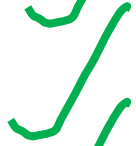


Tutorial overview



1. The ion-qubit: different ion-qubit choices, Ion traps.



2. Qubit initialization.



3. Qubit measurement.

4. Universal set of quantum gates:
single qubit rotations; 
two-ion entanglement gates

5. Memory coherence times

How well???

Benchmarked to current threshold estimates

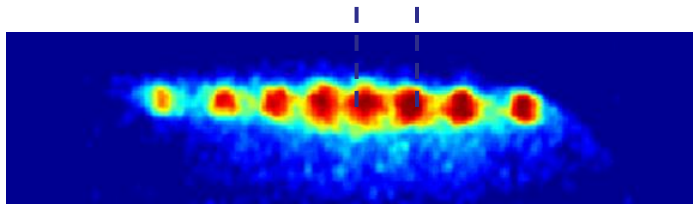
Disclaimer: non exhaustive; focuses on laser-driven gates

Magnetic spin-spin interactions:

- Magnetic dipole: spin. equivalent $\sim ea_0/137$ $ea_0 = \frac{\mu_B}{\alpha c}$
- Dipole-dipole interaction: $\xi = \mu_0(g\mu_B/2)^2/4\pi\hbar d^3$

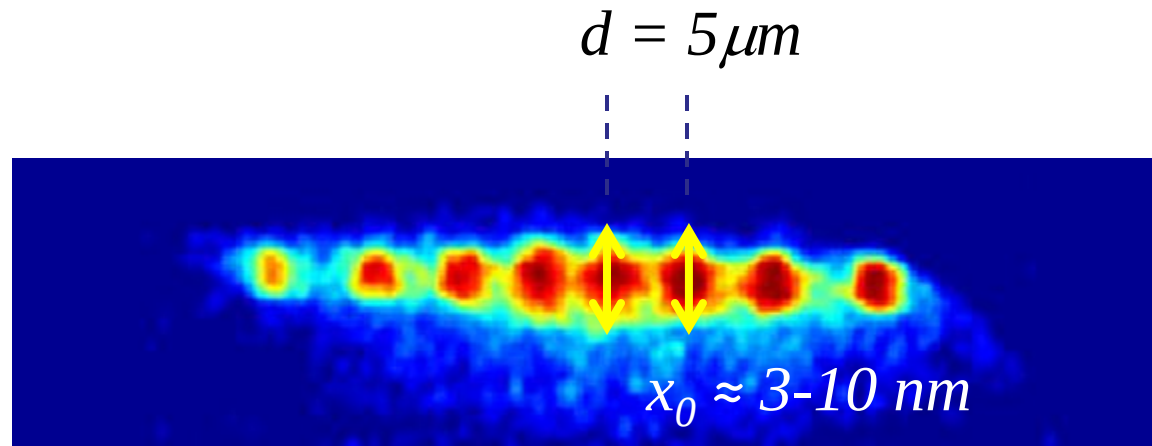
$$\xi/2\pi = \frac{13}{(r/1\mu m)^3} \text{ mHz}$$

$$d = 5\mu m$$



$$\xi/2\pi = 0.1 \text{ mHz}$$

Phonon-mediated (synthetic) - spin-spin interactions



- Spin dependent forces
- Dipole moment $\approx 200 ea_0$
- Dipole-dipole interaction:

$$x_0 = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\hbar\omega_{int} = \frac{e^2}{2\pi\epsilon_0 d^3} x_0^2$$

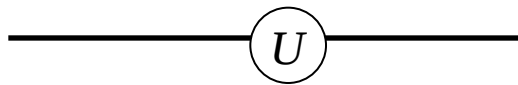
$\approx 10's - 100's \text{ kHz}$

Universal Gate set

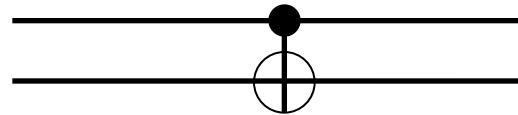
- For N -qubits, and unitary transformation U on a 2^N -dimension Hilbert space.
- A *finite* set of unitary gates that spans any such U .

Single qubit gate

$$U \in SU[2]$$



2-qubit C-not gate

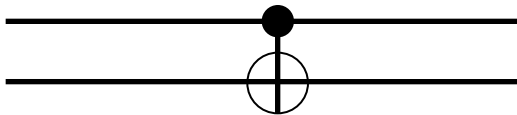


- Barenco et. al. *Phys. Rev. A*, **52**, 3457, (1995).

C-not gates vs. Phase gate

2-qubit C-not gate

$$\alpha|\uparrow\uparrow\rangle + \beta|\uparrow\downarrow\rangle + \gamma|\downarrow\uparrow\rangle + \delta|\downarrow\downarrow\rangle = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}$$



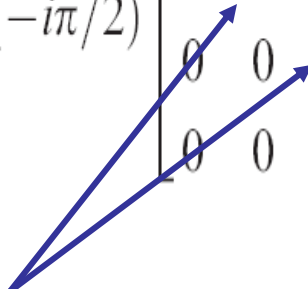
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{bmatrix}$$

A $\pi/2$ Hadamard rotation on the target qubit before and after the gate maps the C-not into a phase gate:

$$[\hat{I}_1 \otimes \hat{H}_2] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} [\hat{I}_1 \otimes \hat{H}_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

C-not gates vs. Phase gate

- A $\pi/2$ phase shift on both qubits:

$$\hat{R}_1(\pi/2, 0, \pi/2) \otimes \hat{R}_2(\pi/2, 0, \pi/2) = \exp(-i\pi/2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$


A phase shift on anti-parallel spin states vs. parallel spin states.

- Changing to the x basis:

$$[\hat{R}_1(\pi/2, 0)] \otimes [\hat{R}_2(\pi/2, 0)] \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} [\hat{R}_1(\pi/2, 0)] \otimes [\hat{R}_2(\pi/2, 0)] = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} -i & 0 & 0 & 1 \\ 0 & i & -i & 0 \\ 0 & -i & i & 0 \\ 1 & 0 & 0 & -i \end{pmatrix}$$

- Collective spin flip: $\hat{U}|\downarrow\downarrow\rangle = \frac{\exp(-i\pi/4)}{2^{1/2}}(|\downarrow\downarrow\rangle + i|\uparrow\uparrow\rangle) \quad \hat{U}^2|\downarrow\downarrow\rangle = |\uparrow\uparrow\rangle$

Phases in QM

(Berry, Aharonov-Anandan)

T- dependent. Schr. Eq. : $\hat{H}(t)|\Psi(t)\rangle = i\hbar(d/dt)|\Psi(t)\rangle$

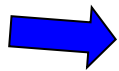
Cyclic motion. : $|\Psi(\tau)\rangle = \exp(i\phi)|\Psi(0)\rangle$

Bare state. : $|\Phi(t)\rangle = \exp[-if(t)]|\Psi(t)\rangle \quad f(\tau) - f(0) = \phi$

Eq. of motion. For f : $\frac{df}{dt} = -\frac{1}{\hbar} \langle \Psi(t) | \hat{H}(t) | \Psi(t) \rangle + i \langle \Phi(t) | \frac{d}{dt} | \Phi(t) \rangle$

$$\phi = -\frac{1}{\hbar} \int_0^\tau \langle \Psi(t) | \hat{H}(t) | \Psi(t) \rangle dt \quad \leftarrow \text{Dynamic phase}$$

