Concepts of Condensed Matter Physics

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Contents

@ Lecture notes taken by Dar Gilboa of the course taught by Prof. Yuval Oreg

References

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[3] Assa Auerbach. *Interacting Electrons and Quantum Magnetism*. Springer-Verlag, 1994.

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1 Introduction - an Overview

2 Spin Models

References: Chapters 1 and 2 in Ref. [3].

2.1 Model Building

2.1.1 First Quantization

We consider Bloch wave-functions which are the single electron solutions of the Hamiltonian

$$H^0\phi_{ks} = \left[\frac{-\hbar^2}{2m}\nabla^2 + V^{\text{ion}}(x)\right]\phi_{ks}(x) = \varepsilon_k\phi_{ks}(x),$$

from which we can define a many electron Hamiltonian:

$$\mathcal{H}^0 = \sum_{i=1}^{N_e} H^0(\nabla_i, x_i).$$

The Hamiltonian $\mathcal{H}^0$ has eigenfunctions and eigenvalues

$$\Psi_K^{\text{Fock}}(x_{1,s_1}, x_{2,s_2}...x_{N_e,s_{N_e}}) = \det_{ij}\left[\phi_{k_i s_i}(x_i)\right] \ , \ E_K = \sum_{i=1}^{N_e}\varepsilon_{k_i},$$

where the ε_k are bounded by the Fermi energy E_F.

The Hamiltonian $\mathcal{H}^0$ describes non-interacting electrons. We add an interaction term and consider

$$H = \mathcal{H}^0 + \frac{1}{2}\sum_{i,j} V^{\text{el}-\text{el}}(x_i, x_j),$$

which we can write

$$H = \sum_{i=1}^{N_e}\left(H^0 + V^{\text{eff}}[x_i, \rho]\right) + \frac{1}{2}\sum_{ij}\tilde{V}(x_i, x_j),$$

where V^{eff} is obtained by in a mean field sense by "freezing" one x_i and summing over all other x_j in $V^{\text{el}-\text{el}}$ and

$$\tilde{V}(x_i, x_j) = V^{\text{el}-\text{el}}(x_i, x_j) - \left(V^{\text{eff}}(x_i) - V^{\text{eff}}(x_j)\right)/N_e.$$

If we neglect $\tilde{V}$ we obtain an effective non interacting theory of Fermions. The theory, however, have compare to $\mathcal{H}^0$ new effective parameters that may be determined, for example, in a self-consistent way. We also assume dynamics which is slower than the plasma frequency.

2.1.2 Second Quantization

We would like now to present the Many body Hamiltonian using the second quantization formalism. For that we define a creation operator $\hat{\psi}_s^\dagger(\vec{r})$. When acting on the vacuum state (a state with zero particles that we denote $|0\rangle$) gives

$$\langle\vec{y},\sigma|\,\widehat{\psi}_s^\dagger(\vec{x})\,|0\rangle = \delta_{s\sigma}\delta(\vec{x}-\vec{y}).$$

(Anti)Commutation relations of ψ
We can choose a basis of the Hilbert space $\{\phi_\alpha\}$ and write

$$\hat{\psi}_s^\dagger(\vec{x}) = \sum_\alpha \phi_{\alpha s}^*(\vec{x}) c_{\alpha s}^\dagger \tag{1}$$

$$\left\{\hat{\psi}_s^\dagger(\vec{x}), \hat{\psi}_\sigma(\vec{y})\right\} = \sum_{\alpha\beta} \phi_{\alpha s}^*(\vec{x})\phi_{\beta\sigma}(\vec{y}) \left\{c_{\alpha s}^\dagger, c_{\beta\sigma}\right\} = \sum_{\alpha\beta} \phi_{\alpha s}^*(\vec{x})\phi_{\beta\sigma}(\vec{y})\delta_{s\sigma}\delta_{\alpha\beta}$$
$$= \sum_\alpha \phi_{\alpha s}^*(\vec{x})\phi_{\alpha\sigma}(\vec{y})\delta_{s\sigma} \underbrace{=}_{\{\phi\}\text{complete}} \delta_{s\sigma}\delta(\vec{x}-\vec{y}). \tag{2}$$

The Hamiltonian is now

$$\hat{\mathcal{H}}^0 = \sum_s \int d^3x \hat{\psi}_s^\dagger(\vec{x}) \left[\frac{-\hbar^2}{2m}\nabla^2 + V^{\text{ion}}(\vec{x}) + V^{\text{eff}}(\vec{x})\right] \hat{\psi}_s(\vec{x})$$

$$\hat{\tilde{V}} = \frac{1}{2}\int d^3x d^3y \tilde{V}(\vec{x},\vec{y}) \left[\hat{\rho}(\vec{x})\hat{\rho}(\vec{y}) - \delta(\vec{x}-\vec{y})\hat{\rho}(\vec{x})\right] \tag{3}$$

where the second term is because we have no self-interaction. Since

$$\hat{\rho}(\vec{x}) = \sum_s \hat{\psi}_s^\dagger(\vec{x})\hat{\psi}_s(\vec{x})$$

$$\hat{\rho}(\vec{x})\hat{\rho}(\vec{y}) = \sum_{s,\sigma} \hat{\psi}_s^\dagger(\vec{x})\hat{\psi}_s(\vec{x})\hat{\psi}_\sigma^\dagger(\vec{y})\hat{\psi}_\sigma(\vec{y}) = -\sum_{s,\sigma} \hat{\psi}_s^\dagger(\vec{x})\hat{\psi}_\sigma^\dagger(\vec{y})\hat{\psi}_s(\vec{x})\hat{\psi}_\sigma(\vec{y}) + \sum_s \hat{\psi}_s^\dagger(\vec{x})\hat{\psi}_\sigma(\vec{y})\delta(\vec{x}-\vec{y})\delta_{s\sigma}$$

$$= \sum_{s,\sigma} \hat{\psi}_s^\dagger(\vec{x})\hat{\psi}_\sigma^\dagger(\vec{y})\hat{\psi}_\sigma(\vec{y})\hat{\psi}_s(\vec{x}) + \hat{\rho}(\vec{x})\delta(\vec{x}-\vec{y})$$

setting this in Eq. (2) the second term cancels giving finally

$$\hat{\tilde{V}} = \frac{1}{2}\sum_{s\sigma}\int d^3x d^3y \tilde{V}(\vec{x},\vec{y})\hat{\psi}_s^\dagger(\vec{x})\hat{\psi}_\sigma^\dagger(\vec{y})\hat{\psi}_\sigma(\vec{y})\hat{\psi}_s(\vec{x}). \tag{4}$$

The ability to neglect $\tilde{V}$ determines whether we can use a Fermi liquid theory with effective parameters or we get more dramatic interaction effects. For a rough estimate of weather $\tilde{V}$ is large or small we note that in materials with an outer electron in the s level the electron wave-functions obey $\langle r\rangle \sim a$ where a is the lattice constant and $\langle r\rangle$ is the average distance of the electron from the nucleolus. For outer electrons in the d,f levels however, $\langle r\rangle \ll a$.

The typical kinetic energy of electrons is

$$E_F = \frac{k_F^2}{2m}, k_F \sim \frac{1}{a} \Rightarrow E_F = v_F k_F \sim \frac{v_F}{a}$$

while the interaction is (with κ being the dielectric constant)

$$U = \frac{e^2}{\langle r\rangle \kappa}.$$

giving a ratio

$$r_s = \frac{U}{E_F} = \frac{e^2}{\kappa v_F}\frac{a}{\langle r\rangle}$$

Hence, in most cases, $\tilde{V}$ will be more important in d, f materials (we know that in many metals $\frac{e^2}{\kappa v_F} \sim 1$).

2.1.3 Effective interactions: Direct (Hartree) Exchange (Fock) and Cooper (Pairing) channels

Inserting Eq. (1) $[\psi_s^\dagger(\vec{x}) = \sum_\alpha c_{s\alpha}^\dagger \phi_\alpha^*(\vec{x})]$ in the interaction term $\hat{\tilde{V}}$ of Eq. (4) and assuming that the wave-function ϕ_α doesn't depend on the spin - no SO coupling for instance. We have

$$\hat{\tilde{V}} = \sum_{\sigma s,\alpha\beta\gamma\delta} \tilde{M}_{\delta\gamma}^{\alpha\beta} c_{\alpha\sigma}^\dagger c_{\beta s}^\dagger c_{\gamma s} c_{\delta\sigma}$$

where

$$\tilde{M}_{\delta\gamma}^{\alpha\beta} = \frac{1}{2}\int d^3x d^3y \phi_\alpha^*(\vec{x})\phi_\beta^*(\vec{y})\phi_\gamma(\vec{y})\phi_\delta(\vec{x}); \quad M_{\delta\gamma}^{\alpha\beta} = \begin{cases} \frac{1}{3}M_{\delta\gamma}^{\alpha\beta} & \text{if} \quad \alpha=\beta=\gamma=\delta \\ M_{\delta\gamma}^{\alpha\beta} & \text{otherwise} \end{cases}$$

Assuming now that we can neglect interaction terms that do not contain at least two identical indexes[1] we can split $\hat{\tilde{V}}$ into three channels

$$\hat{\tilde{V}} = \sum_{\alpha\gamma s\sigma} M_{\alpha\gamma}^{\alpha\gamma} c_{\alpha\sigma}^\dagger c_{\alpha\sigma} c_{\gamma s}^\dagger c_{\gamma s} + M_{\gamma\alpha}^{\alpha\gamma} c_{\alpha\sigma}^\dagger c_{\alpha s} c_{\gamma s}^\dagger c_{\gamma\sigma} + M_{\gamma\gamma}^{\alpha\alpha} c_{\alpha\sigma}^\dagger c_{\alpha s}^\dagger c_{\gamma\sigma} c_{\gamma s},$$

(Notice the factor 1/3 in the diagonal term it is introduced to avoid double counting.) Using the compliances relation of the Pauli matrices

$$\vec{\sigma}_{\sigma\sigma'} \cdot \vec{\sigma}_{s\sigma} = 2\delta_{\sigma\sigma}\delta_{\sigma' s} - \delta_{\sigma\sigma'}\delta_{s\sigma}$$

and the definitions

$$\hat{n}_\alpha = \sum_\sigma c_{\alpha\sigma}^\dagger c_{\alpha\sigma}, \; \hat{\vec{s}}_\alpha = \frac{1}{2}\sum_{\sigma s} c_{\alpha\sigma}^\dagger \vec{\sigma}_{\sigma s} c_{\alpha s}, \; \hat{t}_\alpha^\dagger = c_{\alpha\uparrow}^\dagger c_{\alpha\downarrow}^\dagger$$

we get

$$\hat{\tilde{V}} = \sum_{\alpha\gamma} \Big[\underbrace{\left(M_{\alpha\gamma}^{\alpha\gamma} - M_{\gamma\alpha}^{\alpha\gamma}/2\right)\hat{n}_\alpha \hat{n}_\gamma}_{\text{Hartree (Direct)}} - \underbrace{2M_{\gamma\alpha}^{\alpha\gamma}\hat{\vec{s}}_\alpha \cdot \hat{\vec{s}}_\gamma}_{\text{Fock (Exchange)}} + \underbrace{M_{\gamma\gamma}^{\alpha\alpha}\hat{t}_\alpha^\dagger \hat{t}_\gamma^\dagger}_{\text{Cooper (Pairing)}} \Big]$$

The three terms may be represented by diagrams, where the circles denote the paired indexes.

