Tutorial overview

1. The ion-qubit: different ion-qubit choices, Ion traps.
2. Qubit initialization.
3. Qubit measurement.
4. Universal set of quantum gates:
   single qubit rotations;
   two-ion entanglement gates
5. Memory coherence times

How well???

Benchmarked to current threshold estimates

Disclaimer: non exhaustive; focuses on laser-driven gates

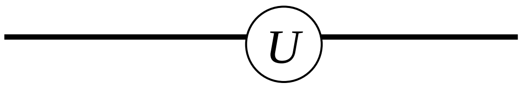
Universal Gate set

- For N-qubits, and unitary transformation U on a 2^N-dimension Hilbert space.

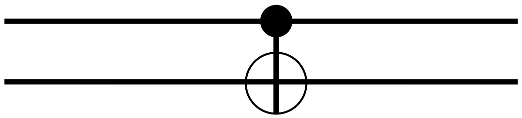
- A *finite* set of unitary gates that spans any such U.

Single qubit gate

$$U \in SU[2]$$



2-qubit C-not gate

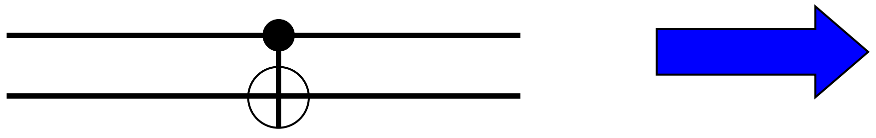


- Barenco *et. al. Phys. Rev. A*, **52**, 3457, (1995).

C-not gates vs. Phase gate

2-qubit C-not gate

$$\alpha|\uparrow\uparrow\rangle + \beta|\uparrow\downarrow\rangle + \gamma|\downarrow\uparrow\rangle + \delta|\downarrow\downarrow\rangle = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{bmatrix}$$

A $\pi/2$ Hadamard rotation on the target qubit before and after the gate maps the C-not into a phase gate:

$$\left[\hat{I}_1 \otimes \hat{H}_2\right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \left[\hat{I}_1 \otimes \hat{H}_2\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

C-not gates vs. Phase gate

- A $\pi/2$ phase shift on both qubits:

$$\hat{R}_1(\pi/2,0,\pi/2)\otimes\hat{R}_2(\pi/2,0,\pi/2)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix} = \exp(-i\pi/2)\begin{bmatrix}1&0&0&0\\0&i&0&0\\0&0&i&0\\0&0&0&1\end{bmatrix}$$

A phase shift on anti-parallel spin states vs. parallel spin states.

- Changing to the x basis:

$$[\hat{R}_1(\pi/2,0)]\otimes\hat{R}_2(\pi/2,0)]\begin{pmatrix}1&0&0&0\\0&i&0&0\\0&0&i&0\\0&0&0&1\end{pmatrix}[\hat{R}_1(\pi/2,0)]\otimes\hat{R}_2(\pi/2,0)] = \frac{e^{i\pi/4}}{\sqrt{2}}\begin{pmatrix}-i&0&0&1\\0&i&-i&0\\0&-i&i&0\\1&0&0&-i\end{pmatrix}$$

- Collective spin flip:

$$\hat{U}|\downarrow\downarrow\rangle = \frac{\exp(-i\pi/4)}{2^{1/2}}(|\downarrow\downarrow\rangle + i|\uparrow\uparrow\rangle) \qquad \hat{U}^2|\downarrow\downarrow\rangle = |\uparrow\uparrow\rangle$$

Phases in QM
(Berry, Aharonov-Anandan)

T- dependent. Schr. Eq. : $\hat{H}(t)|\Psi(t)\rangle = i\hbar(\mathrm{d}/\mathrm{d}t)|\Psi(t)\rangle$

Cyclic motion. : $|\Psi(\tau)\rangle = \exp(i\phi)|\Psi(0)\rangle$

Bare state. : $|\Phi(t)\rangle = \exp[-if(t)]|\Psi(t)\rangle \qquad f(\tau) - f(o) = \phi$

Eq. of motion. For *f* : $\frac{\mathrm{d}f}{\mathrm{d}t} = -\frac{1}{\hbar}\langle\Psi(t)|\hat{H}(t)|\Psi(t)\rangle + i\langle\Phi(t)|\frac{\mathrm{d}}{\mathrm{d}t}|\Phi(t)\rangle$

$$\phi = -\frac{1}{\hbar}\int_0^\tau \langle\Psi(t)|\hat{H}(t)|\Psi(t)\rangle\mathrm{d}t + \mathrm{i}\int_0^\tau \langle\Phi(t)|\frac{\mathrm{d}}{\mathrm{d}t}|\Phi(t)\rangle\mathrm{d}t \equiv \phi_\mathrm{D} + \gamma$$

Dynamic phase

Geometric phase

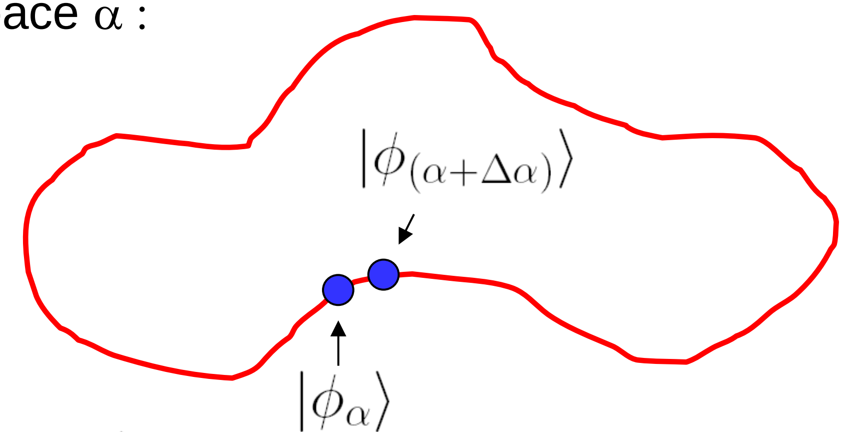
Berry, *Proc. Roy. Soc. Lon.*, **392**, 45, (1984)

