Last time we derived a scaling theory for localization in one dimension starting from the microscopics (i.e. from the single particle SE). The goal of this lecture is to generalize the scaling theory to higher dimensions. Alas this goal is too demanding for a fully microscopic approach. Instead we shall describe a phenomenological scaling theory based on a rather simple (perhaps deceivingly simple) postulate due to the "gang of 4" AALR 1979 (see ref. above).

The idea is to start from a microscopic scale, only somewhat larger than the mean-free path $l$, where we can have some knowledge of how the system behaves, and then gradually extend this knowledge to the behavior of the system at increasing length scales.

The main assumption is that the dimensionless conductance associated with an $L \times L \times L$ box of material is a function of $L$ alone:

$$g = g(L)$$
In other words, we assume that the conductance of a superblock of size $(2L)^d$ depends only on the conductance of the sub-blocks of size $L$, and not, for example independently on $L$, or on the strength of the disorder).

\[ g(L) \quad \quad \quad g(2L) \]

We can therefore describe the change of the conductance with the scale $L$ in the form:

\[ \frac{d \ln g}{d \ln L} = \beta \left[ g(L) \right] \]

Note that the $\beta$ function depends on $L$ only through $g$. It is a direct function of only one variable, $g$. The above assumption and the resulting scaling equation are called the "one parameter scaling" hypothesis. This hypothesis is deceptively simple. In fact it is extremely powerful and for the rest of this lecture we shall study its consequences.
To get a feeling for the form of the $\beta$ function let us, following AALR, examine its behavior in the limits of small and large values of $g$.

\[ g \gg 1 : \text{ the classical theory of conduction should work and we expect Ohm's law to hold:} \]

\[ R = \frac{P}{A} = \frac{P}{L^{d-1}} \rightarrow \sigma \sim \sigma_0 L^{2-d} \]

(for $g \gg 1$)

Differentiating this w.r.t. $L$ we get

\[ \frac{d \ln g}{d \ln L} = \frac{L}{g} \frac{d g}{d L} = d-2 \]

\[ \Rightarrow \lim_{g \rightarrow \infty} \beta[g] = (d-2) \]
$g < 1$ : Anderson’s strong disorder perturbation theory should work. Therefore we expect localized states and the conductance to drop off exponentially with length-scale.

\[ g = e^{-L/\xi} \quad (g < 1) \]

\[ \Rightarrow \quad \frac{d \ln g}{d \ln L} = -\frac{L}{\xi} = \ln g \]

\[ \lim_{g \to 0} \beta [g] = \ln g \]
Now we can plot the β function in the two limits and assume the simplest possible interpolation between them.

\( g \ll 1 \)

\( g \gg 1 \)

\[ \beta(g) \]

\[ s \ln \left( g / g_c \right) \]

\[ d = 3 \]

\[ d = 2 \]

\[ d = 1 \]

plot taken from Y. Imry (Book)

Note the point where the flow begins depends the conductance \( g_0 \) at a microscopic scale.
$d = 1$

We see that in 1d the β function is always negative. Therefore the system always flows to the localized behavior, in agreement with our microscopic analysis from the previous lecture.

Let us try to estimate the 1d localization length from the phenomenological scaling theory. Consider the $g > 1$ limit of the β function:

$$\frac{d \ln g}{d \ln L} = -1 \quad \Rightarrow \quad g(L) \approx g_0 / L$$

The localization length is estimated as the scale at which $g$ is suppressed to order 1:

$$g(\xi_{loc}) = g_0 / \xi_{loc} \sim 1 \quad \Rightarrow \quad \xi_{loc} \sim g_0$$

Again this is in perfect agreement with the microscopic theory presented in the last lecture.
In two dimensions the situation is more interesting.
As we argued before, the limit \( g \rightarrow \infty \) is

\[
\lim_{g \rightarrow \infty} \beta(g) = 0 \quad \text{this is a marginal case}
\]

To decide where things flow one needs to understand how \( \beta \) behaves as a function of the small parameter \( 1/g \). The most plausible behavior is

\[
\beta(g) = -\frac{c}{g}
\]

where \( c \) is some constant of order 1.

In the plot above we assumed that the connection is negative \((c>0)\) because this gives the simplest interpolation with the limit \( g \ll 1 \) where \( \beta(g) \propto \ln g < 0 \).

But can we rule out a more complicated scenario such as this?

\[
\begin{array}{c}
\beta \\
\downarrow
\end{array}
\begin{array}{c}
\ln g
\end{array}
\]
The answer is yes. To do it convincingly one should undertake a rather complicated microscopic calculation, perturbative in the strength of the disorder, and show that the correction to $g_0$ at length scales greater than the m.f.p is negative. Such a calculation is well beyond the scope of this course. But it is easy to understand the essential idea.

The probability for an electron to go from point A to point B by "quantum diffusion" within a block larger than the m.f.p, involves the sum of the amplitudes of all the many paths the electron can take in the disordered potential.

