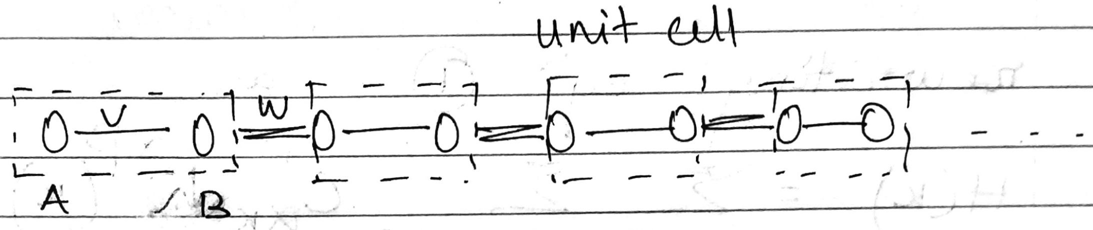
16/04/19

Tutorial : SSH Model

→ Objective : basic concepts of topology phase.
→ Hydrogen atom of topological systems.

→ Toy model of 'spinless' fermions on 1D lattice.



→ Each unit cell has 2 sites A and B (1 orbital per atom)

→ This model was originaly used to describe organic insulating molecule polyacetylene.

The hamiltonian of the chain is

$$H = v\sum_{m=1}^{N} \underbrace{c^{\dagger}_{Am} c_{Bm}}_{\text{intra cell}} + w\sum_{n=1}^{N-1} \underbrace{c^{\dagger}_{nB} c_{An+1}}_{\text{inter cell}} \quad -(1)$$

→ Real hopping amplitude v, w for simplicity.
phase can be gauged away by change of basis set.
→ Zero chemical potential + No onsite potential.

Bulk Hamiltonian (N→∞ thermodynamic limit)

using P.B.C and translational symmetry we plug in bloch states to ①

$$c_{Am} = \sum_{k \in B.Z} c_{Ak} e^{imk.a}$$

rewriting eq^n ① as

$$H(k) = \sum_{k \in B.Z} \sum_{\alpha,\beta \in A,B} c^{\dagger}_{\alpha k} h^{\alpha\beta}(k) c_{\beta k}$$

where $$h^{\alpha\beta}(k) = \begin{bmatrix} 0 & v+we^{ik.a} \\ v+we^{-ik.a} & 0 \end{bmatrix}$$

$$h(k) = \underbrace{(v + w\cos k.a)}_{d_x} \sigma_x + \underbrace{(w \sin k.a)}_{d_y} \sigma_y$$

$$= \vec{d}(k) \cdot \vec{\sigma}$$

→ a useful compact representation of the system!

→ describes a generic 2 band model.

→ Energy eigen values given by the $\vec{d}(k)$

$$E(k) = |\vec{d}(k)| = \sqrt{v^2 + w^2 + 2vw\cos k \cdot a}$$

eigen wave function $|\Psi_\pm\rangle = \begin{pmatrix} \pm e^{-i\phi(k)} \\ 1 \end{pmatrix}$