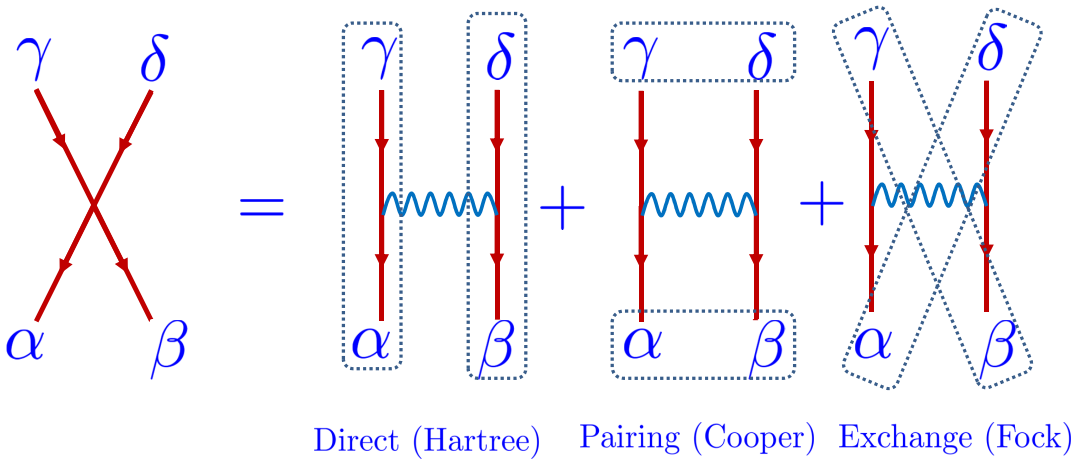


Figure 1: The direct (Hartree), exchange (Fock) and pairing (Cooper) channels.

[1] This will be a good approximation, for example, when the interaction is short-range and the wave function ϕ are fairly localized but demand justification in other cases.

In cases where the off diagonal Cooper terms are neglected the Hamiltonian reduces to

$$\hat{\tilde{V}} = \sum \tilde{U}_{ii'} n_i n_{i'} - 2J_{ii'} \vec{s}_i \cdot \vec{s}_{i'}$$

$$\tilde{U}_{ii'} = \delta_{ii'} U_{ii} + (1 - \delta_{ii'})(U_{ii'} - \frac{J_{ii'}}{2})$$

and $U_{i,i'} = M^{ii'}_{ii'}, J_{i,i'} = M^{ii'}_{i'i}$.

2.1.4 Definition of Heisenberg's Model

The Heisenberg model neglects the pairing and direct terms and assumes that the dominant contribution is from the second term, that mean

$$H = -2 \sum_{ij} J_{ij} \vec{s}_i \cdot \vec{s}_j$$

we will further assume

$$J_{ij} = \begin{cases} J_0 & \text{nearest neighbours} \\ 0 & \text{otherwise} \end{cases}$$

and $J_0 > 0$ for a ferromagnetic interaction, $J_0 < 0$ for an anti-ferromagnetic interaction.

In the ferromagnetic case $J_0 > 0$ case spins will prefer to be aligned. that happens when the overlap between the i and j orbitals then (similar to the case of Hund's rule) electron will tend to aline in parallel spin so that they avoid each other due to the Pauli principle. However if we have opposite spins sitting in neighboring atoms then it can be energetically preferable for one to tunnel, which is only possible if the spins are reversed, hence in this situation $J_0 < 0$.

2.2 Mean Field Solution of the Heisenberg Model – Spontaneous Symmetry Breaking

Before discussing the mean field solution of the Heisenberg Model let us discuss the general mean field approach:

2.2.1 General Mean Field approximation

The general approach is to take a Hamiltonian of the form

$$H = A + B + AB$$

and simplify the interaction term by replacing the operator A by $A \to \langle A \rangle + \Delta A$. For consistency we need to check in the resulting solution that we have

$$\Delta A \ll \langle A \rangle .$$

After a similar substitution for B

$$H = A + B + A(B - \Delta B) + (A - \Delta A)B + \Delta A \Delta B - (A - \Delta A)(B - \Delta B)$$

$$= A + B + A \langle B \rangle + \langle A \rangle B + \Delta A \Delta B - \langle A \rangle \langle B \rangle .$$

Assuming the variations are small we define the mean field Hamiltonian

$$H_{\mathrm{MF}} = A + B + A \langle B \rangle + \langle A \rangle B - \langle A \rangle \langle B \rangle .$$

2.2.2 Mean Field solution of the Heisenberg model

Coming back to the Heisenberg case we have

$$H_{\mathrm{MF}} = -2\sum_{ij} J_{ij} \langle s_i \rangle s_j - 2\sum_{ij} J_{ij} s_i \langle s_j \rangle + 2\sum_{ij} J_{ij} \langle s_i \rangle \langle s_j \rangle$$

since we can rotate all the spins together we expect that without symmetry breaking we will have $\langle s_i \rangle = 0$. In the case of symmetry breaking we look for a state where $\langle s_i \rangle = \langle s_z \rangle \hat{e}_z$. We define the magnetization

$$m = 2\sum_{ij} J_{ij} \langle s_z \rangle \hat{e}_z = 2nJ_0 \langle s_z \rangle \hat{e}_z$$

where n is the number of neighbors. Then if the total number of spins is N,

$$H_{\mathrm{MF}} = -2\sum_{i} \vec{m} \cdot \vec{s}_i + |\vec{m}| N \langle s_z \rangle$$

We write the partition function

$$Z_{\mathrm{MF}} = Tr\left(e^{-\beta H_{\mathrm{MF}}}\right) = \left(e^{\beta m} + e^{-\beta m}\right)^N e^{\beta m N \langle s_z \rangle} = \left(\left(e^{\beta m} + e^{-\beta m}\right) e^{\beta m^2 / 2nJ_0}\right)^N$$

We want to minimize the free energy with respect to m to find the equilibrium value:

$$\frac{\partial F}{\partial m} = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial m} = N\frac{e^{\beta m} - e^{-\beta m}}{e^{\beta m} + e^{-\beta m}} + N\frac{m}{nJ_0} = 0$$

defining $a = \frac{m}{nJ_0}, b = nJ_0\beta$ we obtain

$$a = \tanh(ab).$$

If we consider a as an order parameter which is zero in the disordered phase (since it is proportional to m), we can assume that it is small and expand:

$$a \cong ba - \frac{1}{3}(ba)^3$$

we can look at different cases:

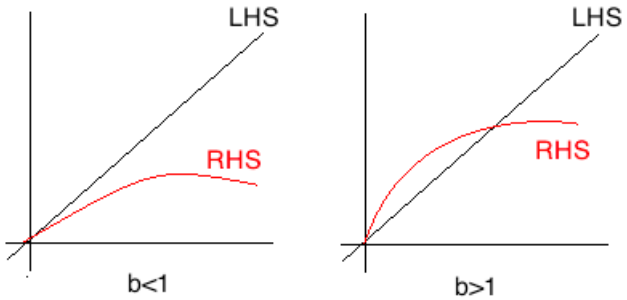


Figure 2: for $b < 1$ the to lines do not cross so there is no solution while for $b > 1$ there is a solution. The value $b = 1 = nJ_0/T_c$ defines the phase transition temperature T_c.

from which we can see that we get a non-zero solution only if $b > 1$. The value of b which separates the two regimes define a critical temperature:

$$b = 1 \Rightarrow kT_c = nJ_0.$$

For small a we can solve giving

$$a = \frac{1}{b}\sqrt{\frac{3(b-1)}{b}} \Rightarrow m = nJ_0\sqrt{3\frac{T_c - T}{T_c}}$$

taking $T \to 0 \Rightarrow b \to \infty$ we get $a \to 1$ which means $m \to nJ_0$ giving the following phase diagram

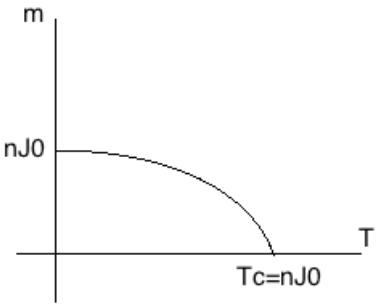


Figure 3: The magnetization as a function of the temperature

2.3 Goldstone Modes (Magnons)

The fact that when continuous symmetry is broken a soft Goldstone mode arises is a general phenomenon. To be explicit we will work on the Heisenberg model and consider the 1D case although similar results hold for higher dimensions.

We would like to find the low excitation of the problem. The ground state is $|G\rangle = |\uparrow\uparrow ... \uparrow\rangle$, i.e. all the spin pointing up.

$$H|G\rangle = -2J\sum_{ij}\vec{s}_i \cdot \vec{s}_j |G\rangle = -2J_0\sum_{\langle ij\rangle} s_i^z s_j^z |G\rangle - J_0\sum_{\langle ij\rangle} s_i^+ s_j^- + s_i^- s_j^+ |G\rangle = -2J_0 N s^2 |G\rangle = E_G |G\rangle .$$

Notice that the term S_i^+ destroy the state as the spin is already in its maximal value and in the last equation the pre-factor is correct only in one dimension as each spin is interact with its left and right spin but we have to avoid double counting.

In looking for the energy of the excited states it is reasonable to assume that the first excited steaks will be build of linear coherent combination of a single spin flip.