Geometric phase

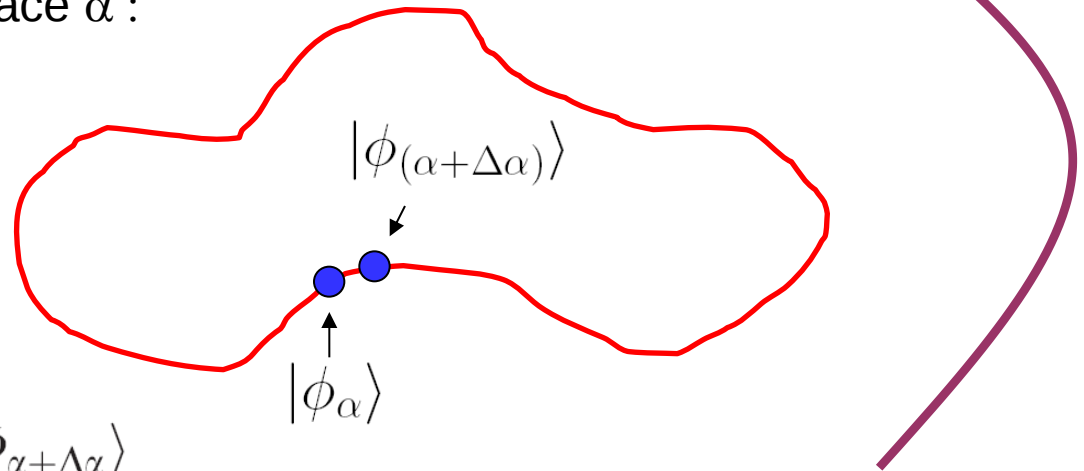


$$+ i \int_0^\tau \langle \Phi(t) | \frac{d}{dt} | \Phi(t) \rangle dt \equiv \phi_D + \gamma$$

The Geometric phase

$$|\Phi(t)\rangle \equiv |\Phi_\alpha\rangle \quad \gamma = i \int_0^\tau \left\langle \Phi(t) \left| \frac{d}{dt} \right| \Phi(t) \right\rangle dt = i \oint_c \langle \Phi_\alpha | \nabla_\alpha | \Phi_\alpha \rangle$$

A curve in parameter space α :



$$\exp(-i\Delta\gamma) = \frac{\langle \Phi_\alpha | \Phi_{\alpha+\Delta\alpha} \rangle}{|\langle \Phi_\alpha | \Phi_{\alpha+\Delta\alpha} \rangle|} \quad \longrightarrow \quad -i\Delta\phi = \langle \phi_\alpha | \vec{\nabla}_\alpha | \phi_\alpha \rangle \cdot d\vec{\alpha}$$

Vector potential: $\vec{A}_\alpha = i \langle \phi_\alpha | \vec{\nabla}_\alpha | \phi_\alpha \rangle \quad \vec{B}_\alpha = \vec{\nabla} \times \vec{A}_\alpha$

Stokes: $\gamma = - \int \int_S \vec{B} \cdot \vec{n} dS$

Encircling an area in parameter space is a necessary condition for $\gamma \neq 0$

Examples for different proposed phase gates.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A (Pseudo) spin attached to a harmonic oscillator

- Cirac and Zoller

Cirac and Zoller, *PRL*, **74**, 4091 (1995).

Geometric,
Spin d.o.f

- Spin dependent dipole forces - “Push” gate

Cirac and Zoller, *Nature*, **404**, 579 (2000).

Dynamic,
Harmonic Oscillator. d.o.f

- Spin dependent dipole forces – near resonant drive

Sorensen and Molmer, *PRL*, **82**, 1971 (1999).

Munro, Milburn, Sanders, *PRA*, **62**, 052108 (2000).

Solano, Santos, Milman, *PRA*, **64**, 024304 (2000).

Geometric + Dynamic combo,
Harmonic Oscillator. d.o.f

Off-resonantly Driven harmonic oscillator

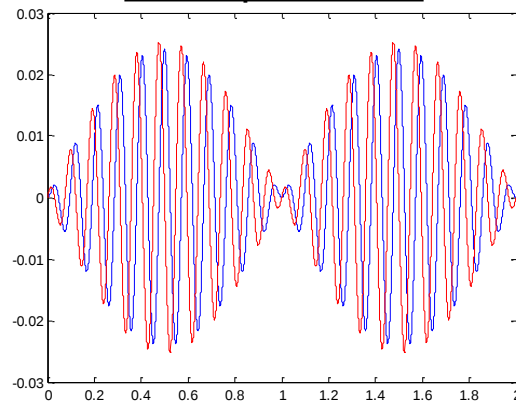
- Classical oscillator.

- Oscillation amplitude grows until the force is out-of-phase with motion, then damped.

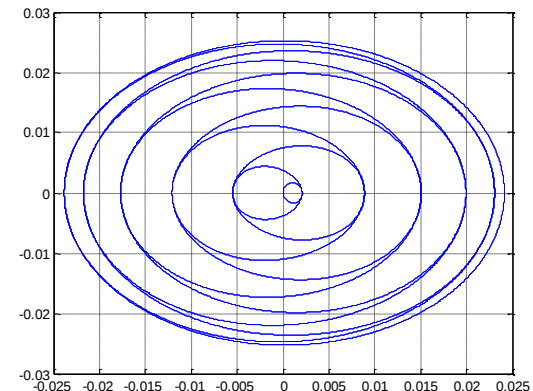
$$x(t) = \frac{F_0/m}{\omega^2 - \omega_m^2} [\cos(\omega t) - \cos(\omega_m t)]$$

$$p(t) = \frac{F_0}{\omega^2 - \omega_m^2} [\omega_m \sin(\omega_m t) - \omega \sin(\omega t)]$$

x and p vs. time



Phase space (x vs. p)



Off-resonantly Driven harmonic oscillator

Normalizing by $2x_0$ and $p_0 \equiv \hbar/x_0$ and moving to a frame rotating at ω_m

$$x'(t) = \frac{x(t)}{2x_0} \cos(\omega_m t) - \frac{p(t)}{p_0} \sin(\omega_m t) \quad p'(t) = \frac{x(t)}{2x_0} \sin(\omega_m t) + \frac{p(t)}{p_0} \cos(\omega_m t)$$

Neglecting terms of order δ/ω_m

$$\equiv \frac{F_0 x_0}{2\hbar\delta} [1 - \cos(\delta t)]$$

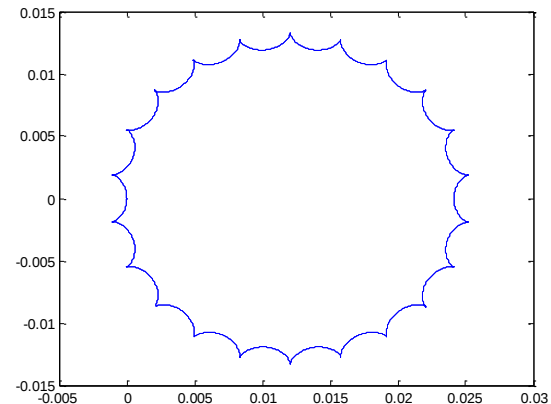
$$\equiv \frac{F_0 x_0}{2\hbar\delta} [\sin(\delta t)]$$

$$\alpha(t) = x'(t) + ip'(t)$$

$$\alpha(t) = \frac{F_0 x_0}{2\hbar\delta} [1 - \exp(i\delta t)]$$

$$S = \pi \left[\frac{F_0 x_0}{2\hbar\delta} \right]^2$$

Phase space, rotating frame (x' vs. p')



