}_2$

(In each part of superposition we first align the two ancilla qubits with the information qubit, and then measure it flipped.)

if qubits parallel $\Rightarrow$ eig = +1

if qubits anti-parallel $\Rightarrow$ eig = -1

notice that $\{\hat{S}_1, \hat{S}_2\} = 0$

and that the spin flip errors:

$E_1 = X\,I\,I$

$E_2 = I\,X\,I$

$E_3 = I\,I\,X$

$\{E_1, S_1\} = 0 \quad \{E_2, S_1\} = \{E_2, S_2\} = 0 \quad \{E_3, S_2\} = 0$

- S_1 & S_2 are the generators of the Abelian group whose members are $\pm\{ZZI, ZIZ, IZZ\}$

S_1 & S_2 are the code stabilizers.

For the 9-qubit code, 8 stabilizers:

$$\hat{S}_1 = ZZIIIIIII$$
$$= ZZ$$
$$= ZZ \quad I$$
$$= ZZ$$
$$= I \quad ZZ$$
$$= ZZ$$
$$= XXXXXXIII$$
$$\hat{S}_8 = IIIXXXXXX$$

again $[S_i, S_j] = 0$ are the generators of an Abelian group and each error-

$I \otimes I \otimes \ldots \underset{\substack{X \\ Y \\ Z}}{} \otimes \ldots I$ anti-commutes with at least one $\hat{S}_i$

$$[X \otimes X, Z \otimes Z] = 0$$

$$[X \otimes Z, Z \otimes X] = 0$$

General properties of QEC codes:

n physical qubits $\mathcal{H}$ a Hilbert space of dimension 2^n

C is a QEC code: a subspace of $\mathcal{H}$

C encodes k logical qubits $\Rightarrow$ has $D = 2^k$

G is the group of all possible operators generated

by multiplication of tensor product $I_i, X_i, Y_i, Z_i \quad i = 1 \to n$ (4^n)

(Pauli group)

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$I|a\rangle = |a\rangle \quad X|a\rangle = |a \oplus 1\rangle \quad Z|a\rangle = (-1)^a |a\rangle \quad Y|a\rangle = i(-1)^a |a \oplus 1\rangle$

$\oplus$ exclusive or

$X^2 = Y^2 = Z^2 = I \qquad XY = iZ \quad XZ = -iY \Leftrightarrow [A, B] = 2i\varepsilon_{ABC} C$

$AB = i\varepsilon_{ABC} C$

all anti-commute: $\{X, Y\} = \{X, Z\} = \{Y, Z\} = 0$

$\Rightarrow$ number of elements in G (order) $= 4^n$ (up to a $\pm 1, \pm i$ phase)

$\Rightarrow 4^{n+1}$ elements.

Properties of elements in G

- All elements in G square to ± 1
- If $A, B \in G$ then either $[A,B] = 0$ or $\{A,B\} = 0$
- The weight of an element in G = number of entries in the tensor product $\neq \hat{I}$.

Since the Pauli group is a basis for all $2^n \times 2^n$ unitary matrices $\Rightarrow$ All unitary operation on at most t qubits can be written as linear combinations of $\{E^+\}$ = all members of G of weight t or less.

example:

$$\exp\left[-i\frac{\hat{\sigma}\cdot\hat{n}\theta}{2}\right] = \cos\left[\frac{\theta}{2}\right]\hat{I} - i\hat{\sigma}\hat{n}\sin\left[\frac{\theta}{2}\right]$$

General single qubit rotation

Theorem: If code C corrects errors $\hat{A}$ & $\hat{B}$ it corrects any linear combination of $\alpha\hat{A} + \beta\hat{B}$

(Non-) proof: in QEC we project the resulting superpositions onto syndromes. (can be the same or diff for A & B).

For the same reason, if C corrects against $\{E_i\}$ will correct against the non unitary $\hat{\rho}_{out} = \sum_i p_i E_i \rho_i E_i^\dagger$

$\Rightarrow$ Conclusion: to correct against all t-qubit errors need a code the corrects all $\{E_j^t\} \in G$

Remark: Hamiltonian contains only E^+, but operates on $n > t$ qubits $\Rightarrow$ $E_j^{t>t}$ will operate due to $\hat{U}(t) = e^{i\hat{H}t/\hbar}$

Solution: correct at a rate $\gg \langle H \rangle / \hbar$ $\Rightarrow$ correction to 1^{st} order in $\frac{\langle H \rangle t}{\hbar}$.

example:

$$\bigotimes_9 (I + \varepsilon E_i) = I + \varepsilon \left(E_1 \otimes I^{\otimes 8} + I \otimes E_2 \otimes I^{\otimes 7} + \dots \right) + O(\varepsilon^2)$$

C is $[n, k, t]$ code. corrects all errors in $\{A\}$

n: Phys qubits; k: Log. qubits; t: Errors max weight.

What do we require of the code words?