$$\tan\phi(k) = \frac{d_y}{d_x}$$

→ we immediately notice that for the system to be gapless

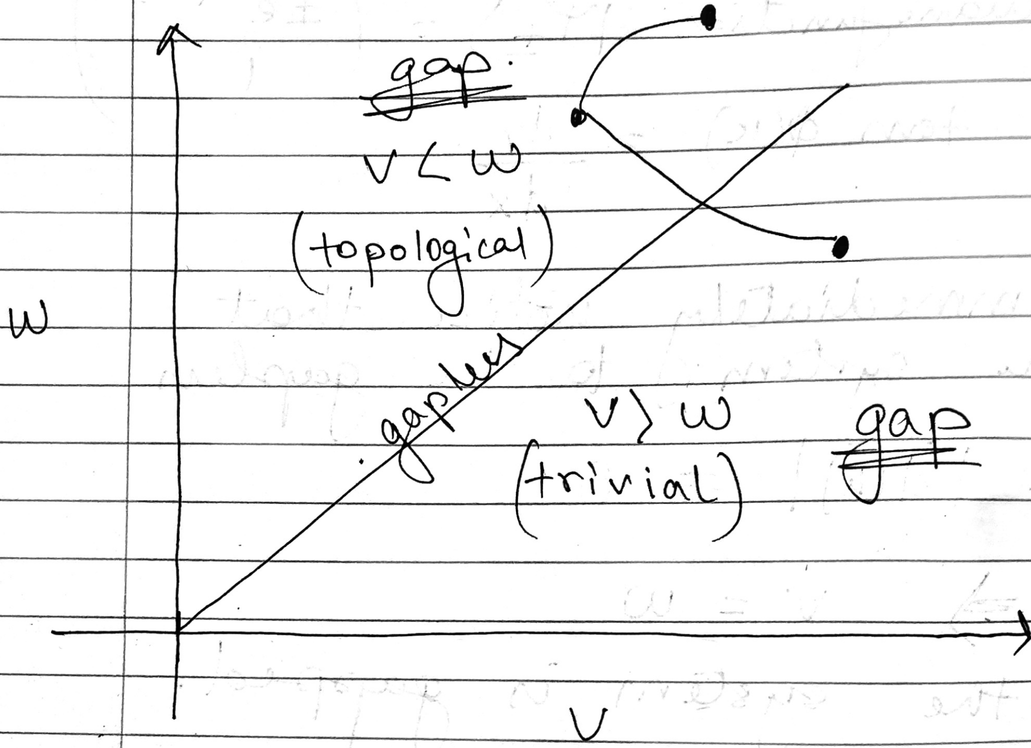
$$E(k) = |\vec{d}(k)| = 0$$

$$\Rightarrow \quad v = w$$

else the system is gapped.

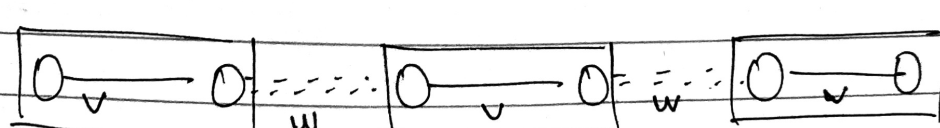
$$\Delta = \min E(k) = |v - w|$$

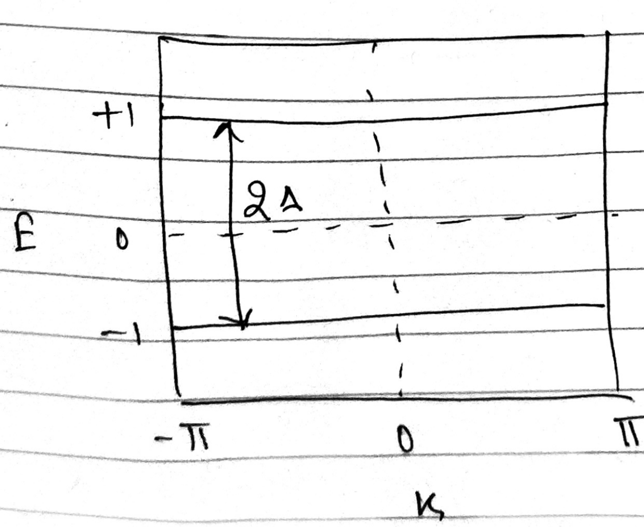
Phase diagram



How does the system look like in the fully dimerized limit

case : $v = 1$, $w = 0$





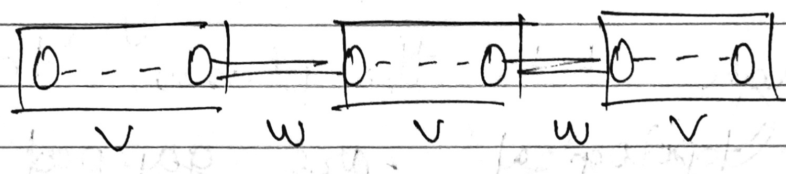
$E(k) = \pm 1$

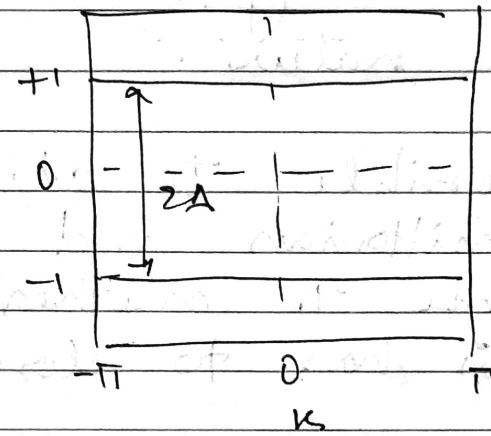
$d_x = 1$

$d_y = 0$

'trivial'

