Quantum Operations:

Gates etc. are great, but what happens if we do not implement them perfectly?

We have to find a way to describe imperfect i.e. Non-unitary operations.

maps → $\mathcal{E}(\rho_{in}) = \rho_{out}$

Three approaches:

operator sum rep. $\mathcal{E}(\rho_{in}) = \sum_k E_k \rho E_k^\dagger$

1. In analogy with density matrices

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k| \leftarrow \text{statistical mixture of pure states } |\psi_k\rangle$$

$$\mathcal{E}(\rho) = \sum_k p_k U_k \rho U_k^\dagger \leftarrow \text{statistical mixture of unitary operations } U_k \rho U_k^\dagger$$

$$\equiv \sum_k E_k \rho E_k^\dagger$$

$$- \sum_k p_k = 1$$

$$0 \leq \mathrm{tr}(E_k \rho E_k^\dagger) = p_k \leq 1$$

2. Taking the analogy with Density matrices further.

$$\rho_A = \mathrm{Tr}_B \left[|\psi_{AB}\rangle\langle\psi_{AB}| \right]$$

non-pure iff $|\psi_{AB}\rangle$ entangled (Schmidt # > 1)

Any ρ_A can be written in the form above.

i.e. any stat. mixture (Schmidt decomposition + transfer in B-basis places)

Can we write any $\mathcal{E}_A$ as a Unitary in $A \otimes B$ + P. trace B?

For operations let's choose a basis $|e_k\rangle$ for system B (environment)

and assume w.l.o.g. the initial state of the environment is separable and $|e_0\rangle\langle e_0|$

$$\mathcal{E}(\rho) = \sum_k \langle e_k| U [\rho \otimes |e_0\rangle\langle e_0|] U^\dagger |e_k\rangle =$$

partial trace   Unitary sys. + environ.

$$= \sum_k E_k \rho E_k^\dagger \quad \text{where} \quad E_k = \langle e_k| U |e_0\rangle$$

$\sum_{ij} |e_i\rangle|\psi_j\rangle \langle\psi_j| \langle e_i|$

$$E_k \rho E_k^\dagger = p_k U_k \rho U_k^\dagger$$

$\leftarrow$ prob. p_k to measure env. in $|e_k\rangle$ and then $\rho_{out} = U_k \rho U_k^\dagger$

and $p_k = \mathrm{Tr}\left[E_k \rho E_k^\dagger \right]$

$\mathcal{E}(\rho)$ is non-unitary only if $\hat{U}$ is entangling operator.

If $\mathcal{E}(\rho)$ is trace preserving

(not necessary → loss of a qubit etc. but used here throughout.)

$$1 = Tr(\mathcal{E}(\rho)) = tr\left(\sum_k E_k \rho E_k^\dagger\right) = tr\left(\sum_k E_k^\dagger E_k \rho\right)$$

since true for any ρ.

$$\Rightarrow \sum_k E_k^\dagger E_k = \hat{\mathbb{I}}$$

3. Axiomatic approach:

what is the most general Quantum map of ρ?

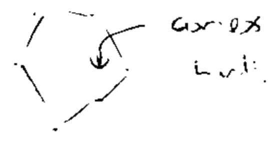
Demand 1. $\mathcal{E}$ is Trace-preserving. $tr(\mathcal{E}(\rho)) = tr(\rho) = 1$

(more general: $Tr(\mathcal{E}(\rho))$ is the prob. that $\mathcal{E}$ occurs)

Demand 2. $\mathcal{E}$ is a convex-linear map.

convex: $p_i \geq 0 \ \forall i$ & $\sum_i p_i = 1$

$$\mathcal{E}\left(\sum_i p_i \rho_i\right) = \sum_i p_i \mathcal{E}(\rho_i)$$



Demand 3. complete positivity

$\mathcal{E}$ maps density matrices to density matrices.

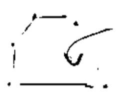
$\mathcal{E}(\rho)$ is positive if ρ is positive

moreover $(I_B \otimes \mathcal{E}_A)(\rho_{AB})$ - positive if ρ_{AB} positive.