Denoting the single spin states:$|i\rangle = |\uparrow\uparrow ... \uparrow \underbrace{\downarrow}_{i^{\text{th}}\text{ position}} \uparrow\uparrow\rangle$ and projecting on these single spin states is done by writing

$$H = \sum_{ij} |i\rangle \langle i| H |j\rangle \langle j| .$$

The term diagonal term $\langle i| H |i\rangle = E_G + 2JS^2$ as two bonds are flipped (we assume that the spins are not at the edge of the sample). Due to the definition of the Heisenberg model the of diagonal terms exist only for nearest neighbor and given by$\langle i| H |i+1\rangle = JS^2$ combing these we have (for the 1D case):

$$H = \left(E_G + 2JS^2\right)\sum_i |i\rangle \langle i| - JS^2\sum_i \left(|i+1\rangle \langle i| + |i\rangle \langle i+1|\right)$$

which can be solved by a fourier transform: $|q\rangle = \sum_j e^{iqaj} |j\rangle$, with a being the lattice constant. It diagonalize the Hamiltonian and gives a spectrum :

$$E_q = E_G + 2JS^2\left[1-\cos(qa)\right].$$

Taking $q \to 0$ the excited state energies are

$$E_q - E_G \to Ja^2q^2.$$

These are the soft modes known as the Goldstone modes - their energy goes to 0 as the wave number goes to 0.

Generally the Goldstone modes are linear in q and not quadratic in q as in the ferromagnetic case. The ferromagnetic case as the order parameter $\vec{M} \propto \vec{S}^{\text{tot}} = \sum_\alpha \vec{s}_\alpha$ commutes with the Hamiltonian. Indeed:

$$\sum_{\alpha\beta\gamma}\left[S^i_\alpha S^i_\beta, S^k_\gamma\right] = \sum_{\alpha\beta\gamma} S^i_\alpha\left[S^i_\beta, S^k_\gamma\right] + \left[S^i_\alpha, S^k_\gamma\right] S^i_\beta = \sum_{\alpha\beta\gamma} i\epsilon_{ikj}\left(S^i_\alpha S^j_\gamma \delta_{\beta\gamma} + S^i_\alpha S^j_\beta \delta_{\alpha\gamma}\right) = \vec{S}^{\text{tot}} \times \vec{S}^{\text{tot}} = 0.$$

Indeed writing the equation of motion in real space we have

$$\int dx\dot{m} = D\nabla^2 m$$

we find that an integration gives a conserved total magnetization

$$\dot{M} = D\oint \nabla m = 0.$$

(Remark: the situation is similar to the diffusion equation of particles that is quadratic in q implying that the total number of particle is conserved04/17/13

2.3.1 Holstein Primakoff

We seek a semi-classical approximation for the Heisenberg model. Considering $\Delta x \Delta p = \langle [x,p] \rangle \sim \hbar$, the spin fluctuations obey

$$\Delta s_i \Delta s_j = \langle [s_i, s_j] \rangle = |\varepsilon_{ijk} \langle s_k \rangle| \leq s$$

$$\frac{\Delta s_i}{s}\frac{\Delta s_j}{s} \leq \frac{1}{s} \underset{s\to\infty}{\to} 0$$

hence quantum fluctuations become negligible when we increase the spin size. Adding a lower index to denote lattice site,

$$[s^k_m, s^l_n] = i\varepsilon_{klj} s^j_n \delta_{mn}$$

and defining $s^\pm_m = s^x_m \pm i s^y_m$ we have

$$[s^+_m, s^-_n] = 2\delta_{mn} s^z_m, [s^z_m, s^\pm_n] = \pm\delta_{nm} s^\pm_m$$

All this has been exact. Defining

$$s^-_m = a^\dagger_m \sqrt{2s - a^\dagger_m a_m}, \quad s^+_m = \sqrt{2s - a^\dagger_m a_m} a_m, \quad s^z_m = s - a^\dagger_m a_m$$

where a_m are bosonic ladder operators, we can check that the new operators obey the same commutation relations. The HP approximation is taking the limit $s \to \infty$ (where s is the spin in each lattice) giving

$$s_m^- \approx \sqrt{2s}a_m^\dagger, s_m^+ \approx \sqrt{2s}a_m$$

in 1D with periodic BC we have

$$H = -2J\sum_m \left(s_m^z s_{m+1}^z + \frac{1}{2}\left(s_m^+ s_{m+1}^- + s_m^- s_{m+1}^+ \right) \right)$$

$$\underset{s\to\infty}{\to} -2JNs^2 - 2Js\sum_m \left(-2a_m^\dagger a_m + a_m^\dagger a_{m+1} + h.c. \right)$$

Using periodic BC $s_{m+N} = s_m \Rightarrow a_{m+N}^\dagger = a_m$ and moving to Fourier space with $a_m = \frac{1}{\sqrt{N}}\sum e^{-ikm}a_k$ giving

$$\hbar\omega_k = 4Js\left(1 - cosk\right) \underset{k\to 0}{\to} 2Jsk^2$$

giving the same dispersion relation we found before in the mean field approach.

2.4 Absence of LRO (Long Range Order) in 1D and 2D with broken continues symmetry - Mermin-Wagner-Berizinskii Theorem

2.4.1 Average magnetization

A series problem appears in the calculation of the average magnetization at finite temperature The deviation from the magnetization at zero temperature where all spin points in the same direction is:

$$\frac{\Delta m}{2Jns} = \frac{1}{N}\left\langle s_{tot}^z \right\rangle - s = -\frac{1}{N}\sum_k n_k$$

where n_k are the number operators, $n_k = \frac{1}{e^{-\omega_k/T}-1}$.

To perform the summation we introduce an IR cutoff $k_0 \sim \frac{1}{L}$ for system size L and assuming $\hbar\omega_{\tilde{k}} < T < Js$ we expand the exponent and set the expression for $\omega_{\tilde{k}}$ that we found

$$\Delta m = -\int_{k_0}^{\tilde{k}} \frac{dkk^{d-1}}{(2\pi)^d}\frac{T}{2Jsk^2} - \frac{1}{N}\sum_{k>\tilde{k}} n_k$$

There is a clear dependence on dimensionality, in one and two dimensions the first term diverges:

$$\Delta m \propto \begin{cases} -\frac{T}{2Js}\frac{1}{k_0} & 1D \\ -\frac{T}{s}\log\left(\frac{\tilde{k}}{k_0}\right) & 2D \end{cases}$$

in 3D we use $\frac{1}{e^{-\omega_k/T}-1} = \sum_{n=1}^{\infty} e^{-n\omega_k/T}$ in things seems to be fine.

$$\Delta m = -\int_{k_0}^{\tilde{k}} \frac{dkk^2}{(2\pi)^3}\sum_{n=1}^{\infty}\exp\left(-\frac{2nk^2Js}{T}\right) \approx -\frac{1}{8}\left(\frac{T}{2Js\pi}\right)^{3/2}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}} = -\frac{1}{8}\left(\frac{T}{2Js\pi}\right)^{3/2}\zeta(\frac{3}{2})$$

2.4.2 Mermin Wagner theorem and Quantum fluctuations

An alternative way to obtain the Mermin Wagner theorem is to consider the following problem. Imagine we invert a spin at time 0 and site 0, calculate the probability for a spin flip at time t and a distance x from site 0. This will give a measure of the correlation of two spins at distance n. In the presence of long range order we expect that the correlation function approaches a constant in an exponential fassion. In the absence of long range order it diverges. To see that formally we note following the tutorial that we can write the correlation $C(x,t) = \langle s^+(x,\tau = it)s^-(0,0) - s^+(0,0)s^-(0,0)\rangle$ in the form:

$$\int d\tau \sum_{\omega_n,\omega_{n'}} \sum_{k,k'} (e^{ikx - i\omega_n \tau} - 1) \left\langle a_{k\omega_n} a^\dagger_{\omega_{n'},k'} \right\rangle = \int d\tau \sum_{\omega_n} \sum_{k} (e^{ikn - i\omega_n \tau} - 1) G(k, i\omega_n) = \int d\tau \sum_{\omega_n} \sum_{k} \frac{e^{ikx - i\omega_n \tau} - 1}{-i\omega_n - \xi_k}$$

We use the functional form of the propagator

$$G(k,i\omega_n)\delta_{k,k'}\delta_{\omega_n,\omega_{n'}} = \left\langle \bar{\psi}(k',\omega'_n)\psi(k,\omega_n)\right\rangle = \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \bar{\psi}_{k'\omega_{n'}}\psi_{k\omega_n} e^{-S(\bar{\psi},\psi)}}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S(\bar{\psi},\psi)}} = \frac{1}{-i\omega_n + \varepsilon_k}\delta_{k,k'}\delta_{\omega_n,\omega_{n'}}$$

from which we can find the spectrum by analytic continuation, where we get poles at energy eigenvalues:

$$G^r(k,\omega) = G(k, i\omega_n \to \omega + i\delta) = \frac{1}{\omega - \varepsilon_k + i\delta}$$

$$Im\,(G^r) = \pi\delta(\omega - \varepsilon_k)$$

a Finite temperature When the temperature is large (what is the meaning of large depends on the model) the contribution from the term with $\omega_n = 0$ will be significantly larger than the other frequencies. That case corresponds to the the "classical limit" where thermal fluctuations are much more important than quantum fluctuations. In this case we get for a massive case:

$$C(x,t) \propto \int_0^{k_c} d^k \frac{(e^{ikx} - 1)k^{d-1}}{k^2 + m^2} = \begin{cases} \pi/m(e^{-mx} - 1) & 1d \\ \log[k_c \max(m, 1/x)] & 2d \end{cases}$$

which diverges in the absence of mass in 1 and 2D leading to absence of long range order.

In case of anti-ferromagnetic (not shown yet) $G = 1/(\omega^2 + k^2)$ leading to the same divergence in the classical limit.

b Zero temperature At zero temperature there are no thermal fluctuations any more, one then has to consider the (usually weaker) quantum fluctuations (due zero point motion) formally we find (in the mass less case) for the 1+1 case

$$C_{AF}(x,t) \propto \int^{k_c} d\omega dk \frac{e^{i(kx - \omega t)} - 1}{\omega^2 + (vk)^2} = 1/2 \log\left[k_c^2(x^2 + (vt)^2)\right].$$

For the ferromagnetic case the integral converges in 1D. Due to diffusive nature of the dispersion (arising from the conservation law) quantum fluctuations are too weak to cause distraction of long range order.