Off-resonantly Driven harmonic oscillator

- Quantum Driven Oscillator: $F(t) = F_0 \cos((\omega_m - \delta)t)$

$$H = \hbar\omega_m(\hat{a}^\dagger\hat{a} + 1/2) + F(t)\hat{x} \equiv H_0 + V(t)$$

With another RWA (interaction representation):

$$V_I = \frac{F_0 x_0}{2}(\hat{a}^\dagger e^{i\delta t} + \hat{a} e^{-i\delta t}) \quad \text{where} \quad i\hbar \frac{d}{dt}|\Psi\rangle_I = V_I|\Psi\rangle_I$$

For short enough times:

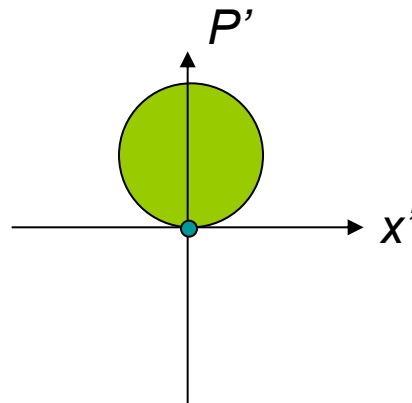
$$|\Psi(t + \Delta t)\rangle_I = e^{-\frac{i}{\hbar}V_I\Delta t}|\Psi(t)\rangle_I = e^{(\Delta\alpha\hat{a}^\dagger + \Delta\alpha^*\hat{a})}|\Psi(t)\rangle_I \equiv \hat{D}(\Delta\alpha)$$

$$\Delta\alpha(t) = -\frac{i}{2\hbar}F x_0 e^{i\delta t} \Delta t \equiv \frac{1}{2x_0}[\Delta x + i\Delta p/m\omega_m]$$

Off-resonantly Driven harmonic oscillator

- Coherent state as displaced vacuum.

$$\hat{D}(\alpha)|0\rangle = |\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$



Phonon Fock basis

$$\Delta\alpha_i = \Delta\alpha(t_i)$$

$$\hat{U}(t=0, t) = \hat{D}(\Delta\alpha_N) \dots \hat{D}(\Delta\alpha_2) \hat{D}(\Delta\alpha_1) \quad \text{where} \quad \sum_{i=1}^N \Delta\alpha_i = \alpha$$

By the end of the gate:

$$\sum_{i=1}^N \Delta\alpha_i = 0$$

Off-resonantly Driven harmonic oscillator

- Two ways of calculating the acquired phase:

1. The direct way:

From
$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}e^{\frac{1}{2}[\hat{A},\hat{B}]}$$

We get:
$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha + \beta)e^{iIm(\alpha\beta^*)}$$

Therefore:

$$\hat{U}(t=0, t) = \hat{D}(\sum_{i=1}^N \Delta\alpha_i) e^{iIm[\sum_{j=2}^N \Delta\alpha_j (\sum_{k=1}^{j-1} \Delta\alpha_k)^*]} = e^{iIm \oint \alpha^* d\alpha}$$

Displacement vs. time:

$$\alpha(t) = \int_0^t \Delta\alpha(t) dt = -\frac{i}{2\hbar} \int_0^t F_0 x_0 e^{i\delta t'} dt' = \frac{F_0 x_0}{2\hbar\delta} (1 - e^{i\delta t})$$

The acquired phase:

$$\gamma = Im \oint \alpha^* d\alpha = \frac{\pi}{2} \left(\frac{F_0 x_0}{\hbar\delta} \right)^2$$

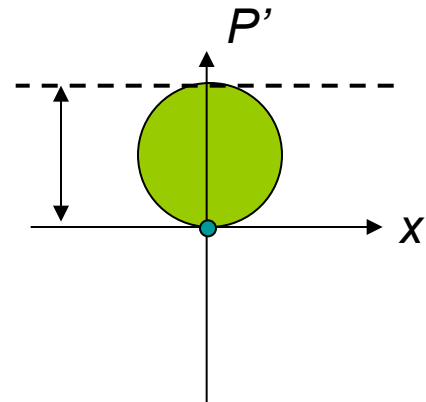
Off-resonantly Driven harmonic oscillator

$$\gamma = \text{Im} \oint \alpha^* d\alpha = \frac{\pi}{2} \left(\frac{F_0 x_0}{\hbar \delta} \right)^2$$

Displacement vs. time:

$$\alpha(t) = \int_0^t \Delta\alpha(t) dt = -\frac{i}{2\hbar} \int_0^t F_0 x_0 e^{i\delta t'} dt' = \frac{F_0 x_0}{2\hbar \delta} (1 - e^{i\delta t})$$

Phase space radius: $2r = \frac{F_0 x_0}{\hbar \delta}$



Phase space area:

$$S = \pi r^2 = \frac{\pi}{4} \left[\frac{F_0 x_0}{\hbar \delta} \right]^2$$



$$\gamma = 2S$$

Off-resonantly Driven harmonic oscillator

- Geometric and dynamic parts:

Dynamic phase:

$$\begin{aligned}\phi_{\text{D}} &= -\frac{1}{\hbar} \int_0^{\tau_{\text{g}}} \langle \alpha(t) | \hat{V}_I(t) | \alpha(t) \rangle dt \\ &= \frac{F_0 x_0}{2\hbar} \int_0^{2\pi/\delta} (\alpha^* \exp(i\delta t) + \alpha^{-i\delta t}) \\ &= \pi \left(\frac{F_0 x_0}{\hbar \delta} \right)^2 = 2\phi.\end{aligned}$$

Off-resonantly Driven harmonic oscillator

- Geometric and dynamic parts:

Geometric phase:

$$\gamma = i \oint_c \langle \alpha | \nabla | \alpha \rangle \cdot d\alpha$$

$$\langle \alpha | \nabla | \alpha \rangle \cdot d\alpha = \langle \alpha | \alpha + \Delta\alpha \rangle - 1$$

The overlap between two coherent states

$$\langle \alpha | \beta \rangle = \exp \left[-\frac{1}{2} (|\alpha|^2 + |\beta|^2) + \alpha^* \beta \right]$$


$$\langle \alpha | \alpha + \Delta\alpha \rangle - 1 \simeq \frac{1}{2} (\alpha^* \Delta\alpha - \alpha \Delta\alpha^*) = i \operatorname{Im}(\alpha^* \Delta\alpha)$$

$$\gamma = -\operatorname{Im} \oint \alpha^* d\alpha = -\phi$$

The acquired phase is a sum of partial canceling dynamic and geometric phases!

Spin dependent forces

- To acquire a spin dependent phase we need a spin dependent force.