Theorem: The map ε satisfies demands 1-3 iff

$$\varepsilon(\rho) = \sum_k E_k \rho E_k^\dagger$$

and $\sum_k E_k E_k^\dagger = \hat{\mathbb{I}}$



if:

1 & 2 obvious: convex linear and (convex: $\sum p_i = 1$ & $p_i \geq 0$ $\forall i$)

$$tr\{\sum_k E_k \rho E_k^\dagger\} = tr\{\sum_k E_k^\dagger E_k \rho\} = tr(\rho) = 1$$

3. completely positive: ; combined system in $A \otimes B$

Define $|\varphi_k\rangle = (I_B \otimes E_k^\dagger)|\psi\rangle$ $\qquad |\psi\rangle \in A \otimes B$

$\langle \varphi_k | \varphi_k \rangle \geq 0$ $\qquad \rho \in A \otimes B$ positive

$$\langle \psi | (I_B \otimes E_k) \rho (I_B \otimes E_k^\dagger) | \psi \rangle = \langle \varphi_k | \rho | \varphi_k \rangle \geq 0$$

by linearity:

$$\sum_k \langle \psi | (I_B \otimes E_k) \rho (I_B \otimes E_k^\dagger) | \psi \rangle \geq 0$$

(w)

Quantum operations: reminder

we were talking about universal gate sets.

to account for imperfections $\Rightarrow$ General, not necessarily unitary operations.

maps $\rho \Rightarrow \mathcal{E}(\rho) \equiv$ super operators.

operator sum rep: $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$

where $\sum_k E_k E_k^\dagger = \hat{\mathbb{1}}$ ~ Kraus operators.

- statistical mixture of unitaries.

- unitary ~ $A \otimes B$ + partial trace over B.

- Every map which satisfies:

  - convex linear: $\mathcal{E}(\sum_i p_i \rho_i) = \sum_i p_i \mathcal{E}(\rho_i)$

  - Trace preserving

  - completely positive.

    i.e. $\mathcal{E}(\rho)$ ← positive $\forall \rho$

    and $I \otimes \mathcal{E}(\sigma_{AB})$ - positive $\forall \sigma$ w.a.

can be written as $\sum E_k \rho E_k^\dagger$ with $\sum_k E_k E_k^\dagger = \hat{I}$ and any such satisfies above

<u>only if:</u> by construction from its spectrum on the $A\otimes B$ states.

$|i\rangle_A$ orthonormal basis to A same d.

$|i\rangle_B$ -"- B

same index.

$$|\Omega\rangle = \sum_i |i\rangle_A |i_B\rangle \leftarrow \text{maximally entangled.}$$

(up to normalization)

look at density matrix.

complete positivity.

$$\sigma = (I_B \otimes \mathcal{E}_A)(|\Omega\rangle\langle\Omega|) =$$

$$= \sum_{i,j} |i\rangle_B {}_B\langle j| \, \mathcal{E}(|i\rangle_A {}_A\langle j|)$$

define $|\psi\rangle_A = \sum_j \psi_j |j\rangle_A \qquad |\tilde{\psi}\rangle_B = \sum_j \psi_j^* |j\rangle_B$

$$\langle\tilde{\psi}|\sigma|\tilde{\psi}\rangle = \langle\tilde{\psi}| \sum_{i,j} |i\rangle_B {}_B\langle j| \, \mathcal{E}(|i\rangle_A {}_A\langle j|) |\tilde{\psi}\rangle =$$

$$= \sum_{i,j} \psi_i \psi_j^* \mathcal{E}(|i\rangle_A {}_A\langle j|) = \mathcal{E}(|\psi\rangle\langle\psi|)$$

look at some decomposition of $\sigma = \sum_i |s_i\rangle\langle s_i|$ unnormalized

$$\langle\tilde{\psi}|\sigma|\tilde{\psi}\rangle = \sum_i \langle\tilde{\psi}|s_i\rangle\langle s_i|\tilde{\psi}\rangle \equiv \sum_i E_i|\psi\rangle\langle\psi|E_i^\dagger$$

$\shortparallel$

$\mathcal{E}(|\psi\rangle\langle\psi|) \qquad E_i|\psi\rangle = \langle\tilde{\psi}|s_i\rangle$

since true for <u>any</u> pure state $|\psi\rangle$ by linearity $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$ ...

Connection to Master eq.