3 Superfluid (Based on Ref. [1])

3.1 Symmetry (Global Gauge symmetry)

We consider a model described by

$$H = \int dr a^{\dagger}(r) \left(\frac{p^2}{2m} - \mu \right) a(r) + \frac{u}{2} \left(a^{\dagger}(r) a(r) \right)^2$$

and we write the partition function as an auxiliary field path integral:

$$Z = Tr\left(e^{-\beta H} \right) = \int D\psi D\bar{\psi}^{-S\left(\psi,\bar{\psi}\right)}$$

where $S(\bar{\psi},\psi) = \int_0^{\beta} d\tau \int d^3 r \left[\bar{\psi}(r,\tau) \left(\partial_\tau - \frac{1}{2m}\nabla^2 - \mu \right) \psi(r,\tau) + \frac{u}{2} \left(\bar{\psi}(r,\tau)\psi(r,\tau) \right)^2 \right]$

$$\psi \left| \psi \right\rangle = a \left| \psi \right\rangle, \left| \psi \right\rangle = e^{-\sum \psi a^{\dagger}} \left| 0 \right\rangle$$

we have a **global** U(1) symmetry $\psi \to e^{-i\varphi}\psi$.

From Noether's theorem we have a conserved current

$$J_\mu = \frac{\delta S}{\delta\left(\partial_\mu \bar{\psi}\right)} \frac{\partial \bar{\psi}}{\partial \varphi} + \frac{\delta S}{\delta\left(\partial_\mu \psi\right)} \frac{\partial \psi}{\partial \varphi}$$

giving (using the relation $\partial\bar{\psi}/\partial\phi = i\bar{\psi}$

$$J_i = -\frac{1}{2mi} \left(\bar{\psi} \nabla_i \psi - \psi \nabla_i \bar{\psi} \right)$$

and the conserved charge is an integral over

$$J_0 = \frac{\delta S}{\delta \dot{\bar{\psi}}} i\psi = \bar{\psi}\psi = \rho.$$

Noether theorem itself gives the continuity equation $\partial_\mu j^\mu = \dot{\rho} - \vec{\nabla} \cdot \vec{J} = 0$ the number of particles is conserved.

3.2 The Bose-Einstein condensation

We consider first the case $u = 0$ so that $H \to H_0 = \int dr a^{\dagger}(r) \left(\frac{p^2}{2m} - \mu \right) a(r)$.

Matsubara frequencies
To represent the free partittion function it is useful to use the imaginary (Matsubara) frequency using the relation:

$$\psi(\tau, r) = \frac{1}{\sqrt{\beta}} \sum_n e^{-i\omega_n \tau} \psi_n(r), \quad \psi_n(r) = \frac{1}{\sqrt{\beta}} \int d\tau e^{i\omega_n \tau} \psi(r,\tau)$$

with

$$\bar{\psi}(\tau, r) = \frac{1}{\sqrt{\beta}} \sum_n e^{i\omega_n \tau} \bar{\psi}_n(r), \quad \bar{\psi}_n(r) = \frac{1}{\sqrt{\beta}} \int d\tau e^{i\omega_n \tau} \bar{\psi}(r,\tau)$$

$$\omega_n = 2\pi n T \text{ for bosons }, \ \omega_n = 2\pi \left(n + \frac{1}{2} \right) T \text{ for fermions}$$

insuring periodic and anti-periodic boundary conditions in $\tau$for boson and fermion respectfully.

Using the Matsubara presebtation for $\psi(r,\tau)$ and the relation $\int_0^\beta e^{-i\omega_n\tau}d\tau = \beta\delta_{\omega_n,0}$ we have $\int\limits_0^\beta d\tau\bar{\psi}(r,\tau)\partial_\tau\psi(r,\tau) = \sum\limits_{\omega_n=2\pi\tau n}\bar{\psi}_n(r)(-i\omega_n)\psi_n(r)$. If we further diagonalize the Hamiltonian and develop the field interms of the eigenfunctionϕ_k,i.e., $\psi_n(r) = \sum_k \phi_k\psi_{kn}$ we can write the partition function$Z$ in terms of the Fourier field ψ_{kn}

$$Z = \int D\bar{\psi}D\psi\exp\left[-S(\bar{\psi},\psi)\right] = \int\prod_{nk}d\bar{\psi}_{kn}d\psi_{kn}\exp\left(-\beta\sum_{kn}\bar{\psi}_{kn}\left(-i\omega_n+\xi_k\right)\psi_{kn}\right) = \prod_k\prod_n\frac{1}{\beta(-i\omega_n+\xi_k)}$$

with $\xi_k = \frac{k^2}{2m} - \mu = \epsilon_k - \mu$. Stability of these integrals demands $\mu \leq 0$. Using the thermodinamic realtion for $N(\mu)$ we find

$$N(\mu) = -T\frac{\partial}{\partial\mu}\log Z = -T\frac{\partial}{\partial\mu}\sum_{n,k}\log\left(\frac{-i\omega_n+\xi_k}{T}\right) = T\sum_{n,k}\frac{1}{i\omega_n-\varepsilon_k+\mu}.$$

We expect to get $\sum_k \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}$ and we can see that this function has poles at $\beta(\varepsilon_k-\mu) = 2\pi ni$ and in our case we require $\varepsilon_k - \mu = 2\pi Tni$ since $\omega_n = 2\pi nT$ hence the two functions have the same poles. Contour integration with few beatiful tricks (see for example Altland Simons page 170) show that shows that indeed:

$$T\sum_n\frac{1}{i\omega_n-\varepsilon_k+\mu} = \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1} \equiv n_B(\xi_k).$$

For a fixed external number of particls N the eqution

$$N \equiv N\left[\mu(T)\right]$$

is an equation for the chemical potential as a function of the temperature T. In three dimensions using

$$N(\mu) = \sum_k n_B(\xi_k) = \Omega\frac{1}{\Omega}\sum_k n_B(\xi_k) = \Omega\frac{1}{(2\pi)^3}\int d^3k n_B(\xi_k) = \Omega\frac{1}{2\pi^2}\int dkk^2 n_B(\xi_k) = \frac{\Omega\mathrm{Li}_{\frac{3}{2}}\left(e^{\beta\mu}\right)}{2\sqrt{2}\pi^{3/2}(\beta m)^{3/2}}$$

with Ω being the system volume. Importantly to satisfy the equation $N(\mu) = N$ we must increase μ as we lower the temperature until at $\mu = 0$. We have

$$N(0) = \Omega c\frac{1}{\lambda_T^3}$$

with

$$\lambda_T = \frac{1}{\sqrt{mT}}$$

known as the particle thermal length and a numerical factor $c = \frac{\zeta\left(\frac{3}{2}\right)}{(2\pi)^{3/2}} = 0.165869$ at the temperature for which $N(0) = \Omega c\frac{1}{\lambda_T^3} = N$ a macroscopic number of particle N_0 must occupy the ground state. Notice that it occurs at the temperature for which the avergae distance between the particles $1/n^{1/3} = (\Omega/N)^{1/3} \approx 0.54\lambda_T$ is equal to the thermal length. This gives

$$T_c = \frac{(cn)^{2/3}}{m} = 0.3\frac{n^{2/3}}{m}.$$

T_c increases for dancer and lighter particles!

Below T_c the we can write teh action in terms of the macroscopic wave filed ψ_0 and we identify the number of particles in the ground state

$$N_0 = \bar{\psi}_0\psi_0$$

then

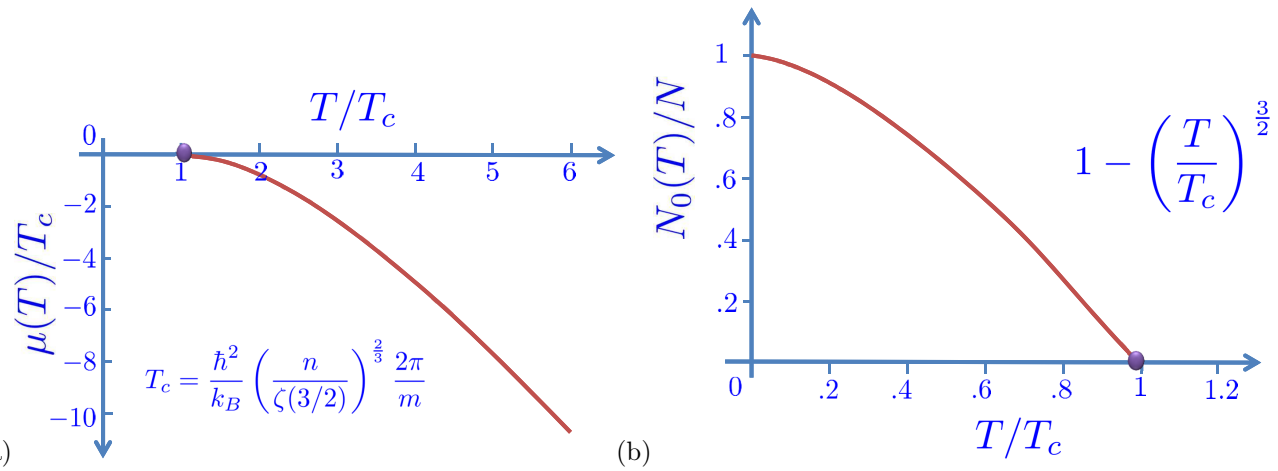


Figure 4: (a)The chemical potentia μas a function of the temperature T. (b) The number of bosons in the ground state.

$$S = \bar{\psi}_0 \beta\mu\psi_0 + \sum_{k\neq 0} \bar{\psi}_{kn}\left(-i\omega_n + \varepsilon_k - \mu\right)\psi_{kn}.$$

Notice that the imaginary time derivative $\partial_\tau \to -i\omega_n$does not appear in the first term, this is already an appromixtion. In the formulation of the path integral the derivative in imaginary time appeard due to the commutation relation of the operators, neglecting them meaning that we ommit quantum effects. In our case it justified to perform this semiclassical approximation for teh operator $a_0$since $a_0^\dagger a_0 \approx N \gg 1$while the commutation relation are $\left[a_0^\dagger, a_0\right] = 1$.
So that in the first term we take into considearation only the thermal fluctuiations (zero Matsubara frequency) and in the second we include also the quantum terms. (We note that the Ginsburg-Landau theory infact neglect the quantum fluctuations)

Using the action S we can write the particle number as

$$N = -\partial_\mu F = \bar{\psi}_0\psi_0 + T\sum_{nk}\frac{1}{i\omega_n - \varepsilon_k} = \bar{\psi}_0\psi_0 + \frac{\Omega}{(2\pi)^3}\int d^3k \frac{1}{e^{\beta k^2/2m}-1}$$

$$= \bar{\psi}_0\psi_0 + \left(\frac{mT}{2\pi}\right)^{3/2}\zeta\left(\frac{3}{2}\right) = N_0 + N\left(\frac{T}{T_c}\right)^{3/2}$$

which give in 3D

$$\frac{N_0}{N} = \left(\frac{T_c - T}{T_c}\right)^{3/2}$$

3.3 Weakly Interacting Bose Gas

3.3.1 Mean Field Solution

We look at the action of the wave function ψ_0 which describes a classical part of the wave function of the BEC.

$$TS(\bar{\psi}_0, \psi_0) = -\mu\bar{\psi}_0\psi_0 + \frac{g}{2L^d}\left(\bar{\psi}_0\psi_0\right)^2.$$

Notice that we take into consideration only the classical part of ψ (we assume that it does not depend on τ) this is justified because $N_0 = \bar{\psi}_0\psi_0 \gg 1$ and the commutation relation of the correspond operators is $\left[a^\dagger, a\right] = 1$.