Necessary & sufficient conditions:

$$\forall\ |\psi\rangle \neq |\phi\rangle \in C \qquad \forall\ A_1, A_2 \in A$$

(1) $$\langle\psi| A_1^\dagger A_2 |\phi\rangle = 0$$

(2) $$\langle\psi| A_1^\dagger A_2 |\psi\rangle = \langle\phi| A_1^\dagger A_2 |\phi\rangle$$

(1) is easy to understand: even after errors occured the resulting states have to be resolved with certainty

$\Rightarrow$ orthogonal.

(2) in a sufficient form: $\langle\psi| A_1^\dagger A_2 |\psi\rangle = 0$

$\Rightarrow$ i.e. different errors on the same codeword result orthogonal states. Not necessary in the 9 qubit code

$$\langle 0| Z_1^\dagger Z_2 |0\rangle = \langle 1| Z_1^\dagger Z_2 |1\rangle = 1$$

conditions 1 & 2 can be written as:

$$\langle \psi_i | A_a^\dagger A_b | \psi_j \rangle = C_{ab} \delta_{ij}$$

C_{ab} is Hermitian and ind. of i,j.

$\Rightarrow$ C_{ab} can be diagonalized $\Rightarrow$ a new basis for A

in which $\langle \psi | A_a^\dagger A_b | \psi \rangle = 0 \quad a \neq b$

$$\langle \psi | A_a^\dagger A_a | \psi \rangle = \langle \phi | A_a^\dagger A | \phi \rangle$$

$\Rightarrow$ A_a can be inverted on code words

If all rows of C are linearly ind. $\Rightarrow$ nondegenerate code

otherwise $\Rightarrow$ degenerate code. $\Leftarrow$ 0 eig. values.

In a non-degenerate code diff errors result linearly ind states. $\Rightarrow$ # of linearly ind states req.

$$\sum_{j=0}^{t} 3^j \binom{n}{j} 2^k \leq 2^n \quad \Leftarrow \text{Quantum Hamming bound.}$$

- An open question in QIP can degenerate QEC codes violate the Q Ham. bound?

no examples are known.

code stabilizes:

C is an $[n,k,t]$ code. A relation to DFS

$$\mathcal{M} = \{ M \in G \text{ s.t. } M|\psi\rangle = |\psi\rangle \ \forall |\psi\rangle \in T \}$$

is the code stabilizer

Suppose $E \in G$ and $\exists M \in \mathcal{M}$ s.t. $\{E, M\} = 0$

$$\Rightarrow \langle\phi|E|\psi\rangle = \langle\phi|EM|\psi\rangle = -\langle\phi|ME|\psi\rangle = -\langle\phi|E|\psi\rangle$$

$$\Rightarrow \langle\phi|E|\psi\rangle = 0.$$

$\Rightarrow$ The set of operators $\{A_i\}$ such that $\forall A_i, A_j$ $\exists M \in \mathcal{M}$

$\{A_i^\dagger A_j, M\} = 0$ or $A_i^\dagger A_j \in \mathcal{M}$ ← Degenerate code

Are the set of errors C corrects.

Properties of $\mathcal{M}$: $\mathcal{M}$ is a group under multiplication

1. $M^2|\psi\rangle = M|\psi\rangle = |\psi\rangle \Rightarrow M^2 = +1$

2. $MN|\psi\rangle = NM|\psi\rangle \Rightarrow [M,N] = 0$

(or C is too trivial)

$\mathcal{M}$ - Abelian group.

3. # of elements in $\mathcal{M}$ (order) $= n-k$

Operators in G that are not connected by C are those that commute with all elements in M but are not in M. $N(S)/S$

Normalizer / centralizer

THEOREM: Let M be an Abelian subgroup of order 2^a of G s.t. $-1 \in S$. If d is the minimum weight of an operator in $N(M)/M$, the the space of states C that are stabilized by M (i.e $\forall m \in M \; m|\psi\rangle = |\psi\rangle$) is an $[n, n-a, (d-1)/2]$ code.

$|\psi\rangle \in C$

$d/2$

How can we construct a stabilizer code?

Given a set of errors $\{A_i\}$ we want to correct (perh. all errors of weight t) against.

1. Find a abelian sub-group M of G s.t. $\forall A_i : \exists M \in \mathcal{M}$ $\{A_i^\dagger A_i, M\} = 0$
(Anti-commutes with all $A \in \mathcal{E}$ up to t weight)

2. Construct the code words. start with a seed $|\phi\rangle$ part of a basis.

$$|\psi_1\rangle = \sum_{M \in \mathcal{M}} M |\phi\rangle$$

choose $|\phi'\rangle$ not in $|\psi_1\rangle$ ⇒ repeat until exhaust all $|\phi'\rangle$'s

3. Syndrome measure in the M basis.

- Relation to parity check matrix

- Take home message:

  - most QEC can be thought of as stab. codes

Advantages:

- Charac. code in terms of syndrom rather than code words.
- An algorithm to construct C out of $\{A_i$'s$\}$
- Analogy to classical EC