Generically the amplitudes would be summed with random phases and therefore add up incoherently. However there are special trajectories that must add up coherently. When point A and point B are the same point, then two of trajectories which are time reversed partners of each other must add up with the same phase due to time reversal symmetry. That leads to enhanced probability to return to the same point, which is the source of the negative correction to the conductance. This is called "weak localization correction."
The important conclusion is that non-interacting electrons in two dimensions are always localized. Let us now go on to estimate the localization length in 2d. As we did in one dimension, we can use the $g \gg 1$ limit of the $\beta$ function (including the leading correction in $1/g$):

$$\frac{d \ln g}{d \ln l} \approx \frac{-c}{g} \quad \text{on} \quad \frac{dg}{d \ln l} \approx -c < 0$$

$$g = g_0 - c \ln \left( \frac{L}{l} \right)$$

(where $l$ is the mean free path and $g_0$ is the "initial condition" at the scale $L$)

Now as before we define the localization length as the scale where $g(l_{loc}) \sim 1$, that is, where the weak disorder expansion must break down:

$$g(l_{loc}) = g_0 - c \ln \left( \frac{\rho_{loc}}{l} \right) \sim 1$$

$$\rho_{loc} \sim l \exp \left[ g_0 / c \right]$$
Let us take a moment to understand the meaning of this result.

For weak disorder $g_0 \gg 1$ while $c$ is a constant of order 1.

So $\frac{g_0}{c}$ is in this case astronomical.

Although strictly speaking the system is localized and insulating, if the disorder is weak we may need a sample larger than our planet in order to see the localized behavior.

$$d = 3$$

The situation in 3d is even more interesting.

For $g \gg 1$ we have $\beta = 1 > 0$, while for $g \ll 1$ $\beta \sim \log \xi < 0$. By continuity there is some $g = g_c$ where $\beta = 0$.

This is an unstable fixed point of the flow, and therefore a critical point marking a 2nd order metal-insulator phase transition.
If \( g_0 \), the conductance at a microscopic scale, is larger than \( g_c \) (\( g_0 > g_c \)), then the system will "flow" to the classical Ohmic behavior at macroscopic scales

\[
g_0 > g_c \Rightarrow g(L) = g_0 L = g_0 (L/\ell_c)
\]

If, on the other hand, \( g_0 < g_c \) then the system will "flow" to localized behavior and the macroscopic conductance will vanish.

Exactly on the critical point (\( g_0 = g_c \)) \( \beta = 0 \) and therefore the system exhibits scale invariance.

To better understand the behavior of the system near the transition point, we linearize the flow around \( g_c \):
$$\beta^* \left[ \frac{g}{g_c} \right] = s \left( \frac{g - g_c}{g_c} \right) = s \cdot s$$

$\beta^*$ is the linearized version of the scaling function $\beta$ and $s$ is the slope of the flow in the $\beta - g$ plane at $g = g_c$

$$\frac{d \ln g}{d \ln L} = \frac{1}{g} \frac{d g}{d \ln L} = s \left( \frac{g - g_c}{g_c} \right)$$

$$\frac{d \ln \left( \frac{g - g_c}{g_c} \right)}{d \ln L} = s \left( \frac{g}{g_c} \right) \frac{g - g_c}{g_c} = s \left( \frac{g}{g_c} \right) + O\left( \frac{g - g_c}{g_c} \right)^2$$

$$\ln \left[ \frac{g(L) - g_c}{g_c} \right] = \ln \left[ \frac{g_0 - g_c}{g_c} \right] + s \ln \frac{L}{L_0}$$

$$\frac{g(L) - g_c}{g_c} = \frac{g_0 - g_c}{g_c} \left( \frac{L}{L_0} \right)^s$$

Here, $L_0$ is the microscopic scale (mean free path) so the conductance at this scale, and $g_c$ the critical conductance.
At scales \( L \) such that \( \frac{|g_L - g_c|}{g_c} \ll 1 \)
our linearized approximation holds and we will observe the above critical scaling behavior. This will break down at a scale \( \xi \) for which \( \left| \frac{g(\xi) - g_c}{g_c} \right| \sim 1 \).

Plugging this into the scaling form we can estimate

\[
\frac{g_0 - g_c}{g_c} \left( \frac{\xi}{L_0} \right)^S \sim 1
\]

\[
\left( \frac{\xi}{L_0} \right)^S \left| \frac{g_0 - g_c}{g_c} \right|^{-1/S} \equiv \left| \frac{g_0 - g_c}{g_c} \right|^{-1/S}
\]

In the localized side of the transition \( (g_0 < g_c) \) we will start observing localized behavior only beyond this scale. We can therefore associate \( \xi \) with the localization length on this side of the transition.

\[
\xi_{loc} \sim L_0 \left| \frac{g_c}{g - g_c} \right|^\nu
\]
In the metallic side of the transition ($g > g_c$) $\xi$ represents the length scale beyond which we would start observing classical ohmic behavior.

$$g(x) \approx \sigma(x) L \quad \text{(in } d=3)$$

Thus the macroscopic conductivity in the metallic side in the critical regime behaves as:

$$\sigma = \sigma(\xi) = g(\xi)/\xi \approx g_c/\xi = \left(\frac{g_c}{L_0}\right) \left| \frac{g - g_c}{g_c} \right|^\nu$$

* The conductivity vanishes continuously at the transition with the critical exponent $\nu$

* Mott's postulate of a minimum metallic conductivity, which would imply a discontinuous transition turned out to be wrong.
Note:

Of course the phenomenological scaling theory cannot predict the value of the critical exponent $\nu$. Numerical experiments indicate that it is close to 1.