The partition function is $Z = \int d\bar{\psi}d\psi_0 e^{-TS(\bar{\psi}_0\psi_0)}$ for $T \ll \mu$ we evaluate Z via saddle point approximation

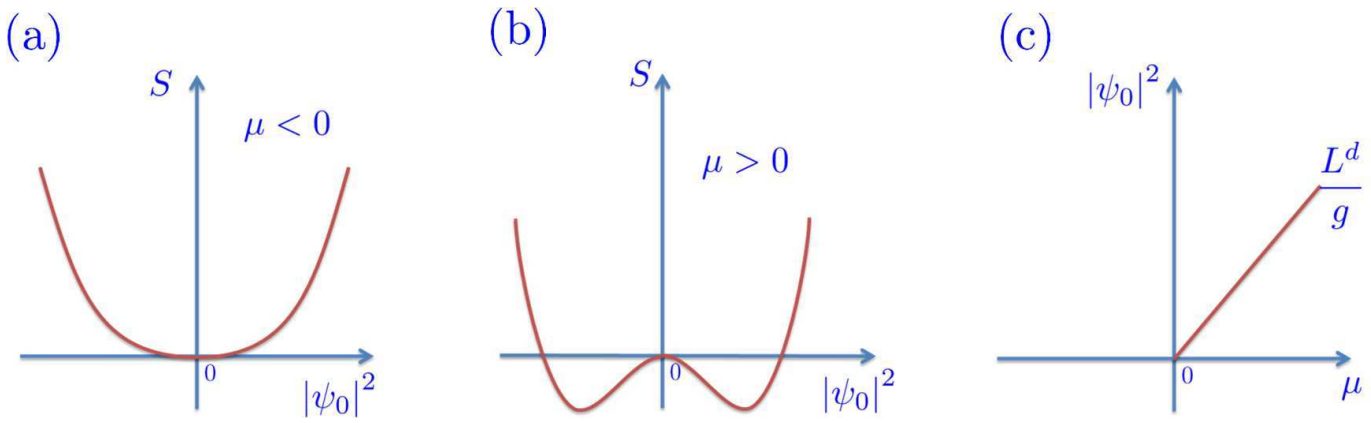


Figure 5: (a)The action S for $\mu < 0$ at the minimum $|\psi_0|^2 = 0$. (b) The action S for $\mu > 0$. (c) The number of boson in the condensate $N_0 = |\psi_0|^2$.

$$\frac{\delta S}{\delta \psi_0} = 0 \Rightarrow \bar{\psi}_0 \left(-\mu + \frac{g}{L^d} \left(\bar{\psi}_0 \psi_0 \right) \right) = 0.$$

$$|\psi_0| = \sqrt{\frac{\mu L^d}{g}}$$

Notice that the total number of particle in the condensate $\bar{\psi}_0 \psi_0$ is proportional to the volume L^d as it should be. Due to the interaction term g we no longer have a condition on the sign of μ, and we get the transition at $\mu = 0$. In the interacting case the number of the particles in the condensate is proportional to μ

3.3.2 Goldstone Modes

We will find now the goldstone modes of the system, to do so we define the average condensate density

$$\rho_0 = \frac{\bar{\psi}_0 \psi_0}{L^d}$$

so that the field ψ_0 is given by:

$$\psi = \sqrt{\rho_0 + \rho(r,t)} e^{i\phi(r,t)}.$$

The amplitude fluctuations are given by $\rho(r,t)$ and the phase of the condensate $\phi(r,t)$.

Substituting in the action we find:

$$S = \int d\tau d^d r \bar{\psi} \left[\left(\partial_\tau - \frac{\nabla^2}{2m} \right) \psi + \frac{1}{2} g \left| \psi \right|^2 \left(\left| \psi \right|^2 - 2\rho_0 \right) \right]$$

this is equivalent to the action discussed above with $\rho_0 g = \mu$, where we added terms which depends on derivatives with respect to time and space to account for the fluctuations. assuming now that $\rho(r,t) \ll \rho_0$ and expand the action we have

$$S \approx \int_0^\beta d\tau d^d r \left[i\rho_0 \dot{\phi} + \underline{i\rho\dot{\phi}} + \frac{\rho_0}{2m} \left(\phi' \right)^2 + \frac{\rho'^2}{8m\rho_0} + \frac{u}{2} \rho^2 \right] + O(\rho^3, (\nabla\phi)^3)$$

In analogy with the lagrangian for a single particle $L = p\dot{q} - H$ with p and q conjugate variables (i.e. quantum mechanically they do not commute) $[p,q] = i$, the underlined terms in S helps us identify ϕ and ρ as conjugate variables, giving

$$[\phi, \rho] = \delta(x - x').$$

We want to integrate over the ρ part to obtain an effective action for ϕ.

For one variable we have

$$S = \int d\tau \left(ip\partial_\tau q - \frac{p^2}{2m} - V(q) \right)$$

integrating over p gives

$$\int d\tau \frac{m \left(\partial_\tau q\right)^2}{2} - V(q).$$

In our case we perform a Fourier transform

$$S = \int_0^\beta d\tau \frac{d^d k}{(2\pi)^d} \left[i\rho_0 \dot{\phi}_0 + i\rho_{-k}\dot{\phi}_k + \frac{\rho_0}{2m} k^2 \phi_k \phi_{-k} + \frac{1}{2g\left(1 + k^2\xi^2\right)} \dot{\phi}_k \dot{\phi}_{-k} \right]$$

$$\xi = 4mg\rho_0$$

and for

$$k\xi \ll 1$$

transforming back gives

$$S = \int_0^\beta d\tau d^d r \left[i\rho_0 \partial_\tau \phi + \frac{1}{2u} \left(\partial_\tau \phi\right)^2 + \frac{\rho_0}{2m} \left(\nabla \phi\right)^2 \right] \quad \text{(valid for } r \gg \xi\text{)}.$$

This is an XY model. The name XY is originated from the fact that the order parameter ψ is a complex function that presented by a real and imaginary part i.e. "live" in an XY plane.

Examine the action we obtain it seems like the same quadratic action we always have, with one important difference:ϕ is compact: $\phi + 2\pi = \phi$. This means that the equations of motion derived from S can be solved by vortices.

A common definition is the superfluid density:

$$\rho_s = \frac{\rho_0}{m}$$

which determines the energetic cost of deforming the condensate phase in space, and a compressibility

$$\kappa = \frac{1}{g}$$

which determines the cost of phase changes in time. Transforming to q, ω space

$$S = \frac{1}{2} \sum_{q,\omega} \left(\kappa\omega_q^2 + \rho_s q^2 \right) \phi_{q\omega} \phi_{-q-\omega}$$

$$\omega_q = cq \Rightarrow c = \sqrt{\rho_s/\kappa} = \sqrt{\frac{\rho_0 g}{m}}.$$

Reminder: we assume $|r| \gg \xi$, meaning we have performed coarse graining.

3.4 Superfluidity

To discuss superfluidity let us add an external chemical potential μ_{ext} then the Hamiltonian

$$H = \frac{1}{2}\int dx \left[\rho_s \left(\nabla\phi\right)^2 + \frac{1}{\kappa}\rho^2 - \mu_{ex}\rho\right]$$

we should understand now ρ and ϕ as operators

$$[\phi(x), \rho(x')] = \delta(x - x')$$

the current operator is

$$J = \frac{1}{2mi}\left(\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}\right) = \frac{\rho(r,t)}{m}\nabla\phi \approx \frac{\rho_0}{m}\nabla\phi = \rho_s\nabla\phi$$

The Hamilton equations are

$$\frac{\partial\rho}{\partial t} = -\frac{\partial H}{\partial\phi} = -\rho_s\nabla^2\phi = -\nabla\cdot J$$

which is the expected continuity equation. The more interesting equation is

$$\frac{\partial\phi}{\partial t} = \frac{\partial H}{\partial\rho} = \mu_{ex} - u\rho \equiv \mu(r,t).$$

This is one of the Josephson relations - the time derivative of the phase depends linearly on an external potential.

To see how this system exhibits superfluidity let assume that $\phi = qx$ then we have

$$J = \frac{\rho_0}{m}\nabla\phi = q\frac{\rho_0}{m}\widehat{x}$$

and the phase vector in x space looks like this:

x

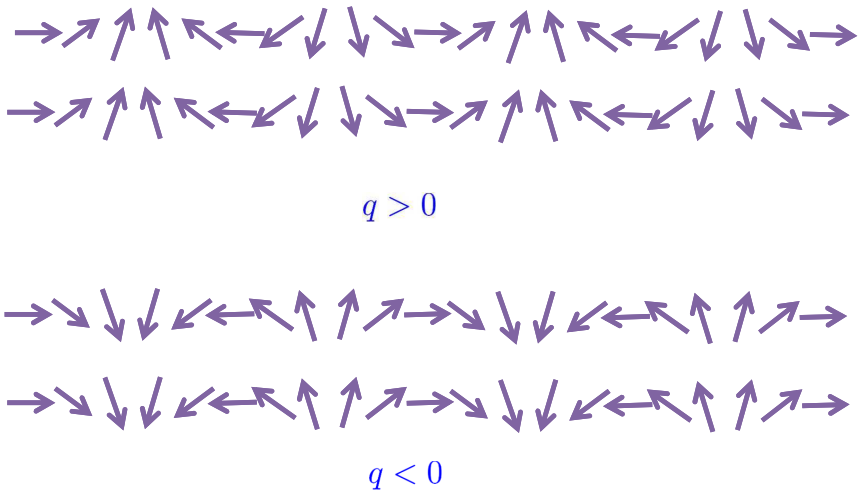


Figure 6: Phase evolution in the presence of current

The surprising thing is that such a current is stable, since we have Goldstone modes (low energy excitations) in the system we could have expected that such a current would excite them.