Aharonov and Anandan, *PRL*, **58**, 1593, (1987)

The Geometric phase

$$|\Phi(t)\rangle \equiv |\Phi_\alpha\rangle \qquad \gamma = \mathrm{i}\int_0^\tau \left\langle \Phi(t)\left|\frac{\mathrm{d}}{\mathrm{d}t}\right|\Phi(t)\right\rangle \mathrm{d}t = \mathrm{i}\oint_c \langle \Phi_\alpha|\nabla_\alpha|\Phi_\alpha\rangle$$

A curve in parameter space α :



$$\exp(-\mathrm{i}\Delta\gamma) = \frac{\langle \Phi_\alpha|\Phi_{\alpha+\Delta\alpha}\rangle}{|\langle \Phi_\alpha|\Phi_{\alpha+\Delta\alpha}\rangle|} \quad \Rightarrow \quad -i\Delta\phi = \langle \phi_\alpha|\vec{\nabla}_\alpha|\phi_\alpha\rangle \cdot d\vec{\alpha}$$

Vector potential: $\vec{A}_\alpha = i\langle \phi_\alpha|\vec{\nabla}_\alpha|\phi_\alpha\rangle \qquad \vec{B}_\alpha = \vec{\nabla} \times \vec{A}_\alpha$

Stokes: $\gamma = -\int\int_S \vec{B}\cdot\vec{n}dS$

Encircling an area in parameter space is a necessary condition for $\gamma \neq 0$

Examples for different proposed phase gates.

A (Pseudo) spin attached to a harmonic oscillator

- Cirac and Zoller
  Cirac and Zoller, *PRL*, **74**, 4091 (1995).

Geometric,
Spin d.o.f

- Spin dependent dipole forces - "Push" gate
  Cirac and Zoller, *Nature*, **404**, 579 (2000).

Dynamic,
Harmonic Oscillator. d.o.f

- Spin dependent dipole forces – near resonant drive
  Sorensen and Molmer, *PRL*, **82**, 1971 (1999).
  Munro, Milburn, Sanders , *PRA*, **62**, 052108 (2000).
  Solano, Santos, Milman, *PRA*, **64**, 024304 (2000).

Geometric,
Harmonic Oscillator. d.o.f

Off-resonantly Driven harmonic oscillator

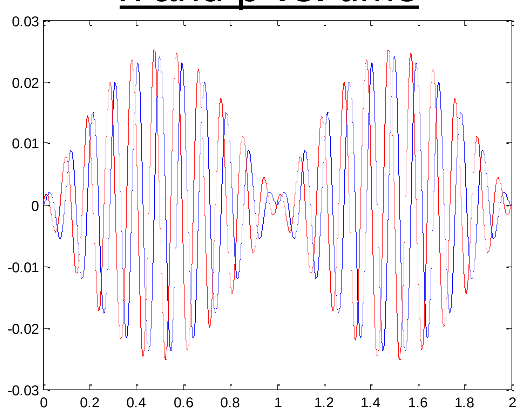
- Classical oscillator.

- Oscillation amplitude grows until the force is out-of-phase with motion, then damped.

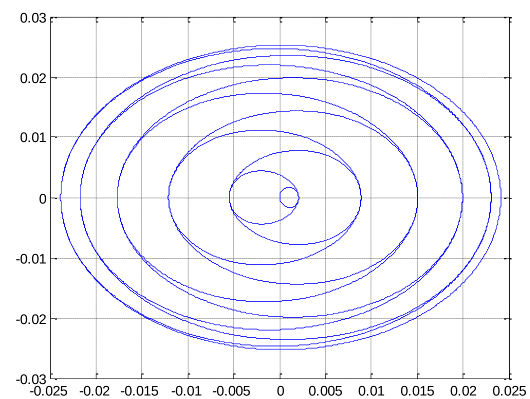
$$x(t) = \frac{F_0/m}{\omega^2 - \omega_{\mathrm{m}}^2}\left[\cos\left(\omega t\right) - \cos\left(\omega_{\mathrm{m}} t\right)\right]$$

$$p(t) = \frac{F_0}{\omega^2 - \omega_{\mathrm{m}}^2}\left[\omega_{\mathrm{m}} \sin\left(\omega_{\mathrm{m}} t\right) - \omega \sin\left(\omega t\right)\right]$$

x and p vs. time



Phase space (x vs. p)



Off-resonantly Driven harmonic oscillator

Normalizing by $2x_0$ and $p_0 \equiv \hbar/x_0$ and moving to a frame rotating at ω_m

$$x'(t) = \frac{x(t)}{2x_0}\cos(\omega_\mathrm{m} t) - \frac{p(t)}{p_0}\sin(\omega_\mathrm{m} t) \qquad p'(t) = \frac{x(t)}{2x_0}\sin(\omega_\mathrm{m} t) + \frac{p(t)}{p_0}\cos(\omega_\mathrm{m} t)$$