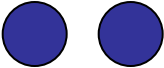
$$V_I = \sum_{m=\uparrow,\downarrow} \frac{F_m x_0}{2} (\hat{a}^\dagger e^{i\delta t} + \hat{a} e^{-i\delta t}) |m\rangle \langle m|$$
$$= (\hat{a}^\dagger e^{i\delta t} + \hat{a} e^{-i\delta t}) \left[\frac{(F_\uparrow + F_\downarrow)x_0}{4} \hat{I} + \frac{(F_\uparrow - F_\downarrow)x_0}{4} \hat{\sigma} \cdot \vec{n} \right]$$

Where $\uparrow, \downarrow$ are eigenstates of $\hat{\sigma} \cdot \vec{n}$

Spin dependent forces: Two ions

- Same analysis applies, only:

Two. Long. motional modes:

1st mode (COM) 

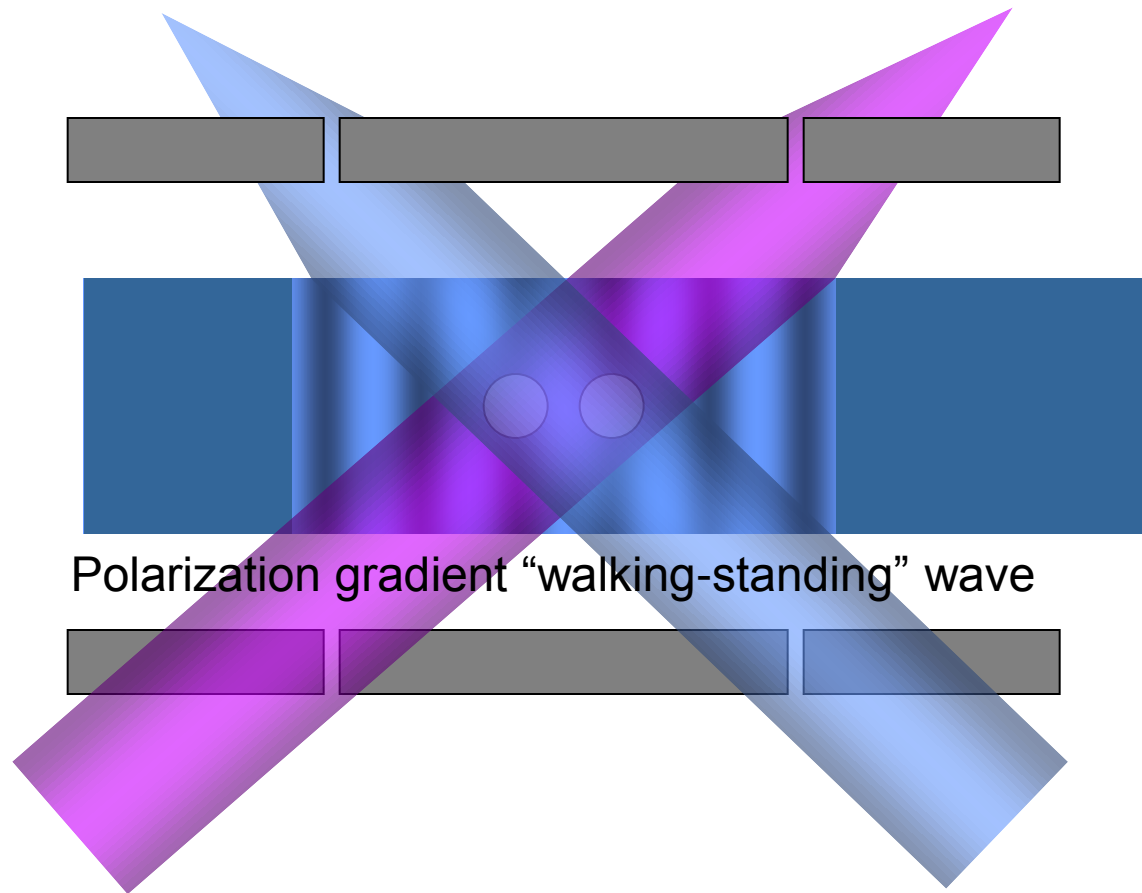
2nd mode (stretch) 

$$\omega_{st} = \sqrt{3}\omega_{cm}$$

$$\hat{x}_i = x_{eq,i} + \frac{x_{cm0}}{\sqrt{2}}(\hat{a}_{cm}^\dagger + \hat{a}_{cm}) \pm \frac{x_{st0}}{\sqrt{2}}(\hat{a}_{st}^\dagger + \hat{a}_{st})$$

- For COM mode: total force = sum of forces on the two ions.
- For Stretch mode: total force = diff. of forces on the two ions.
- For a non-uniform force: forces at eq. positions of the two ions.
- Assume force is uniform over ion w.f. (Lamb-Dicke regime).

Sigma z gate



Polarization gradient “walking-standing” wave

For a Zeeman qubit: $\mathbf{F}_{\downarrow} = -\mathbf{F}_{\uparrow}$

Set ion spacing such that $\Delta k(x_{eq1} - x_{eq2}) = 2\pi N$ and set $\omega_m = \omega_{st}$

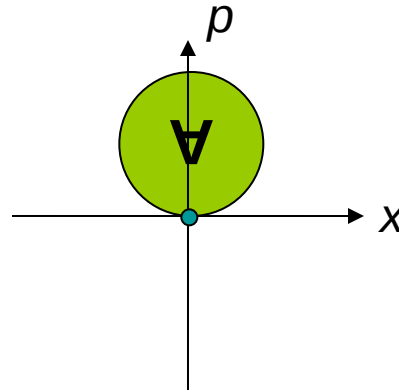
Sigma z gate

- Hyperfine qubit (NIST)

${}^9\text{Be}^+$, $|F=1, m_f = -1\rangle$; $|F=2, m_f = -2\rangle$

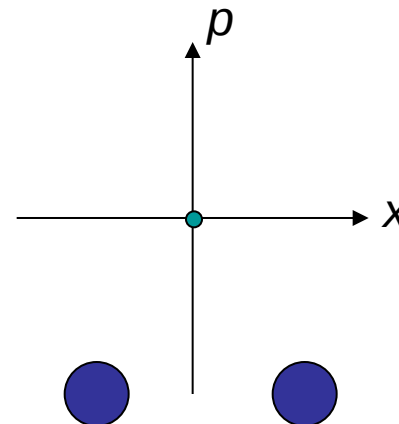
$|\downarrow\uparrow\rangle$ or $|\uparrow\downarrow\rangle$