Case: $V = 0$, $W = 1$





$E(k) = \pm 1$

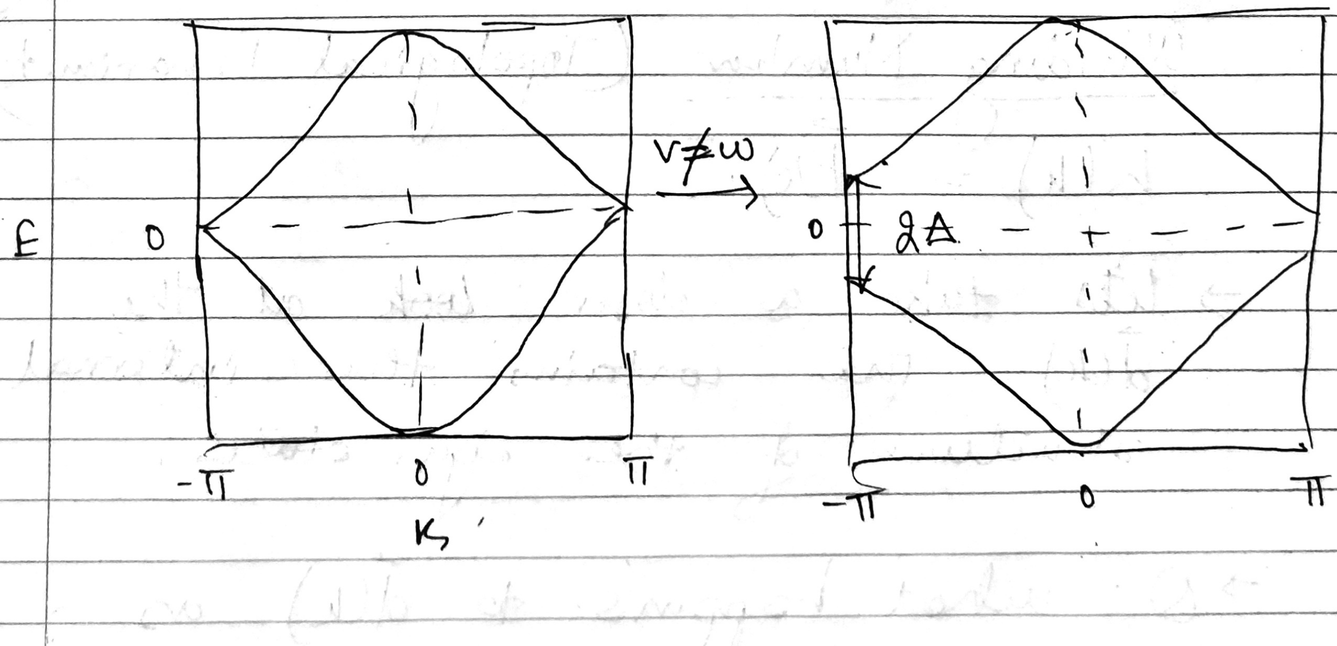
$d_x = \cos k \cdot a$

$d_y = \sin k \cdot a$

→ generally $V \neq W$ due to Pierls instability

→ staggering is favorable as it lowers the energy of the filled states.

Case: $V = W$, $E(k) = \pm V\sqrt{1 + \cos k a}$



→ Refer to the phase diagram.

→ although both the phases 'trivial' and 'topological' are gapped, they are distinct in nature.

→ it is not possible to adiabatically deform the Hamiltonian and go from one phase to another. Since the gap has to close.