Neglecting terms of order δ/ω_m

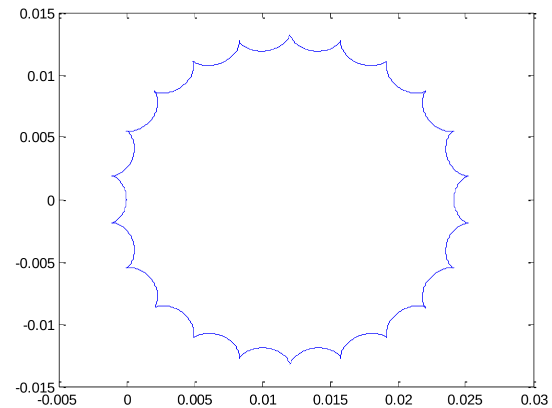
$$\equiv \frac{F_0 x_0}{2\hbar\delta}\left[1 - \cos(\delta t)\right] \qquad \equiv \frac{F_0 x_0}{2\hbar\delta}\left[\sin(\delta t)\right]$$

$$\alpha(t) = x'(t) + \mathrm{i}p'(t)$$

$$\alpha(t) = \frac{F_0 x_0}{2\hbar\delta}\left[1 - \exp(\mathrm{i}\delta t)\right]$$

$$S = \pi\left[\frac{F_0 x_0}{2\hbar\delta}\right]^2$$

Phase space, rotating frame (x' vs. p')



Off-resonantly Driven harmonic oscillator

- Quantum Driven Oscillator: $F(t) = F_0 \cos((\omega_m - \delta)t)$

$$H = \hbar\omega_{\rm m}(\hat{a}^\dagger\hat{a} + 1/2) + F(t)\hat{x} \equiv H_0 + V(t)$$

With another RWA (interaction representation):

$$V_I = \frac{F_0 x_0}{2}(\hat{a}^\dagger e^{i\delta t} + \hat{a}e^{-i\delta t})$$ where $$i\hbar\frac{d}{dt}|\Psi\rangle_I = V_I|\Psi\rangle_I$$

For short enough times:

$$|\Psi(t+\Delta t)\rangle_I = e^{-\frac{i}{\hbar}V_I\Delta t}|\Psi(t)\rangle_I = e^{(\Delta\alpha\hat{a}^\dagger + \Delta\alpha^\star\hat{a})}|\Psi(t)\rangle_I \equiv \hat{D}(\Delta\alpha)$$

$$\Delta\alpha(t) = -\frac{i}{2\hbar}Fx_0e^{i\delta t}\Delta t \equiv \frac{1}{2x_0}[\Delta x + i\Delta p/m\omega_m]$$

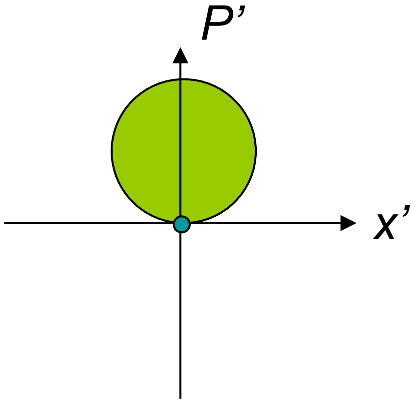
Lee *et. al. J.* of Optics B, **7**, S371, (2005)

Off-resonantly Driven harmonic oscillator

- Coherent state as displaced vacuum.

$$\hat{D}(\alpha)|0\rangle = |\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

Phonon Fock basis



$$\Delta\alpha_i = \Delta\alpha(t_i)$$

$$\hat{U}(t=0,t) = \hat{D}(\Delta\alpha_N)...\hat{D}(\Delta\alpha_2)\hat{D}(\Delta\alpha_1)$$ where $$\sum_{i=1}^{N} \Delta\alpha_i = \alpha$$

By the end of the gate:

$$\sum_{i=1}^{N} \Delta\alpha_i = 0$$

Off-resonantly Driven harmonic oscillator

- Two ways of calculating the acquired phase:

1. The direct way:

From $$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}e^{\frac{1}{2}[\hat{A},\hat{B}]}$$

We get: $$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)e^{iIm(\alpha\beta^\star)}$$

Therefore:

$$\hat{U}(t=0,t) = \hat{D}(\sum_{i=1}^{N}\Delta\alpha_i)e^{iIm[\sum_{j=2}^{N}\Delta\alpha_j(\sum_{k=1}^{j-1}\Delta\alpha_k)^\star]} = e^{iIm\oint\alpha^\star d\alpha}$$

Displacement vs. time:

$$\alpha(t) = \int_o^t \Delta\alpha(t)dt = -\frac{i}{2\hbar}\int_o^t F_0x_0e^{i\delta t'}dt' = \frac{F_0x_0}{2\hbar\delta}(1-e^{i\delta t})$$

The acquired phase:

$$\gamma = Im\oint\alpha^\star d\alpha \quad = \frac{\pi}{2}\left(\frac{F_0x_0}{\hbar\delta}\right)^2$$

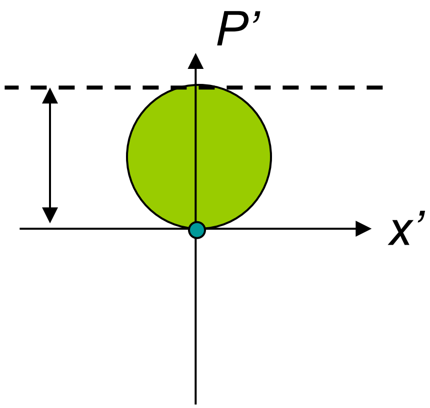
Off-resonantly Driven harmonic oscillator

$$\gamma = Im \oint \alpha^\star d\alpha \quad = \frac{\pi}{2}\left(\frac{F_0 x_0}{\hbar\delta}\right)^2$$