$$\Delta\phi = \pi/2$$

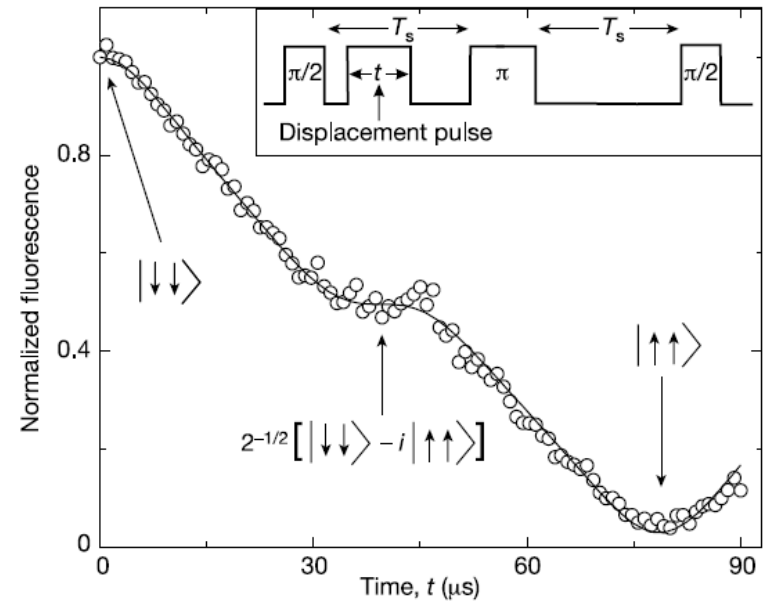


$|\uparrow\uparrow\rangle$ or $|\downarrow\downarrow\rangle$

$$\Delta\phi = 0$$



$$F\downarrow = -2F\uparrow$$



$$F = 0.97; \varepsilon = 0.03$$

Leibfried et. al., *Nature*, **422**, 412 (2003)

Sigma ϕ gate (Sorensen-Molmer)

Used two facts:

1. Both spin states feel a force.
2. Force is different.



This Sigma z gate would not work for:

1. Optical qubit.
2. Clock transition qubit (hyperfine).

$\uparrow, \downarrow$ are eigenstates of $\hat{\sigma} \cdot \vec{n}$ where $\vec{n} = \begin{pmatrix} \cos \phi \\ \sin \phi \\ 0 \end{pmatrix}$

- Since in the meas. Basis looks like collective spin flip the bi-chromatic fields have frequencies $\sim \omega_0$
- How is ϕ determined? – optical phase differences.

Sigma ϕ gate

- Hyperfine qubit (NIST), $F=0.83$

Sacket, *et. al.* Nature, **404**, 256 (2000).

- Clock transition hyperfine qubit (Michigan), $F=0.79$

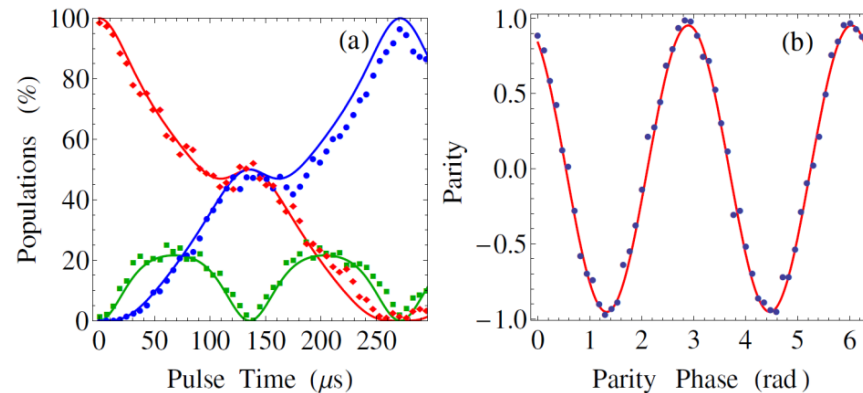
Haljan, *et. al.* PRA, **72**, 062316 (2005).

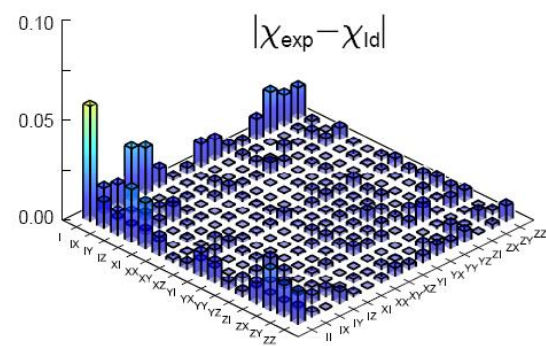
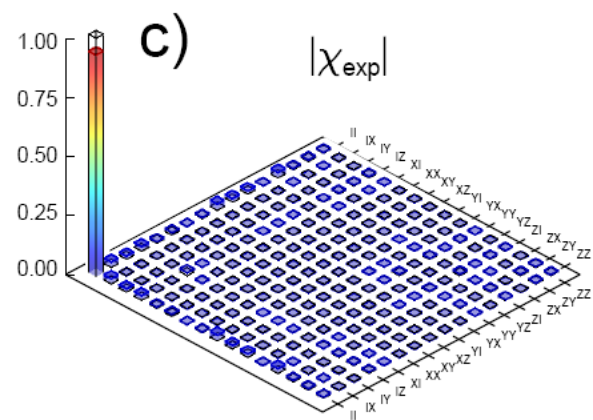
- Optical qubit (Innsbruck), $F=0.993$

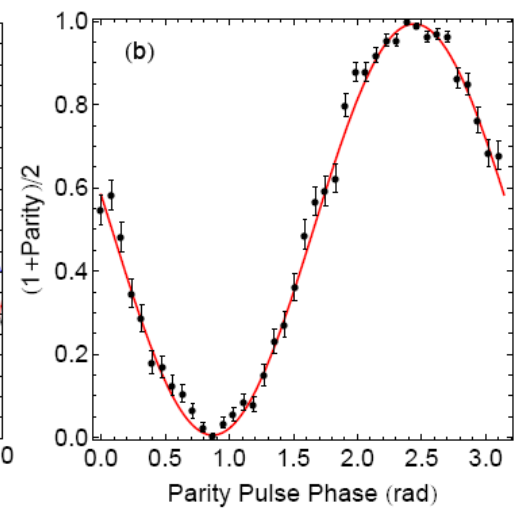
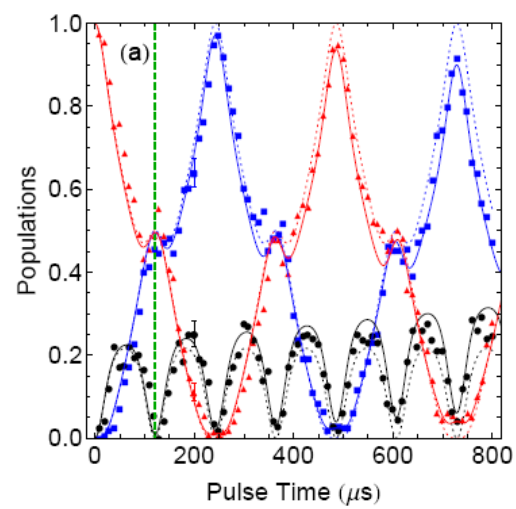
Benhelm, *et. al.* Nature Phys. (2008).