3.5 Landau's Argument

For a fluid moving uniformly in the lab frame the system without excitations has energy $E_0 = \frac{1}{2}mv^2$. In case we have excitations they will move with the system. Moving to the center of mass frame there is no kinetic energy, and the entire energy of the system is the energy of the fluctuations $\varepsilon(p) = c \cdot p$. This means that in the lab frame

$$E_1 = \frac{1}{2}mv^2 + \vec{p} \cdot \vec{v} + \epsilon(p)$$

(the $\vec{p} \cdot \vec{v}$ term is from moving the fluctuation, just from moving something with momentum p with a velocity v)

If there was dissipation as a result of this fluctuation, we would have $E_1 - E_0 < 0 \Rightarrow \vec{p} \cdot \vec{v} + \varepsilon(p) = \vec{p} \cdot \vec{v} + \vec{c} \cdot \vec{p} < 0$ which requires $|\bar{v}| > c$. This means that for low speeds this doesn't hold and there is no dissipation. The fact that we have a linear dispersion relation means we don't always excite the system. Usually we have a quadratic dispersion relation which means we can always excite the system.

3.6 Various Consequences

3.6.1 Quantization of Circulation

Considering

$$J = \frac{\rho_0}{m}\nabla\phi$$

we integrate over a closed path

$$\oint \nabla\phi = \phi(L) - \phi(0) = \phi(2\pi) - \phi(0)$$

and since ϕ is compact ($\psi = \rho e^{i\phi}$) we require

$$\oint \nabla\phi = 2\pi n$$

$$\Rightarrow \int v \cdot dl = \int \frac{J}{\rho_0} \cdot dl = \frac{\hbar}{m} 2\pi n = \frac{hn}{m}$$

where we returned $\hbar$. We have a quantization of superfluid velocity.

3.6.2 Irrotational flow

$$\nabla \times J = \rho_s \nabla \times \nabla\phi = 0$$

3.6.3 Vortexes = Vortices

Defining

$$\phi = n\theta, n \in Z.$$

where θ is an angle in real space. the fact that n is an integer guaranties that after a rotation in real space the field $\psi \propto e^{i\phi}$ is single valued.

Explicitly we have:

$$\phi = n \arctan(\frac{x}{y}).$$

and the current is given by:

$$J = \rho_s \nabla \phi = \rho_s \frac{n}{r} \hat{\theta}.$$

(it is easy to show that $\vec{\nabla} \times \vec{\nabla} \phi = \delta(r)$)

This seems to diverge for small r but we have a cutoff $r > \xi$.

$$v = \frac{\hbar}{m} \frac{n}{r} \hat{\theta}.$$

The proportionality of the velocity field to $1/r$ is very different from a rigid rotation where

$$v = \omega r \widehat{\theta}, \nabla \times v = \omega \widehat{z}$$

4 Superconductivity

4.1 Basic Model and Mean field solution

Exercise number 4.

4.2 The Anderson -Higgs Mechanism

4.2.1 Local gauge symmetry

The essential difference from the case of the superfluid is that the particles are charged so that we should add coupling to the electromagnetic filed, this is done by the "minimal coupling"

$$S = \int \bar{\psi} \frac{\left(\nabla - \frac{ie}{\hbar} A\right)^2}{2m} \psi$$

this changes the symmetry from a global U(1) to local gauge invariance under U(1)

$$\psi \to \psi e^{i\phi(x)},\ A \to A - \nabla\phi(x)$$

and in the polar representation ($e = \hbar = c = 1$)

$$S = \int \sqrt{\rho} e^{-i\phi} \left(\nabla - iA\right) \left(\nabla - ieA/c\right) e^{i\phi} \sqrt{\rho} = \int \sqrt{\rho} e^{-i\phi} \left(\nabla - iA\right) e^{i\phi} \left(\frac{\partial \sqrt{\rho}}{\partial x} + i\sqrt{\rho} \frac{\partial \phi}{\partial x} - iA\sqrt{\rho}\right)$$

(ignoring the fluctuations in $\sqrt{\rho}$)

$$= \int \frac{\rho_0}{2m} \left(\nabla\phi - A\right)^2 .$$

Following steps similar to the ones we performed for the neutral superfluid we will get after integrating out the massive fluctuations of the amplitude ρrestroing the universal factor $e, c, \hbar$ we get:

$$S = \frac{1}{2} \int d\tau \int d^3 r \frac{\left(\partial_\tau \phi\right)^2}{u} + \frac{\rho_0}{m} \left(\nabla\phi - \frac{e}{c\hbar} A\right)^2 .$$

When Quantum fluctuations can be ignored?

To study the Anderson Mechanism we will ignore quantum fluctuations – when is that justified? The GL action is

$$S[\phi, A] = \frac{1}{2}\int d\tau d^3 r \frac{1}{u}\left(\partial_\tau \phi - eA_0\right)^2 + \left(\nabla A_0\right)^2 + \frac{\rho_0}{m}\left(\nabla\phi - A\right) + \left(\nabla \times A\right)^2$$

Looking at the terms which contain A_0 in fourier space, we have

$$\frac{1}{u}\omega^2\phi^2 + \underbrace{\frac{2}{u}e\omega\phi}_{a} A_0 + \underbrace{\left(\frac{e^2}{u} + q^2\right)}_{b} A_0^2$$

integrating over A_0 gives

$$\left[\frac{1}{u}\omega^2 - \underbrace{\frac{\left(\frac{1}{u}e\omega\right)^2}{\left(\frac{e^2}{u}+q^2\right)}}_{a^2/4b}\right]\phi^2 = \frac{\omega^2 q^2}{\frac{e^2}{u}+q^2}$$

Assuming no magnetic field and minimizing the free energy of the full action we have

$$\frac{\omega^2 q^2}{\frac{e^2}{u}+q^2} = \frac{\rho_0}{m}q^2$$

which for small q gives

$$\omega^2 = \frac{\rho_0}{m}\frac{e^2}{u} = \frac{4\pi e^3 n}{m} = \omega_p^2$$

This is the plasma frequency. It is related to the dielectric constant of a material by the following development (see p. 18 of Ref. [2])

$$\frac{d\rho}{dt} = eE, \frac{dj}{dt} = -\frac{e^2 nE}{m};\;\; i\omega j = -\frac{en}{m}E \equiv \sigma(\omega)E \text{ with } \sigma(\omega) = \frac{ie^2 n}{m\omega}$$

using Maxwel's equations we find:

$$-\nabla^2 E = \nabla\times\nabla\times E = i\frac{\omega}{c}\nabla\times H = i\frac{\omega}{c}\left(\frac{4\pi}{c}j - \frac{i\omega}{c}E\right) = \frac{\omega^2}{c^2}\left(1 + \frac{4\pi i\sigma}{\omega}\right)E \equiv \frac{\omega^2}{c^2}\varepsilon(\omega)E \tag{5}$$

and by definition

$$\varepsilon(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

When ε is real and negative $\omega < \omega_p$ the solution of Eq. (5) decay in space, i.e. electric field can not propagate in the material, for $\omega > \omega_p$ radiation can propagate in the metal and it become transparent.
The q^2 term came from a 3D fourier transform of a Coulomb interaction $V(q)$. When the material is confined 2D (and the electric field lines can propagate in 3D) $v(q) \propto q$, giving $\omega \sim \sqrt{q}$. In 1D metal of width a when $V(q) \propto \log qa$ given $\omega \sim q$ and quantum fluctuations can not be ignored.

Taking the classical approximation (no τ dependence) and adding a term for the action of the magnetic field

$$S[A,\phi] = \frac{\beta}{2}\int d^3 r \frac{\rho_0}{m}\left(\nabla\phi - \frac{e}{c\hbar}A\right)^2 + \left(\nabla\times A\right)^2 = \frac{\beta}{2}\int d^3 r \frac{\rho_0}{m}\left(\nabla\phi - \frac{e}{c\hbar}A\right)^2 + \frac{\beta}{2}\int d^3 r \left|B\right|^2.$$

In momentum space

$$S[A,\phi] = \frac{\beta}{2}\sum_q \frac{\rho_0}{m}\left(i\vec{q}\phi_{\vec{q}} - \vec{A}_{\vec{q}}\right)\left(-i\vec{q}\phi_{-\vec{q}} - \vec{A}_{-\vec{q}}\right) + \left(\vec{q}\times\vec{A}_{\vec{q}}\right)\left(-\vec{q}\times\vec{A}_{-\vec{q}}\right)$$

$$= \frac{\beta}{2} \sum_q \frac{\rho_0}{m} \left[q^2 \phi_{\vec{q}} \phi_{-\vec{q}} - 2i\vec{q} \cdot \vec{A}_{-\vec{q}} \phi_{\vec{q}} + \vec{A}_{\vec{q}} \cdot \vec{A}_{-\vec{q}} \right] + \left(\vec{q} \times \vec{A}_{\vec{q}} \right) \left(-\vec{q} \times \vec{A}_{-\vec{q}} \right)$$

We break up $\vec{A}$ into a longditudinal and transverse part: $\vec{A}_{\vec{q}} = \underbrace{\vec{A}_{\vec{q}} - \frac{\vec{q}\left(\vec{q} \cdot \vec{A}_{\vec{q}}\right)}{q^2}}_{A_\perp} + \underbrace{\frac{\vec{q}\left(\vec{q} \cdot \vec{A}_{\vec{q}}\right)}{q^2}}_{A_\parallel}$.