Displacement vs. time:

$$\alpha(t) = \int_o^t \Delta\alpha(t) dt = -\frac{i}{2\hbar}\int_o^t F_0 x_0 e^{i\delta t'} dt' = \frac{F_0 x_0}{2\hbar\delta}(1 - e^{i\delta t})$$

Phase space radius: $2r = \frac{F_0 x_0}{\hbar\delta}$



Phase space area:

$$S = \pi r^2 = \frac{\pi}{4}\left[\frac{F_0 x_0}{\hbar\delta}\right]^2 \quad \Rightarrow \quad \gamma = 2S$$

Off-resonantly Driven harmonic oscillator

- Geometric and dynamic parts:

Dynamic phase:

$$
\begin{aligned}
\phi_{\mathrm{D}} &= -\frac{1}{\hbar}\int_0^{\tau_{\mathrm{g}}} \langle\alpha(t)|\hat{V}_I(t)|\alpha(t)\rangle \mathrm{d}t \\
&= \frac{F_0 x_0}{2\hbar}\int_0^{2\pi/\delta} (\alpha^{\star}\exp(\mathrm{i}\delta t) + \alpha^{-\mathrm{i}\delta t}) \\
&= \pi\left(\frac{F_0 x_0}{\hbar\delta}\right)^2 = 2\phi.
\end{aligned}
$$

Off-resonantly Driven harmonic oscillator

- Geometric and dynamic parts:

Geometric phase:

$$\gamma = \mathrm{i} \oint_c \langle \alpha | \nabla | \alpha \rangle \cdot \mathrm{d}\alpha$$

$$\langle \alpha | \nabla | \alpha \rangle \cdot \mathrm{d}\alpha = \langle \alpha | \alpha + \Delta\alpha \rangle - 1$$

The overlap between two coherent states

$$\langle \alpha | \beta \rangle = \exp\left[-\frac{1}{2} \left(|\alpha|^2 + |\beta|^2 \right) + \alpha^\star \beta \right]$$

$$\Rightarrow \quad \langle \alpha | \alpha + \Delta\alpha \rangle - 1 \simeq \frac{1}{2} \left(\alpha^\star \Delta\alpha - \alpha \Delta\alpha^\star \right) = \mathrm{i}\, \mathrm{Im}(\alpha^\star \Delta\alpha)$$

$$\gamma = -\mathrm{Im} \oint \alpha^\star \, \mathrm{d}\alpha = -\phi$$

The acquired phase is a sum of partial canceling dynamic and geometric phases!

Spin dependent forces

- To acquire a spin dependent phase we need a spin dependent force.

$$V_I = \sum_{m=\uparrow,\downarrow} \frac{F_m x_0}{2}(\hat{a}^\dagger e^{i\delta t} + \hat{a} e^{-i\delta t})|m\rangle\langle m|$$

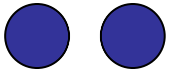
$$= (\hat{a}^\dagger e^{i\delta t} + \hat{a} e^{-i\delta t})\left[\frac{(F_\uparrow + F_\downarrow)x_0}{4}\hat{I} + \frac{(F_\uparrow - F_\downarrow)x_0}{4}\hat{\sigma}\cdot\vec{n}\right]$$

Where ↑,↓ are eigenstates of $\hat{\sigma}\cdot\vec{n}$

Spin dependent forces: Two ions

- Same analysis applies, only:

Two. Long. motional modes:

1st mode (COM) 

2nd mode (stretch) 

$$\omega_{st} = \sqrt{3}\omega_{cm}$$

$$\hat{x}_i = x_{eq,i} + \frac{x_{cm0}}{\sqrt{2}}(\hat{a}_{cm}^{\dagger} + \hat{a}_{cm}) \pm \frac{x_{st0}}{\sqrt{2}}(\hat{a}_{st}^{\dagger} + \hat{a}_{st})$$