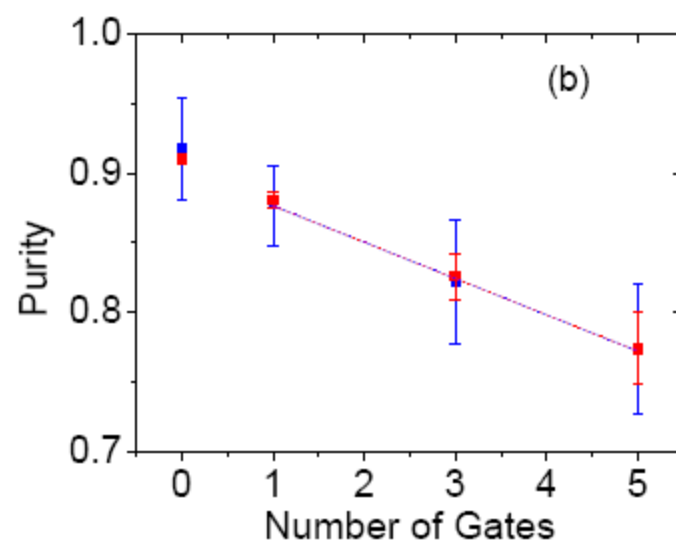
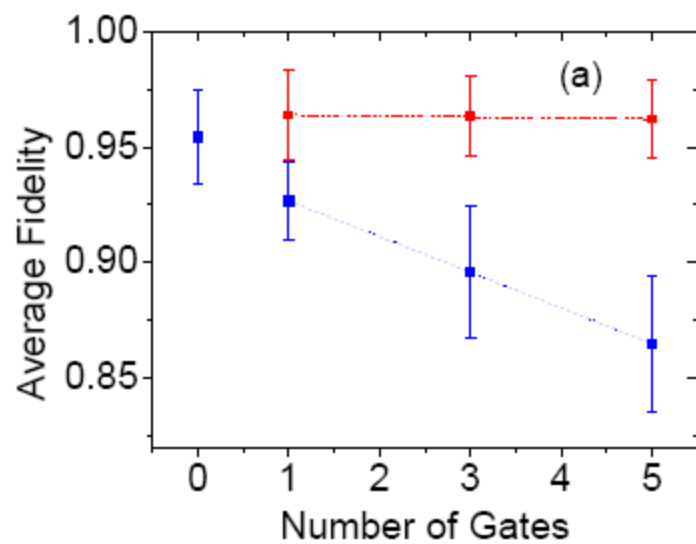
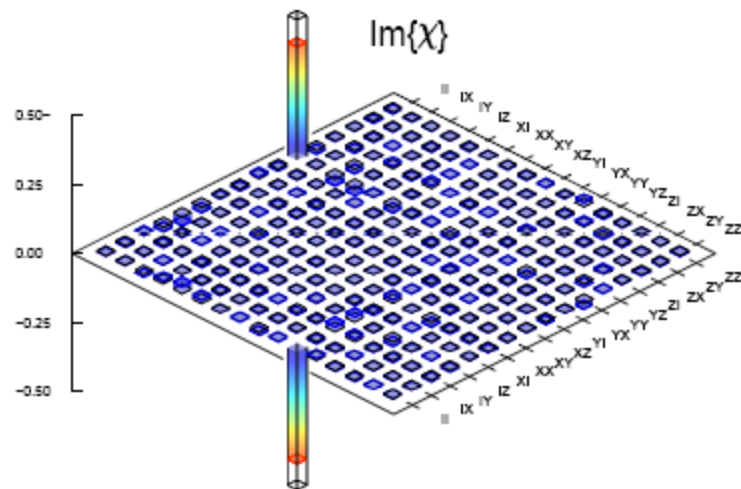
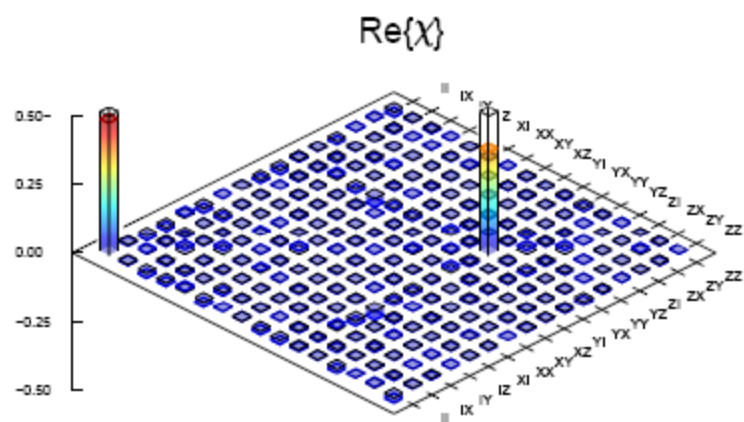
- Optical qubit (Weizmann), $F=0.985$

Navon, *et. al.* arXiv:1309.4502 (2013).





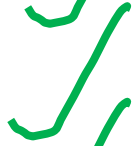




Tutorial overview



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2. Qubit initialization.