Notice that only the transverse part will contribute to the magnetic field, since $\vec{B}_{\vec{q}} = \vec{q} \times A^\perp_q$(since $\vec{q} \times \vec{q} = 0$). Performing a gaussian integral on the ϕ degrees of freedom $\left(\int e^{-x^2 + yx} \sim e^{-y^2} \right)$we have

$$S[A] = \frac{\beta}{2} \sum_q \frac{\rho_0}{m} \left[\vec{A}_{\vec{q}} \cdot \vec{A}_{-\vec{q}} - \frac{\left(\vec{q} \cdot \bar{A}\right) \left(\vec{q} \cdot \vec{A}\right)}{q^2} \right] + \left(\overline{q} \times \overline{A}_{\overline{q}} \right) \left(-\overline{q} \times \overline{A}_{-\overline{q}} \right).$$

$$= \frac{\beta}{2} \sum_q \left(\frac{\rho_0}{m} + q^2 \right) A^\perp_q A^\perp_{-q}.$$

The equation of motion is

$$\left(\frac{\rho_0}{m} - \nabla^2 \right) A_\perp = 0. \tag{6}$$

The mechanism that we encounter here, namely

1. A symmetry breaking that we find through a mean field solution
2. The appearance of Goldstone soft modes ϕin the superconducting case
3. Coupling betweeen the Goldstone mode and the gauge filed $A$and the Goldstone mode
4. The gauge field acquires a mass

is known as the Anderson Higgs mechanism.

4.3 London Equations (Phenomenology of Superconductivity)

Taking a curl of Eq.(6) we get (The first equation of London)

$$\text{The First London Equation:} \quad \left(\frac{\rho_0}{m} - \nabla^2 \right) B = 0 \tag{7}$$

This is the first London equation. It shows that the field decays inside the superconduction where

$$\lambda = \sqrt{\frac{m}{\rho_0}} = \sqrt{\frac{mc^2}{4\pi n_s e^2}}$$

with n_s representing the density of particles in the superconducting phase.
Since

$$\nabla \times \nabla \times A = \frac{4\pi}{c} j$$

if we choose the London gauge

$$\vec{q} \cdot \vec{A}_{\vec{q}} = 0$$

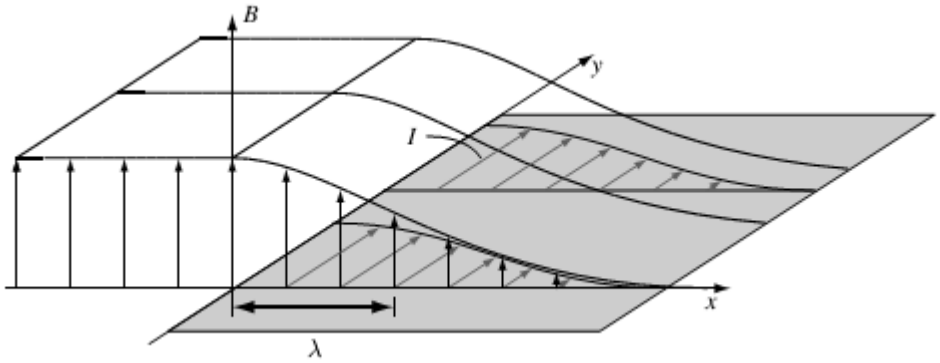


Figure 7: The Meissner effect: due to the Anderson-Higgs mechanism. An external field outside the superconductor induced diamagnetic supercurrent inside the superconductor – these current generates a counter field that diminishes the external field.

we get (since $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot A) - \nabla^2 A$)

$$q \times q \times A_\perp = -q^2 A_\perp$$

$$\nabla^2 A = \frac{4\pi}{c} j$$

But from the first London equation we find that $\nabla^2 A = \rho_0 / mA$ this gives:

$$\text{The second London Equation:} \quad j = \frac{n_s e^2}{mc} A. \tag{8}$$

Eq. (8) is known as the second London equation. It presents a perfect diamagnetism (notice that it is not gauge invariant).

Physically the Meisenner effect consist of current that due the the Biot-Savart low creates magnetic field that cancels the external one, see figure .

4.4 Vortexes in Superconductor

4.4.1 The magnetic field penetration depth

To studey the behavior of magnetic field in area were there is superconductivty we will start from the Ginsburg-Landau theory. The Ginzburg Landau theory can be obtained by introducing a Habburd-Stratanovich field Δthat deacoples the interaction term $g\psi^4 \to \Delta\psi\psi + \Delta^2/g$ the action is then quadratic in ψso we can integrate out the ψfield and expand the action assume that $\Delta$is small we the get (See [7] for details):

$$\mathcal{F}_s = \int f_s d^3x$$

$$f_s = f_n + \alpha |\Psi|^2 + \frac{\beta}{2} |\Psi|^4 + \frac{1}{2m} \left| \left(\frac{\hbar}{i} \nabla - \frac{e^*}{c} A \right) \Psi \right|^2 + \frac{B^2}{8\pi}$$

were we switch the notation $\Delta \to \Psi$here Ψrepresent the pair function. A microscopic theory gives

$$\alpha = \alpha' \frac{T - T_c}{T_c}, \quad \alpha' = \frac{12\pi^2 m T_c^2}{\zeta(3) P_F^2}, \quad \beta = \frac{\alpha'}{n_e}, \quad e^* = 2e, \quad m^* = 2m$$

where n_e is the electron density. f_n describes the electrons which aren't in the superconducting phase. B is the magnetic field. For $\alpha < 0$ we have a minimum at

$$f_s - f_n = \frac{-\alpha^2}{2\beta} + \frac{B^2}{8\pi}$$

We expect that the field will eventually be too strong to be repulsed by the superconductor, and the critical magnetic field is found from the above equation

$$\frac{H_c^2}{8\pi} = \frac{\alpha^2}{2\beta} = \frac{n_e m T_c^2}{P_F^2} = m P_F T_c^2$$

mP_F is the density of states at E_F.

This results should not be suprising as in the nornal state we have electrons near the Fermi level, while in the superconduting state a gap of size Δ is formed the energy of the electrons that "repeled" from the Fermi surface is $\sum_{i,\epsilon_i<\Delta} \epsilon_i \sim \nu\Delta^2$. (See excersize)

We can identify a magnetic length by comparing terms in $\mathcal{F}$

$$\frac{B^2}{8\pi} = \frac{(\nabla \times A)^2}{8\pi} \propto \frac{1}{\lambda^2}\frac{A^2}{8\pi} \sim \frac{e^{*2}}{c^2}\frac{1}{2m^*}\left|\psi^2\right| A^2$$

$$\lambda_{\text{eff}}^2 = \frac{m^* c^2}{4\pi \left|\psi\right|^2 e^{*2}}$$

$$\left|\Psi_\infty\right|^2 = -\frac{\alpha}{\beta} = n_s^* = \frac{n_e}{2} = \frac{m^* c^2}{4\pi e^{*2}} = \frac{m^* c^2}{8\pi e^2 \lambda_{\text{eff}}^2}.$$

Notice that the units in the last equation are right as $\hbar/mc$ has unitis of length and $\hbar c/e^2$is dimless so that $|\Psi_\infty|^2$has units of 1/volume as it should. Finially we can identify

$$\alpha(T) = \frac{2e^2}{mc^2} H_c^2(T)\lambda_{\text{eff}}^2(T).$$

These relations are usefull as H_c and $\lambda$are experimentallty measurable quantitiies even if we do not have a microscopic theory.

05/07/13

4.4.2 The Coherence Length

We discussed the following free energy for the superconductor

$$f_s = f_n + \alpha\left|\Psi\right|^2 + \frac{\beta}{2}\left|\Psi\right|^4 + \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\nabla - \frac{e^*}{c}A\right)\Psi\right|^2 + \frac{B^2}{8\pi}$$

and found the magnetic length

$$\lambda_{\text{eff}}^2 = \frac{m^* c^2}{4\pi \left|\psi\right|^2 e^{*2}}$$

We can also define another length scale - the coherence length.

$$\frac{\delta f}{\delta \psi^*} = 0 \Rightarrow \alpha\psi + \beta\psi^*\psi^2 - \frac{1}{2m^*}\psi''$$

which can be solved by a stationary solution

$$|\psi_\infty|^2 = -\frac{\alpha}{\beta}$$

$$\frac{\psi}{\psi_\infty} - \left(-\frac{\beta}{\alpha}\right)\frac{\psi^*}{\psi_\infty}\psi^2 - \frac{1}{2m^*\alpha}\frac{\psi''}{\psi_\infty}$$

and assuming ψ is real and defining $f = \frac{\psi}{\psi_\infty}$ we have

$$f - f^3 + \frac{\hbar^2}{2m^*|\alpha|}f'' = 0$$

The coefficient of the last term has units of 1/area, which allows us to define a coherence length:

$$\xi^2(T) = \frac{\hbar^2}{2m^*|\alpha|} \propto \frac{T_c}{T_c - T}$$

$$\xi^2(T=0) \propto \frac{1}{m\alpha} \underbrace{\propto}_{\text{microscopic expression for } \alpha} \frac{p_f^2}{m^2 T_c^2} = \frac{v_f^2}{T_c^2} \Rightarrow \xi = \frac{v_f}{T_c}$$

For dirty systems with diffusion we derive a length scale from the diffusion equation which contains the mean free path due to diffusion:

$$\frac{D}{\xi_d^2} = T_c \Rightarrow \xi_d = \sqrt{\frac{D}{T_c}} = \sqrt{\frac{v_p l}{T_c}} = \sqrt{\xi l}$$

We can look at small deviations from ψ_∞ by defining $f = g + 1$ and linearizing the equation

$$(1+g) - (1+g)^3 + \xi^2 g'' = 0$$

$$\xi^2 g'' = 2g$$

$$g \sim e^{\pm\sqrt{2}x/\xi}.$$

4.4.3 Two types of superconductors

The relation between the two length scales defines two types of superconductors.