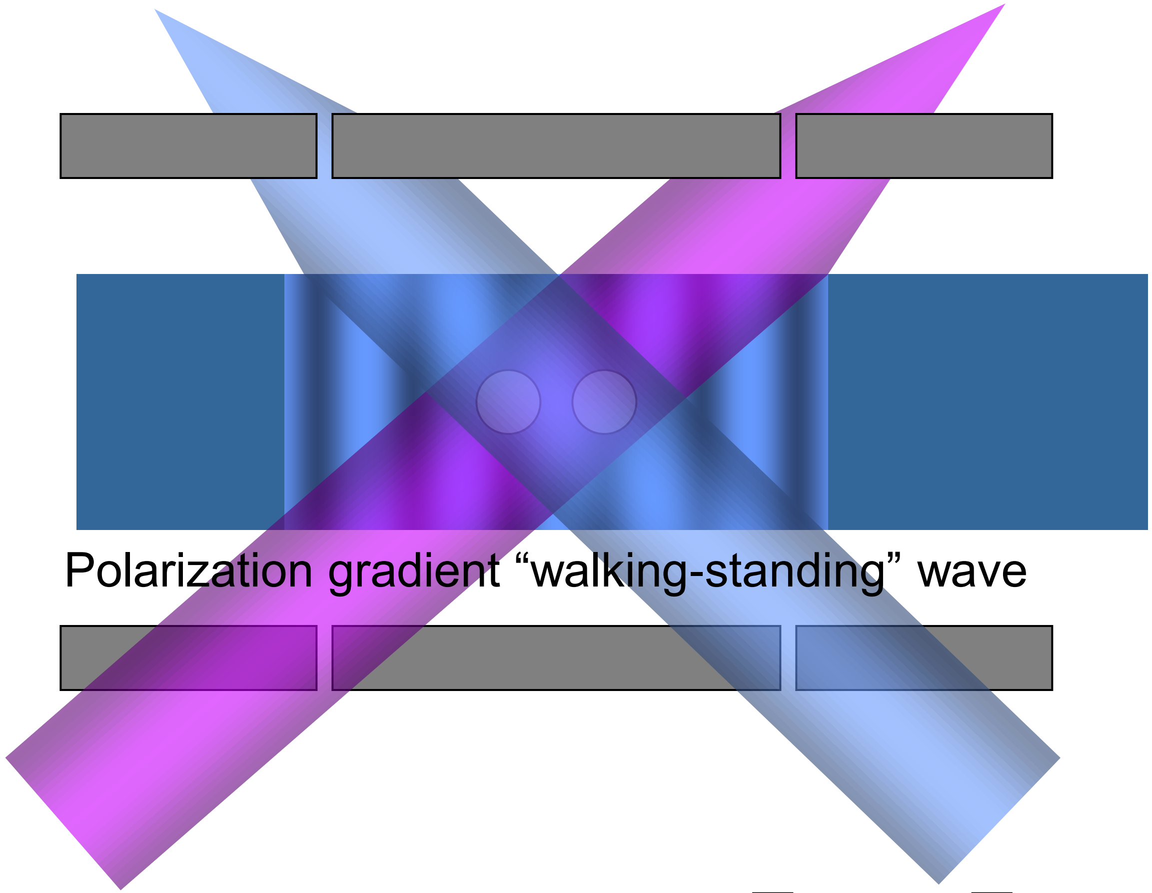
- For COM mode: total force = sum of forces on the two ions.

- For Stretch mode: total force = diff. of forces on the two ions.

- For a non-uniform force: forces at eq. positions of the two ions.

- Assume force is uniform over ion w.f. (Lamb-Dicke regime).

Sigma *z* gate



For a Zeeman qubit: $\boldsymbol{F}_{\downarrow} = -\boldsymbol{F}_{\uparrow}$

Set ion spacing such that $\Delta k(x_{eq1}-x_{eq2})=2\pi N$ and set $\omega_m = \omega_{st}$

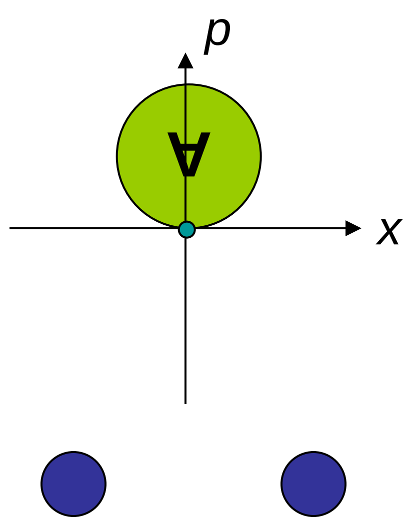
Sigma *z* gate

- Hyperfine qubit (NIST)

$^9Be^+$, $|F=1, mf = -1\rangle$; $|F=2, mf = -2\rangle$

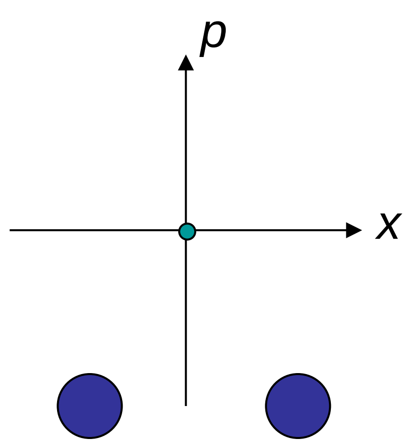
$\boldsymbol{F}_{\downarrow} = -2\boldsymbol{F}_{\uparrow}$

$|\downarrow\uparrow\rangle$ or $|\uparrow\downarrow\rangle$

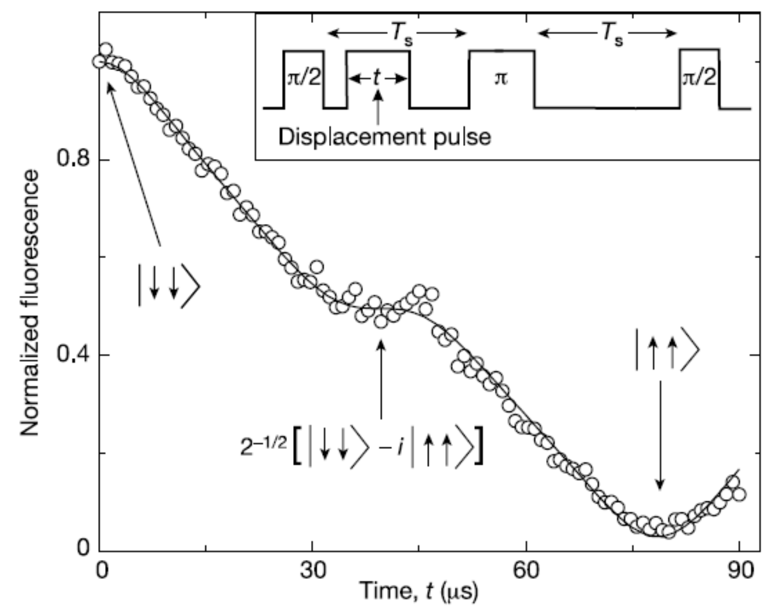


$$\Delta\phi = \pi/2$$

$|\uparrow\uparrow\rangle$ or $|\downarrow\downarrow\rangle$



$$\Delta\phi = 0$$



F = 0.97; ε = 0.03

Leibfried *et. al.*, *Nature*, **422**, 412 (2003)