3. Qubit measurement.

4. Universal set of quantum gates:
single qubit rotations;
two-ion entanglement gates



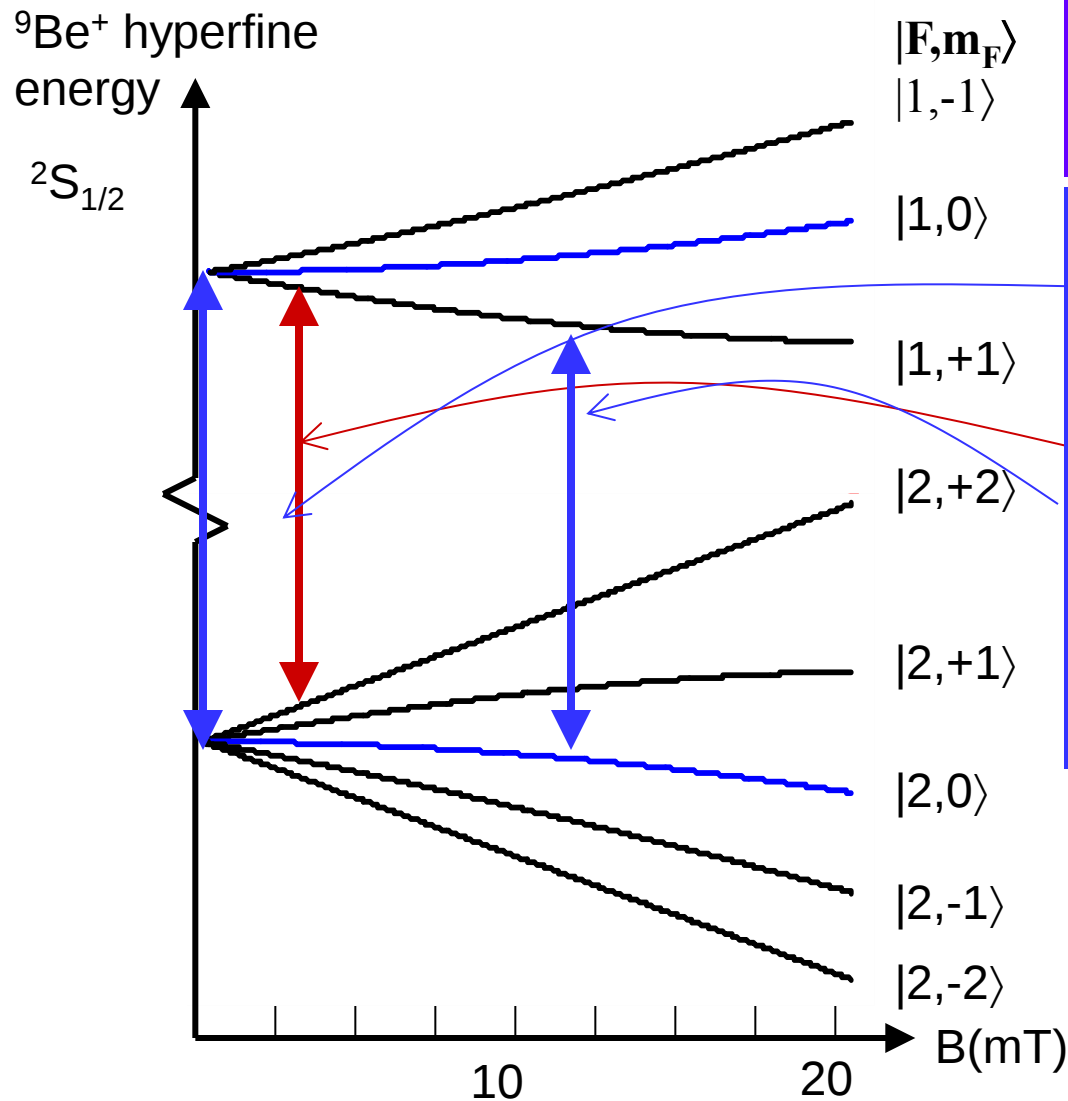
5. Memory coherence times

How well???

Benchmarked to current threshold estimates

Disclaimer: non exhaustive; focuses on laser-driven gates

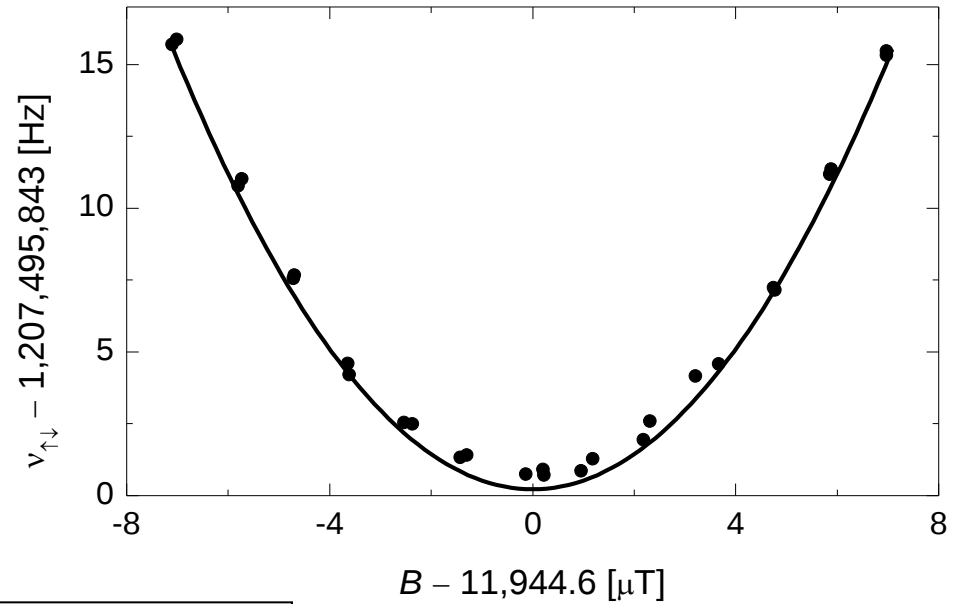
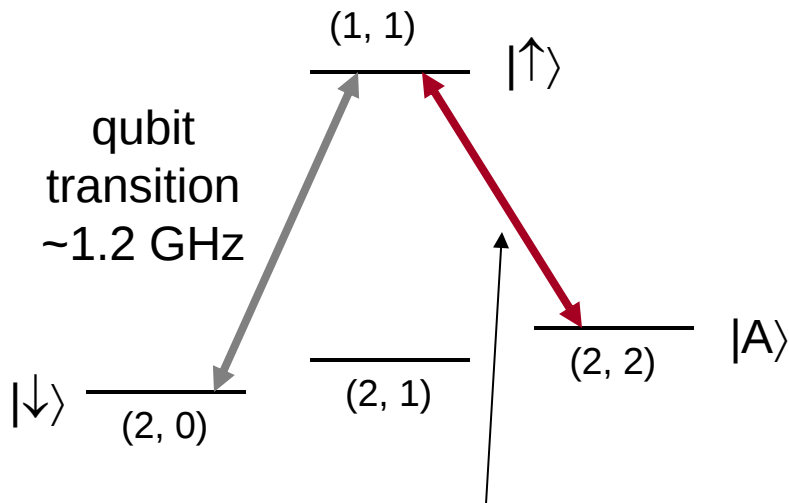
Field independent qubits



$|2, 0\rangle \leftrightarrow |1, 0\rangle - (g_I I + g_J J) B$
 field-independent at $B = 0$
 Selection rules are not applicable

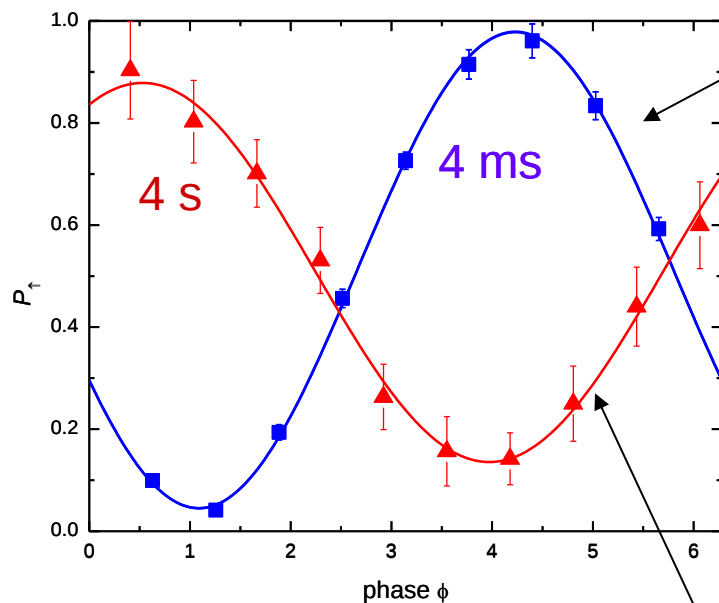
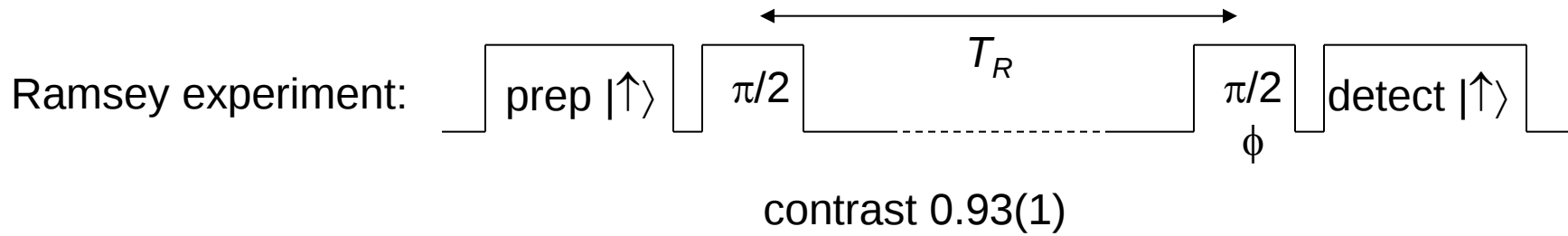
$|2, 0\rangle \leftrightarrow |1, 0\rangle \tau_2 \cong 1.7 \text{ ms}$
 $\delta f / \delta B = 940 \text{ Hz}/\mu\text{T} @ 1.3 \text{ mT}$
 $|2, 2\rangle \leftrightarrow |1, 1\rangle \tau_2 \cong 76 \mu\text{s}$
 $|2, 2\rangle \leftrightarrow |1, 1\rangle \tau_2 \cong 76 \mu\text{s}$
 $|2, 0\rangle \leftrightarrow |1, 1\rangle \tau_2 \cong 53 \text{ s}$
 $|2, 0\rangle \leftrightarrow |1, 1\rangle \tau_2 \cong 53 \text{ s}$
 $\tau_2(\delta\phi = 1 \text{ rad}) \cong 53 \text{ s}$