Diff. eq. approach:

Unitary evolution: $\dot{\rho} = -\frac{i}{\hbar}[\mathcal{H},\rho]$ - Schrödinger's eq.

non-unitary evolution for Markovian processes.

$$\dot{\rho} = -\frac{i}{\hbar}[\mathcal{H},\rho] + L\rho L^{+} - \frac{1}{2}(\rho L^{+}L + L^{+}L\rho)$$

Lindblad eq.

Connection: Lindblad operators are generators of Q. operators
+ Hamiltonian

to see this, lets look at maps that are close to $\hat{\mathbb{I}}$.

$$\Rightarrow \quad \mathcal{E}(\hat{\rho}) = \hat{\rho} + \Delta\hat{\rho}$$

write $\quad \mathcal{E}(\hat{\rho}) = A\rho A^{+} + B\rho B^{+}$

where $\quad AA^{+} + BB^{+} = \hat{\mathbb{I}}$

$A = \hat{\mathbb{I}} + \delta\hat{a} \qquad \delta \in \mathbb{R}$

$A^{+}A + B^{+}B = \hat{\mathbb{I}} + O(\delta^{2})$

let's null 1^{st} order

$\Rightarrow \quad B = \sqrt{\varepsilon}\, b + O(\delta)$

$(\mathbb{I} + \delta a^{+})(\hat{\mathbb{I}} + \delta a) + \varepsilon b^{+} b = \mathbb{I} + O(\delta^{2})$

$$\Rightarrow \quad a^{+} + a + b^{+} b = 0$$

can only work if $\quad a = i\mathcal{H} - \frac{1}{2} b^{+} b$

and $\mathcal{H}$ - hermitian.

$$\mathcal{E}(\hat{\rho}) = \hat{\rho} + \delta\hat{\rho} = \left[\hat{\mathbb{1}} + \delta(i\mathcal{H} - \tfrac{1}{2} b^{+} b)\right] \hat{\rho} \left[\hat{\mathbb{1}} + \delta(-i\mathcal{H} - \tfrac{1}{2} b^{+} b)\right]$$

$$+ \delta\, b \rho b^{+} =$$

$$\cong \hat{\rho} + \delta(i\mathcal{H} - \tfrac{1}{2} b^{+} b)\rho + \delta \rho(-i\hat{\mathcal{H}} - \tfrac{1}{2} b^{+} b) + \mathcal{O}(\delta^{2}) + \delta b \rho b^{+}$$

$$= \frac{\mathcal{E}(\rho) - \hat{\rho}}{\delta} = i[\mathcal{H}, \hat{\rho}] + b \rho b^{+} - \tfrac{1}{2} \rho b^{+} b - \tfrac{1}{2} b^{+} b \rho = \frac{d\rho}{d\delta}$$

the master eq. is a direct consequence of

req. 1-3.

- freedom is determining decision between $\mathcal{H}$ and b.

Freedom in the operator-sum representation:

E.g. $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^{\dagger}$ $\quad E_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$E_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

stochastic π rotation around $\hat{z}$.

$\mathcal{F}(\rho) = \sum_i F_i \rho F_i^{\dagger}$ $\quad F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$F_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Projection along $\hat{z}$.

$\mathcal{F}(\rho) = \mathcal{E}(\rho) \Rightarrow$ easy to see 1. invariant state.

2. $\mathcal{F}(\rho) = \frac{(E_1+E_2)\rho(E_1+E_2)}{2} + \frac{(E_1-E_2)\rho(E_1-E_2)}{2} =$

$= E_1\rho E_1^{\dagger} + E_2\rho E_2^{\dagger} = \mathcal{E}(\rho)$

Freedom can be seen e.g. by the choise of $\sigma = \sum_i |s_i\rangle\langle s_i|$

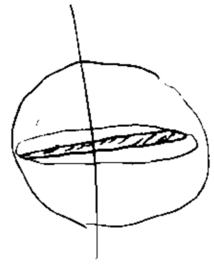
arb. choise of $|s_i\rangle$'s

Different Quantum Channels: (non-unitary).

For a single spin. map from Bloch sphere surface to volume

Bit flip: $E_1 = \sqrt{p}\,\hat{I} \quad E_2 = \sqrt{1-p}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sqrt{1-p}\,\hat{X}$

$\hat{\sigma}_x$



$|\pm x\rangle$ invariant.

phase flip $\quad E_1 = \sqrt{p}\,\hat{I} \quad E_2 = \sqrt{1-p}\,\hat{Z} \leftarrow$ Dephasing.