Sigma ϕ gate
(Sorensen-Molmer)

Used two facts:

1. Both spin states feel a force.
2. Force is different.

→

This Sigma z gate would not work for:

1. Optical qubit.
2. Clock transition qubit (hyperfine).

$\uparrow,\downarrow$ are eigenstates of $\hat{\sigma}\cdot\vec{n}$ where $\vec{n} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$

- Since in the meas. Basis looks like collective spin flip the bi-chromatic fields have frequencies $\sim\omega_0$

- How is ϕ determined? – optical phase differences.

Sigma ϕ gate

- Hyperfine qubit (NIST), *F=0.83*

Sacket, *et. al.* Nature, **404**, 256 (2000).

- Clock transition hyperfine qubit (Michigan), *F=0.79*

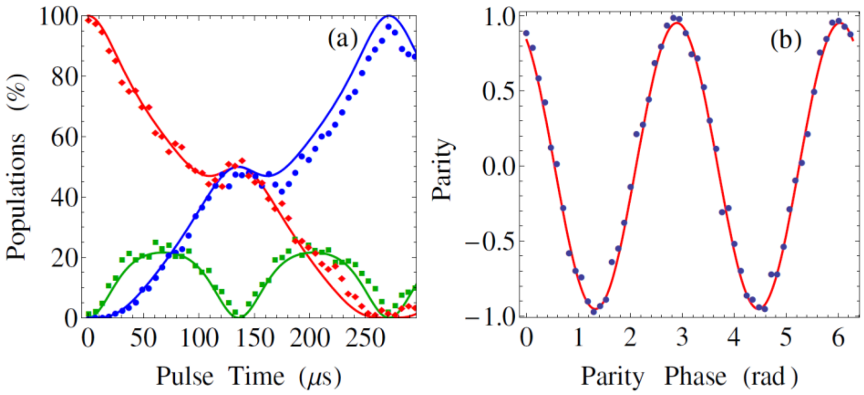
Haljan, *et. al.* PRA, **72**, 062316 (2005).

- Optical qubit (Innsbruck), *F=0.993*

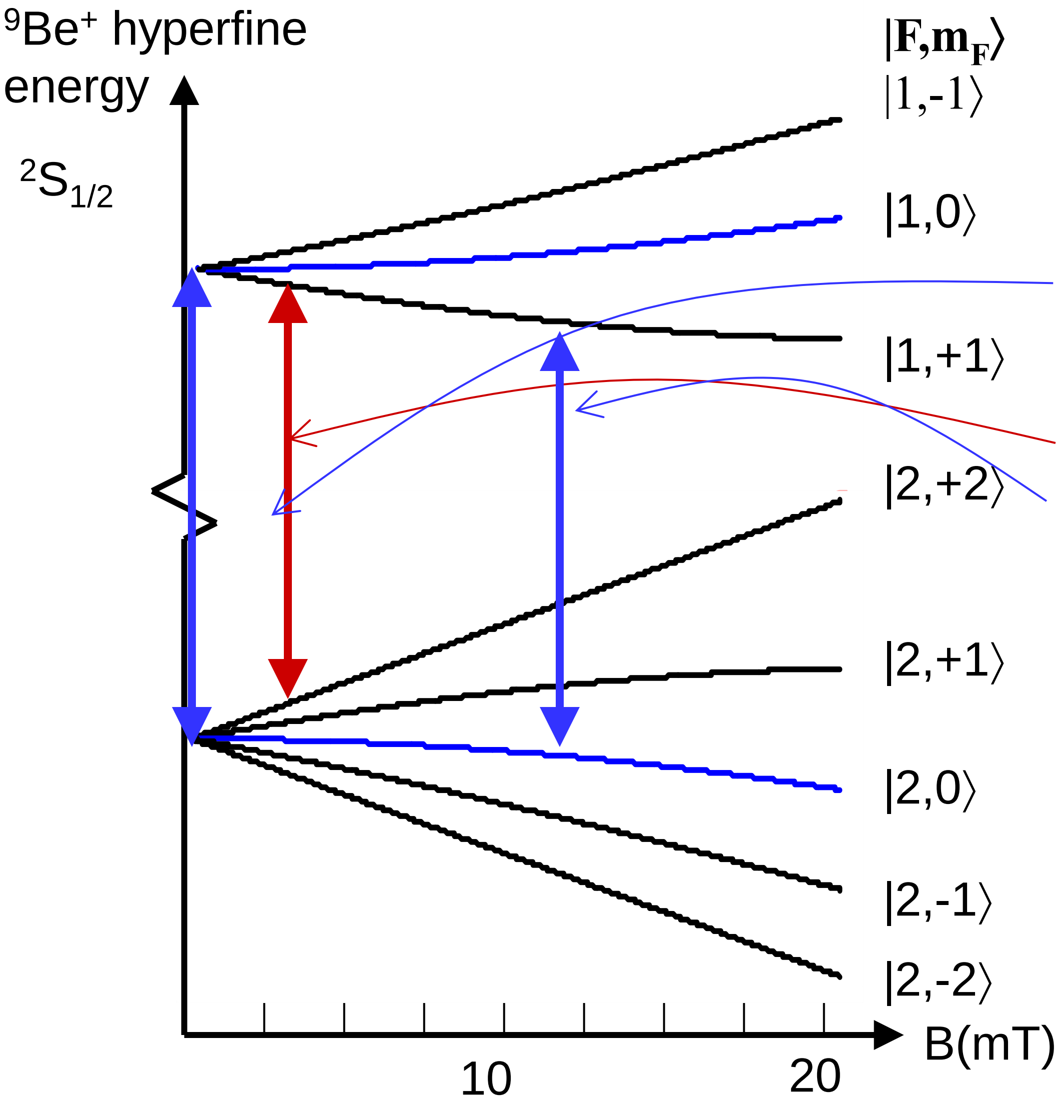
Benhelm, *et. al.* Nature Phys. (2008).