Characterizing the field-independent qubit

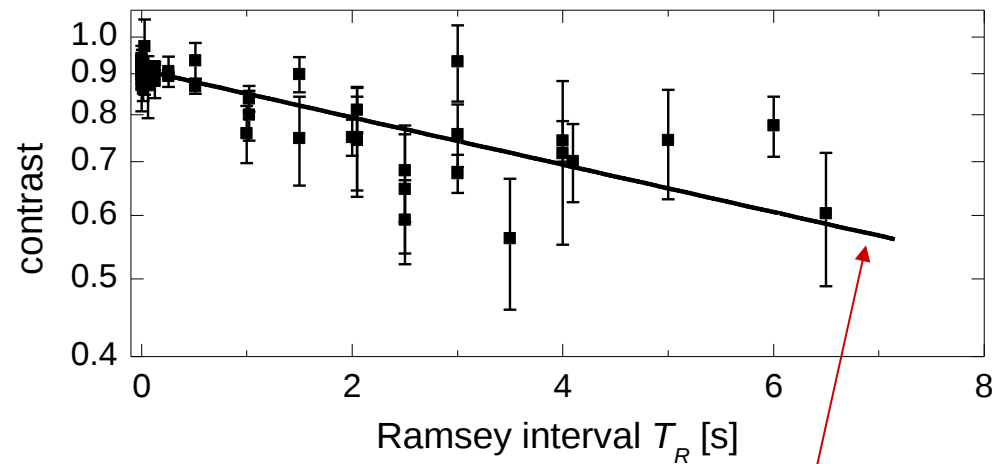


transition frequency $|A\rangle \leftrightarrow |\uparrow\rangle$ is linearly dependent on the magnetic field:
 $\nu - \nu_0 = 17.6 \text{ kHz}/\mu\text{T} \times (B - B_0)$.
 Probing this transition measures the magnetic field.

Qubit coherence time



contrast 0.74(4)



Coherence time 14.7 ± 1.6 s

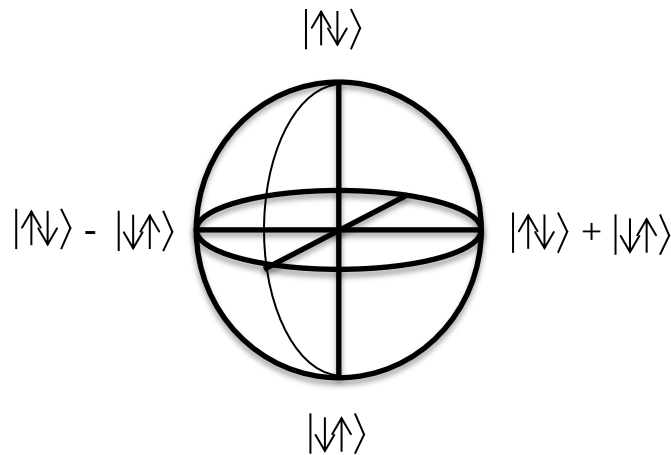
During detection: $\varepsilon \sim 10^{-5}$

Decoherence-Free Subspaces

All the states in the sub-space, spanned by the two states,

$$|\Psi+\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2} \quad (\text{triplet; } m=0)$$

$$|\Psi-\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2} \quad (\text{singlet})$$



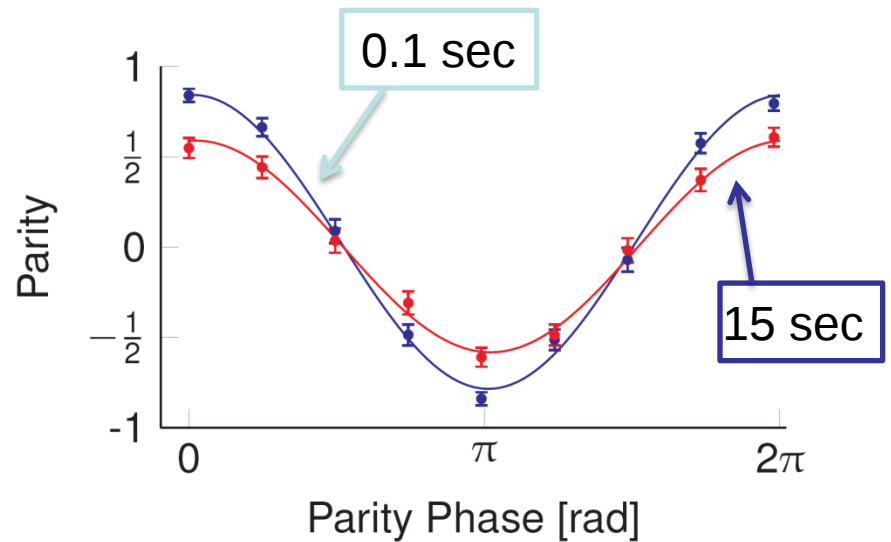
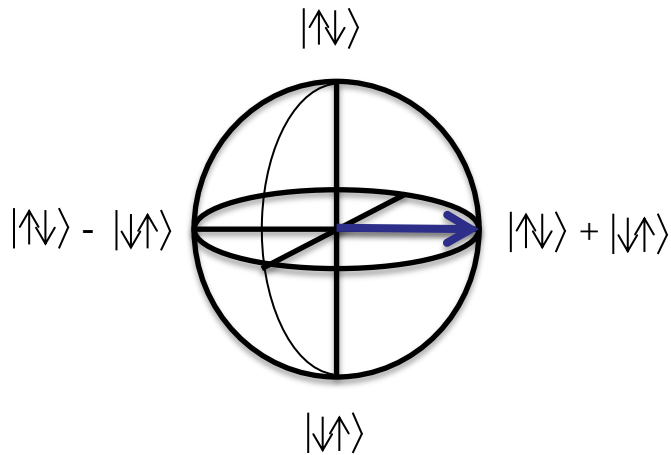
- Remain coherent under collective magnetic field noise.

Haffner *et. al.* Appl. Phys. B, **81**, 151 (2005)

Langer *et. al.* Phys. Rev. Lett., **95**, 060502 (2005)

Decoherence-Free Subspaces

Coherence time = 44 Seconds



High Fidelity Universal Set of Quantum Operations