Adiabatic equivalence:

gap should not close.

symmetries should not be broken.

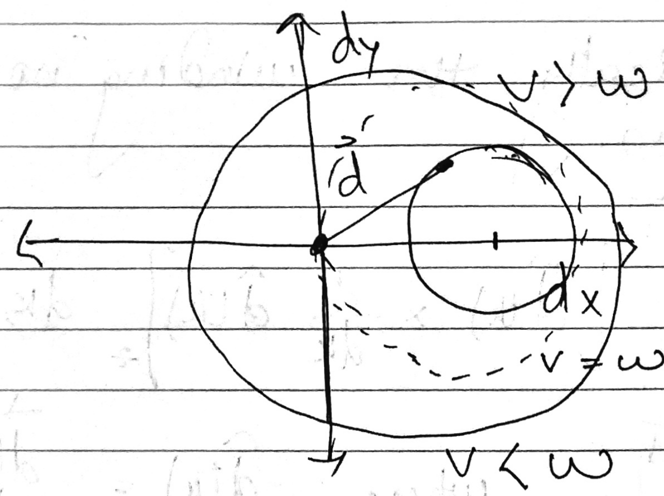
→ How to characterize these two different phases?

Winding Number (Topological Invariant)

$h(k) = \vec{d}(k) \cdot \vec{\sigma}$

→ lets take a closer look at the $\vec{d}(k)$. This contains the internal structure of the eigen states.

→ Q: what happens to $\vec{d}(k)$ as swept across the B.Z. $k = 0 \to 2\pi$



$d_x = v + w \cos ka$
$d_y = w \sin ka$

dimer limit

$v = 1,\ w = 0$
trivial

$v = 0,\ w = 1$
topological.

→ the origin in the d plane corresponds to gappless spectrum.

→ Just like we saw in the phase diagram, it is not possible to adiabatically deform the gapped H from one phase to another without closing the gap.

→ this realization resulted in a new type of classification of matter.

→ the two loops (different gapped phase) can be characterized by an integer called winding number

= # times the loop winds around the origin.

→ Mathematically the winding no. is expressed as.

$$\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\hat{d}(k) \times \frac{d}{dk} \hat{d}(k) \right]_z dk$$

where $\hat{d}(k) = \frac{\vec{d}(k)}{|\vec{d}(k)|}$

→ Its easy to check the winding number in the fully dimerized limit.

case : trivial $v = 1, \; w = 0$

$d_x = 1, \; d_y = 0$

$\nu = 0$

case : topological $v = 0, \; w = 1$

$\underbrace{d_x = \cos k.a}_{= h_x} \quad \underbrace{d_y = \sin k.a}_{= h_y}$

$\nu = 1.$

map the problem to complex plane

$$\nu = \frac{1}{2\pi} \oint_c \frac{h_x \, dh_y - h_y \, dh_x}{|h|^2}$$

$\cancel{\frac{h_y}{h_x}} = \cancel{\tan}$ define $\frac{h_x}{|h|} = \cos \alpha$

$$\nu = \frac{1}{2\pi} \oint_c d\alpha \quad , \quad \alpha = \tan^{-1}\left(\frac{h_y}{h_x}\right)$$

→ the same as the visual interpretation of the winding number.

→ So ν can change from
$0 \longrightarrow 1$ only by gap closing.

~~iff~~

the ring is not lifted off the $d_x - d_y$ plane.
i.e $d_z = 0$.

→ Q: How to ensure $d_z = 0$

Chiral Symmetry

→ the SSH model has the following unitary symmetry

$$S H S^{\dagger} = -H$$

known as chiral symmetry.

→ easily verify: $S = \sigma_z$

$$\sigma_z H \sigma_z^{-1} = -H$$

$$\Rightarrow d_z = 0$$

→ This is also called sub lattice symmetry. i.e interchange of the sites in the unit cell.

→ it ensures : only off diagonal term in H

i.e no onsite potential
no hopping between same specie.

→ The energy spectrum is symmetric.

$$H|\psi_n\rangle = E_n|\psi_n\rangle$$

$$HS|\psi_n\rangle = -SH|\psi_n\rangle$$

$$= -E_n \underbrace{S|\psi_n\rangle}_{\text{orthogonal eigen state.}}$$

for $E_n = 0$, zero energy state are their own chiral partner.