- Optical qubit (Weizmann), *F=0.985*

Navon, *et. al.* In preparation. (2008).



Field independent qubits



$$H = A\,\mathbf{I}\cdot\mathbf{J} - (g_I\,\mathbf{I} + g_J\,\mathbf{J})\cdot\mathbf{B}$$

Solution: Breit-Rabi formula

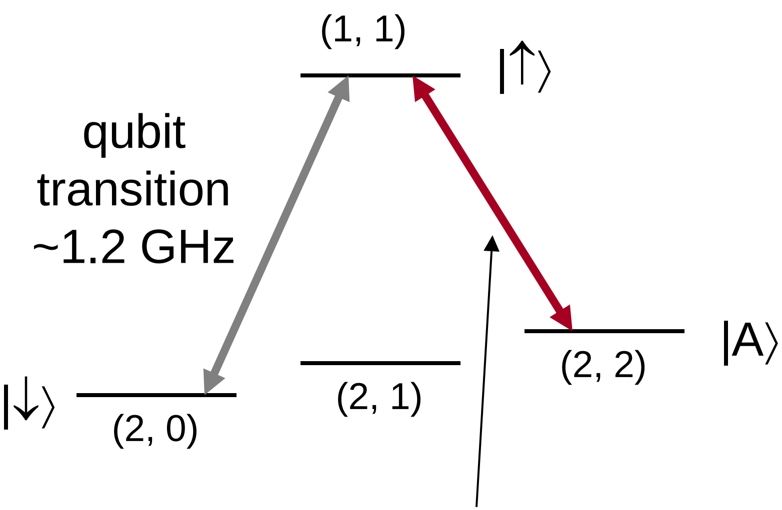
$|2,0\rangle \leftrightarrow |1,0\rangle$ $\tau_2 \cong 1.7$ ms

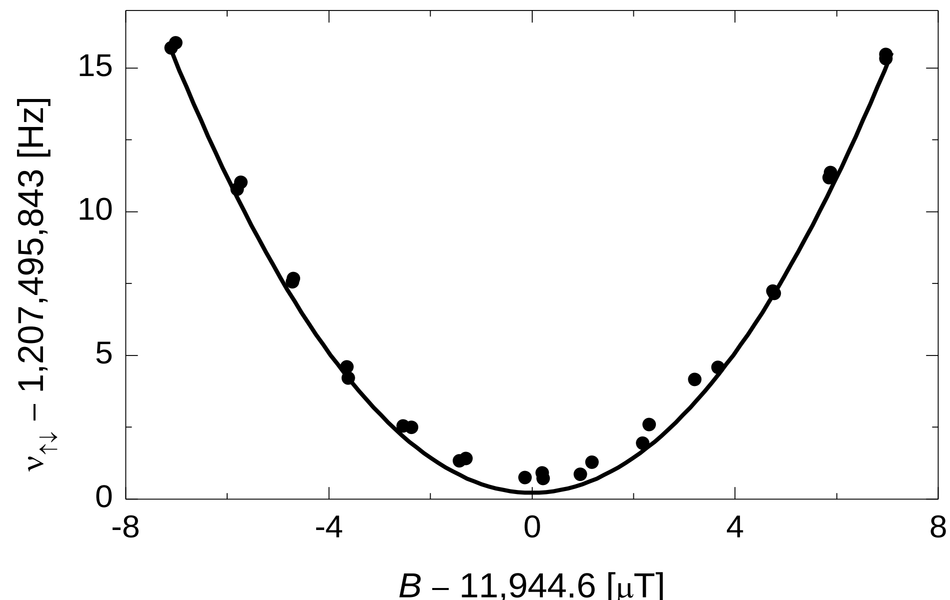
$|2,2\rangle \leftrightarrow |1,1\rangle$ $\tau_2 \cong 76$ μs

$|2,0\rangle \leftrightarrow |1,1\rangle$ $\tau_2 \cong 53$ s

$|2,1\rangle \leftrightarrow |1,0\rangle$
$\frac{1}{2}\,\delta^2 f/\delta B^2 = 0.3$ Hz/μT^2
<u>for $\delta B = 0.1$ μT</u>
$\tau_2(\delta\phi = 1 \text{ rad}) \cong 53$ s

Characterizing the field-independent qubit



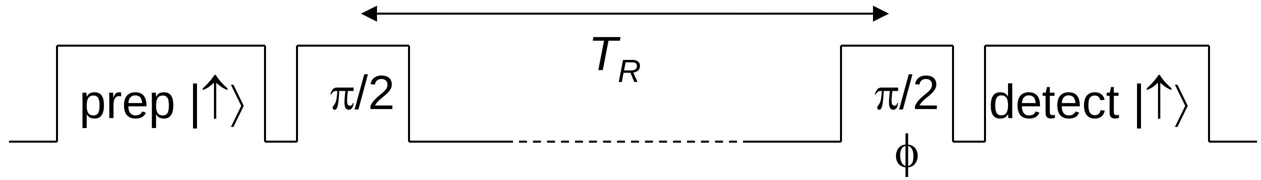


transition frequency |A⟩ ↔ |↑⟩ is linearly dependent on the magnetic field:
$\nu - \nu_0$ = 17.6 kHz/μT × ($B - B_0$).
Probing this transition measures the magnetic field.