Bit-phase flip $\quad E_1 = \sqrt{p}\,\hat{I} \quad E_2 = \sqrt{1-p}\,\hat{Y}$

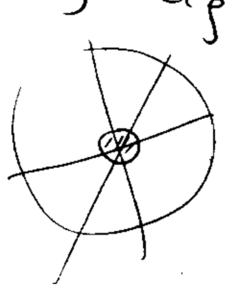
Depolarization channel:

$$\mathcal{E}(\rho) = \frac{p}{2}\hat{I} + (1-p)\hat{\rho}$$

$$\frac{I}{2} = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4}$$

scalar

$$\Rightarrow \mathcal{E}(\rho) = (1-p)\,\hat{I}\rho\hat{I} + \frac{p}{3}\left(\hat{X}\rho\hat{X} + \hat{Y}\rho\hat{Y} + \hat{Z}\rho\hat{Z}\right)$$

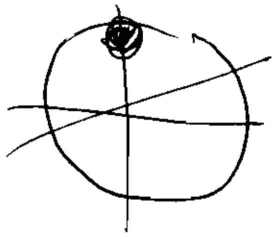


Amplitude damping channel:

$$\mathcal{E}(\rho) \; : \quad E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\beta} \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & \sqrt{\beta} \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \to (1-p) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + p \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



Quantum process tomography:

How can you tell which Quantum operation you are applying?

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$$

$$E_i = \sum_m e_{im} \tilde{E}_m \leftarrow \text{basis } d^2\text{-elements}$$

e.g. $\hat{I}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ and $\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}$ any 2x2 matrix can be written as a sum of above.

$$\Rightarrow \mathcal{E}(\rho) = \sum_{mn} \tilde{E}_m \rho \tilde{E}_n^\dagger \chi_{mn}$$

where $\chi_{mn} = \sum_i e_{im} e_{in}^* \leftarrow \chi\text{-matrix.}$

# of ind. real parameters: $d^2 \times d^2$ − # of constraints.

$d = 2^n$

$$\sum_i \hat{E}_i^\dagger \hat{E}_i = \hat{I}$$

$$\Rightarrow d^4 - d^2$$

⇒ # of meas. required

for a single qubit: $16 - 4 = 12$

two qubits: $256 - 16 = 240$

inefficient, i.e. exp. with n.

In the experiment.

choose a linearly ind. basis for ρ. $\{\rho_k\text{'s}\}^{d^2}$

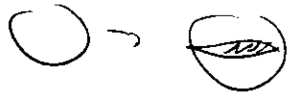
use them as input states, and measure $\mathcal{E}(\rho_k)$'s

using state tomog.

write $\mathcal{E}(\rho_j) = \sum_k \lambda_{jk}\rho_k$

Since any ρ is $\rho = \sum_j \alpha_j \rho_j$

by linearity, $\mathcal{E}(\rho) = \sum_{jk} \lambda_{jk}\alpha_k \rho_j$ ⇒ draw map



to reconstruct χ_{mn}

$$\tilde{E}_m \rho_j \tilde{E}_n^{\dagger} = \sum_k \beta_{jk}^{mn} \rho_k$$

$$\Rightarrow \mathcal{E}(\rho_j) = \sum_{mn} \chi_{mn} \sum_k \beta_{jk}^{mn} \rho_k = \sum_k \lambda_{jk} \rho_k$$

since ρ_k are linearly ind.

$$\Rightarrow \sum_{mn} \chi_{mn} \beta_{jk}^{mn} \rho_k = \lambda_{jk} \rho_k \quad \text{for each } k.$$

↑

a set of linear eq.'s — d^2 k's, d^2 j's

⇒ can be algebraically solved. — d^4 eq.

Remark:

- in state tomog. ρ is reconstructed with $\frac{1}{\sqrt{N}}$ accuracy. (even ideally)

$\Rightarrow$ Algebraically reconstructed χ_{mn} is often unphysical. i.e. lead to a non trace preserving or non completely-positive maps.

$\Rightarrow$ maximum likelihood. $\in$ other methods. — what is the physical χ_{mn} with the largest prob. of meas. {data}?