<u>Physical Symmetry</u>.

TR : $T H(k) T^{+} = H(-k)$

Inversion : $I H(k) I^{+} = H(-k)$

$$I = \sigma_x$$

these ensure:

d_x even in k

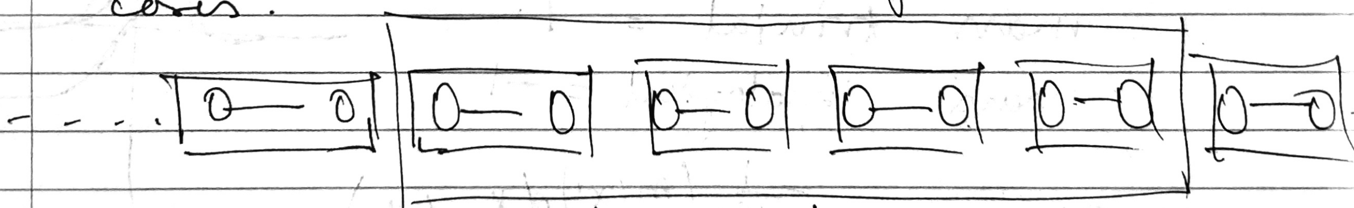
d_y odd in k

$\boxed{d_z = 0}$

Bulk Boundary Correspondence

Edge states

→ Lets examine the fully dimerized cases.

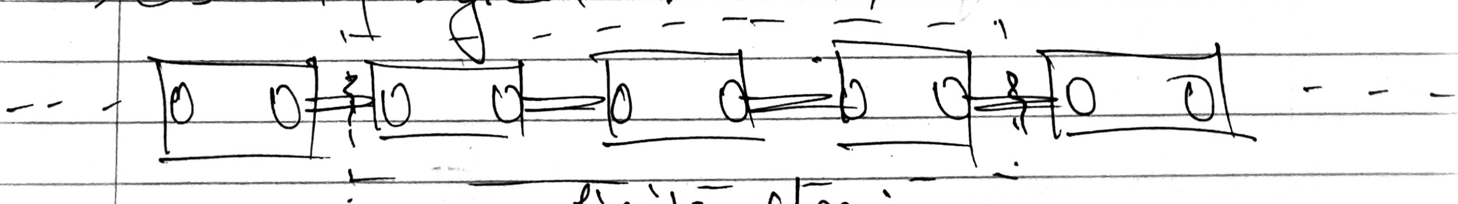


finite chain

→ case 'trivial' : $v = 1$, $w = 0$

All electrons are paired and form bonds
No zero energy state

→ case 'topological' : $v = 0$, $w = 1$



finite chain

→ Unpaired electrons at the edge of the chain.

→ No onsite potential ⇒ zero energy state.

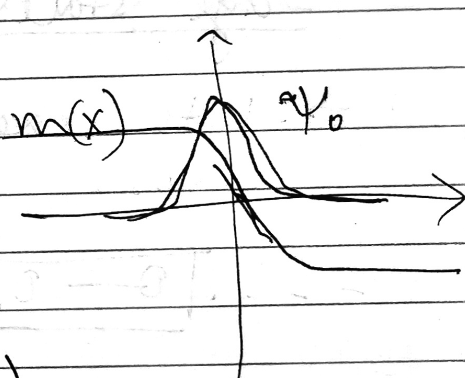
→ low energy continuum theory of the TB model gives

$H = -i v_F \sigma_x \partial_x + m \sigma_y$

This is obtained by $k \to \pi + q$
$q \to -i\partial_x$

$E(q) = \pm\sqrt{(v_F q)^2 + m^2}$

$m = v - w \quad , \quad v_F = wa$



mass trivial $= 1$
mass topological $= -1$

$\psi_0 = e^{-\frac{1}{v_F}\int m(x')\,dx'} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

This is the Jackiw - Rebbi solution.
The zero mode of SSH model.

→ Reiterate the importance of Bulk - Boundary correspondance.