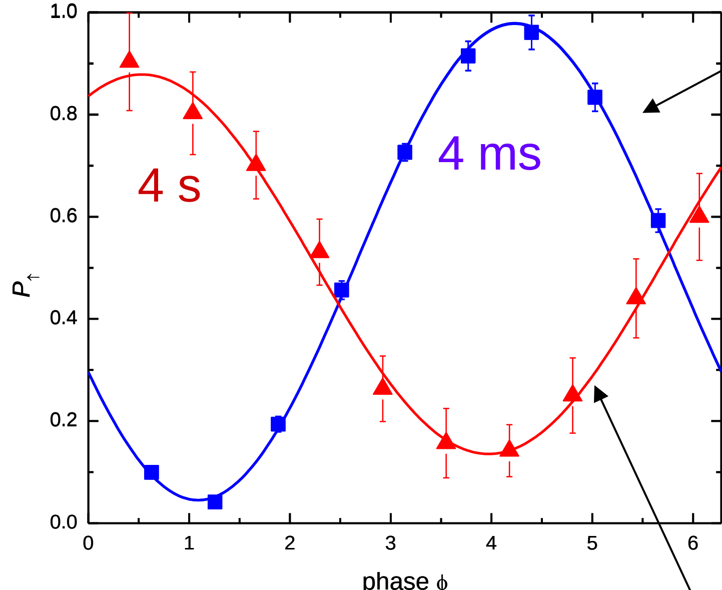
States labeled (F, m_F)

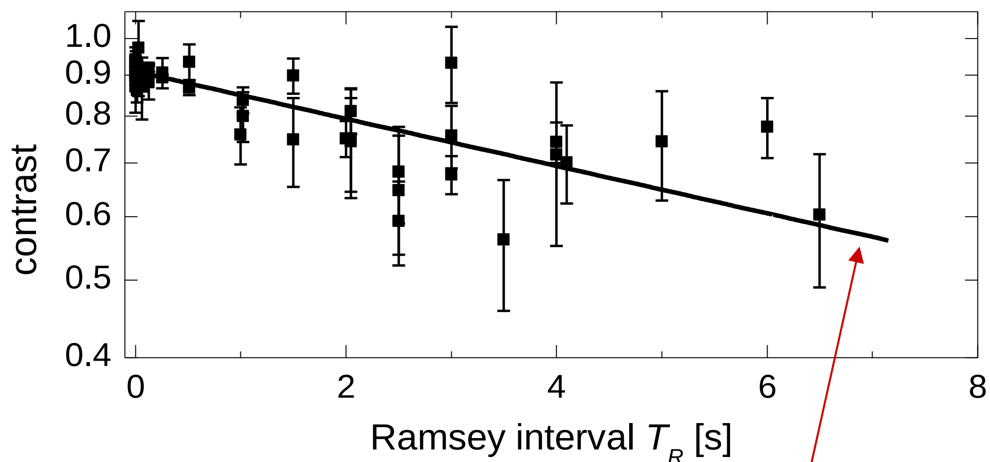
Qubit coherence time

Ramsey experiment:



contrast 0.93(1)





contrast 0.74(4)

Coherence time 14.7 ± 1.6 s

During detection: $\varepsilon \sim 10^{-5}$

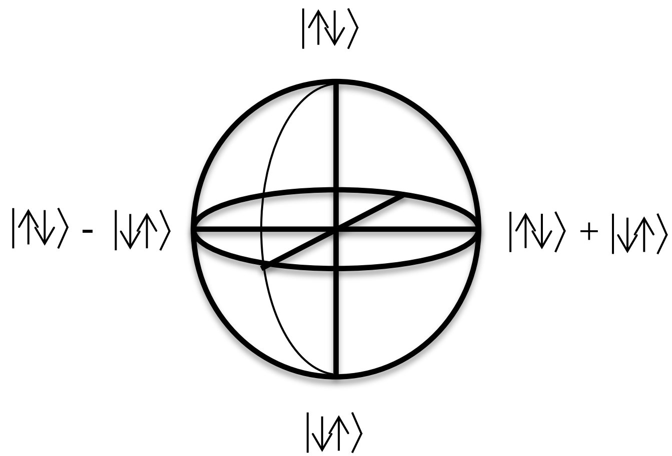
C. Langer, et. al. Phys. Rev. Lett. **95** 060502 (2005).

Decoherence-Free Subspaces

All the states in the sub-space, spanned by the two states,

$$|\Psi+\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$$ (triplet; m=o)

$$|\Psi-\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$$ (singlet)



- Remain coherent under collective magnetic field noise.

Haffner *et. al.* Appl. Phys. B, **81,** 151 (2005)
Langer *et. al.* Phys. Rev. Lett., **95,** 060502 (2005)

Decoherence-Free Subspaces

Coherence time = 44 Seconds