In type I materials, $\lambda \ll \xi$ we expel the magnetic field which costs energy, but are not yet in the superconducting phase which is beneficial energetically, hence such a boundary costs energy

$$F_b > 0$$

In type II materials, $\lambda \gg \xi$

In this case we obtain the reduced energy of entering the superconducting phase without having to expend much energy on expelling the field. We then have

$$F_b < 0$$

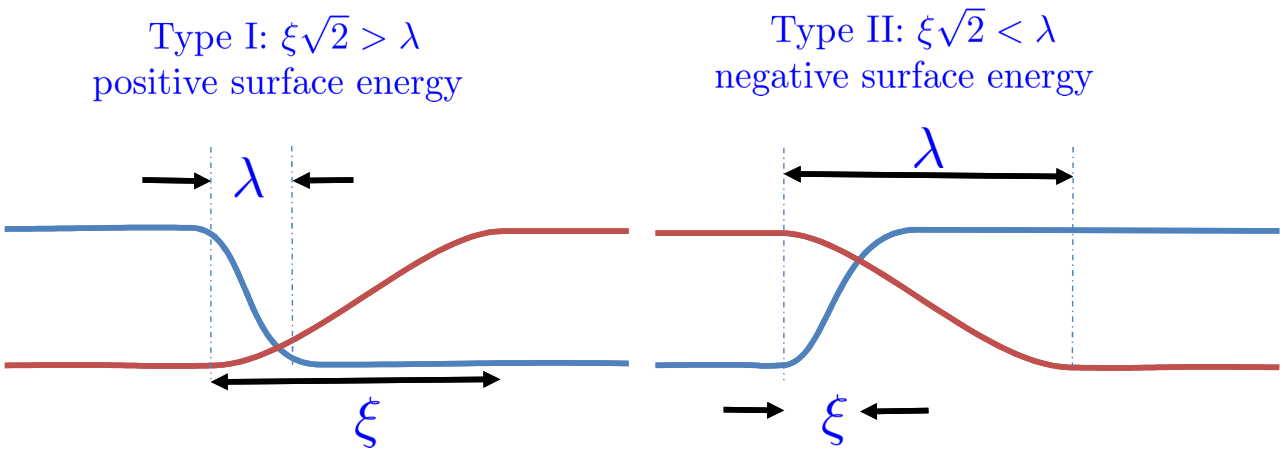


Figure 8: The surface energy is positive when $\lambda < \xi$ we "pay" (the free energy is positive) for repealing magnetic field and still do not gain energy from the superconductor. For $\lambda > \xi$ we have superconductor and still do not repel the magnetic field hence "gain" from both, the surface energy is negative.

and it becomes energetically favorable to increase the boundary as much as possible, which is done by generating vortexes.

The phase diagram for type II looks like this (see Fig 9)

We recall that each vortex carries at least Φ_0 of flux. Hence if the flux through the entire model is less than Φ_0 there can be no vortexes and we have a perfect Meissner effect. The next phase allows creation of vortexes. Eventually the vortexes are so common that they coalesce, leaving the superconducting phase only on the boundary.

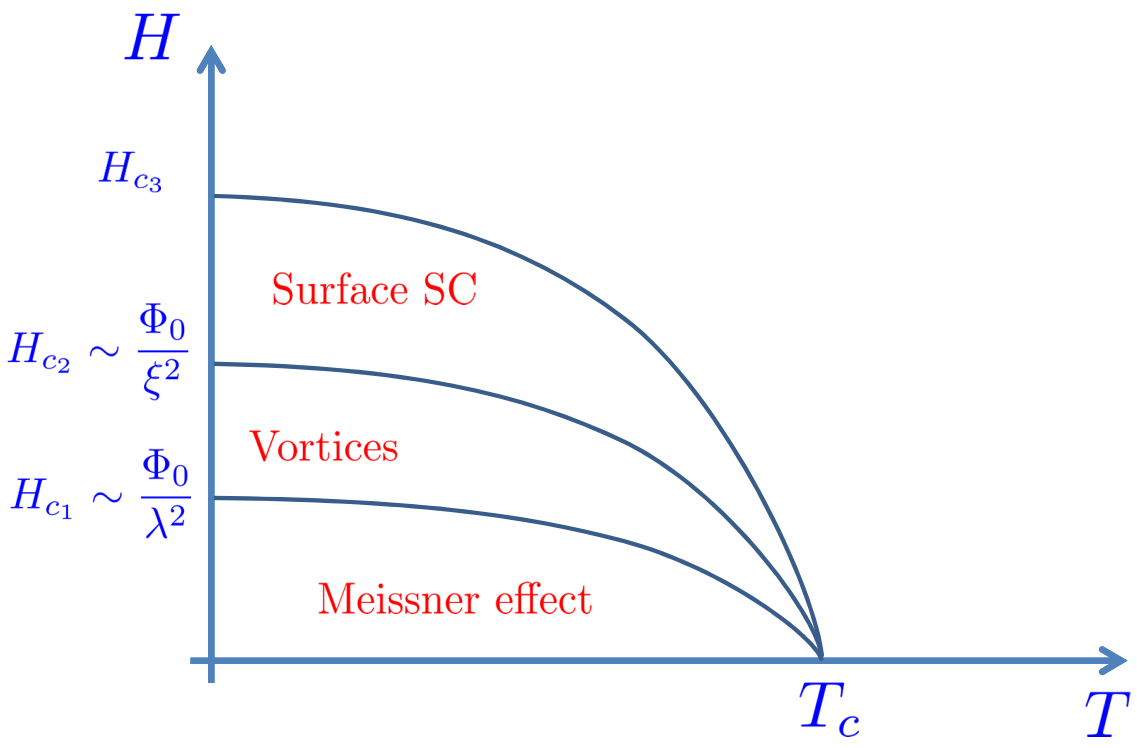


Figure 9: H_{c1} the critical filed when the first vortex penetrates. H_{c2} when the vortices core overlap and H_{c3} when the residue superconductivity on the surface disappears

4.4.4 Vortexes in Type II superconductor

Looking at a type II superconductor near H_{c1} where we have a single vortex. The energ is

$$F = \int d^3r \frac{1}{8\pi} \left[h^2 + \lambda^2 \left(\nabla \times h \right)^2 \right]$$

the last term is the kinetic energy which was derived using $j = en_s v = \nabla \times h \frac{4\pi}{c}$ and $\lambda = \sqrt{\frac{mc^2}{4\pi n_s}}$

$$E_{kin} = \frac{1}{2} m v^2 n_s = \frac{1}{2} \frac{m}{n_s e^2} j^2 = \lambda^2 \left(\nabla \times h \right)^2$$

Taking a variation we derive the equation of motion:

$$h + \lambda^2 \nabla \times \nabla \times h = 0$$

We want to solve for a single vortex hence we add a term to ensure the integration over the surface gives the correct flux.

$$h + \lambda^2 \nabla \times \nabla \times h = \Phi_0 \delta(r)$$

integrating over the surface of the model

$$\int dsh + \lambda^2 \int ds \nabla \times \nabla \times h = \int dsh + \lambda^2 \oint_c dl \nabla \times h = \Phi_0$$

For $r \gg \lambda$, the field is effectively uniform along the integration contour and the second term drops giving

$$\int dsh = \Phi_0$$

while if $\xi < r < \lambda$ the first term is negligible

$$\lambda^2 \oint_c dl \nabla \times h = 2\pi r \lambda^2 \nabla \times h = \Phi_0$$

$$\frac{dh}{dr} = \frac{\Phi_0}{2\pi\lambda^2}\frac{1}{r}$$

$$h = \frac{\Phi_0}{2\pi\lambda^2}\left(\log\left(\frac{\lambda}{r}\right) + const\right)$$

while for $r \gg \lambda$ we have $h \sim e^{-r/\lambda}$.

We can also solve exactly by a fourier transform:

$$h + \lambda^2 h'' = \Phi_0 \delta(r)$$

$$\left(1 + \lambda^2 k^2\right) h_k = \Phi_0$$

$$h(r) = \frac{\Phi_0}{(2\pi)^2}\int \frac{e^{ikrcos\theta} k dk d\theta}{1 + \lambda^2 k^2} = \frac{\Phi_0}{2\pi}\int \frac{kdk}{1+\lambda^2 k^2} J_0(kr) = \frac{\Phi_0}{2\pi\lambda^2} K_0(r/\lambda)$$

The free energy of a vortex of a model of height L is given by

$$F_{vor} = \int dV \frac{h^2 + \lambda^2 \left(\nabla \times h\right)^2}{8\pi} = \frac{\Phi_0}{8\pi}\frac{\Phi_0}{2\pi\lambda^2}\log\left(\frac{\lambda}{\xi}\right) L$$

The critical field for creating separated N vortex is thus given by

$$0 = F_{vor} - \int \frac{B H_{c1}}{4\pi} = N\left(F_{vor} - \frac{\Phi_0 H_{c1}}{4\pi}\right)$$

where the second term is the multiplication of the internal B field by the external one (see Ref. [4] (pages 33 50 and 66)).

$$H_{c1} = \frac{\Phi_0}{4\pi\lambda^2}\log\left(\frac{\lambda}{\xi}\right)$$

We ignored the energetic cost of destroying the superconducting phase in this calculation. The energy we gained per unit volume of superconductor is

$$\frac{\alpha^2}{\beta} = \frac{H_c^2}{8\pi} \propto m p_F T_c^2$$

(H_c is defined for type 1 superconductors)

and this the energy loss due to the creation of the core of the vortex is

$$E_{core} = C\nu T_c^2 \xi^2 L$$

where ν is the density of states, and C is some parameter which contains the details of the shape of the core.

5 XY Model Kosterlitz-Thoeless Beresinskii pahse transition

Here we follow [B. I. Halperin, Superfluidity, Melting and Liquid Crystal Phases in Two-Dimensions, in Proceeding of Kyoto Summer Institute 1979- Physics of Low Dimensional Systems, edited by Y. Nagaoka and S. Hikami (Publications Office, Progress of Theoretical Physics, Kyoto, 1979), p. 53.]

This model features a complex scalar order parameter

$$\psi = S_x - iS_y$$

Superconductor with complex order parameter that have a phase and an amplitude are one example but there are many other examples, e.g, spin that are confined to be in plane, having only x and y components.

We already saw that in 2D we do not have long range order. Indeed if we define

$$\psi = \psi_0 e^{i\phi(r)}.$$

and the Hamiltonian which describes the system is

$$H = \frac{1}{2}\int \kappa_0 \left(\nabla\phi\right)^2 dr \tag{9}$$

then the correlation function is

$$\langle\psi^*(r)\psi(0)\rangle = \left\langle \psi_0 e^{i\phi(r)}\psi_0 e^{-i\phi(r)}\right\rangle = |\psi_0|^2 e^{-\frac{1}{2}\left\langle(\phi(r)-\phi(0))^2\right\rangle} \tag{10}$$

with

$$\left\langle\left(\phi(r)-\phi(0)\right)^2\right\rangle = \frac{T}{\kappa_0}\int \frac{d^2k}{(2\pi)^2}\left|e^{i\vec{k}\cdot\vec{r}}-1\right|^2\left\langle\phi_k^2\right\rangle.$$

The last equation follows from the Fourier substitution:

$$\phi(\vec{r}) = \frac{1}{(2\pi)^2}\int e^{i\vec{k}\cdot\vec{r}}d^2k\phi_{\vec{k}}$$

and the relation:

$$\int dr^2 e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} = \delta^2(\vec{k}-\vec{k}')$$

Using the free Hamiltonian in Eq. (9) and the relation

$$\frac{1}{(2\pi)^2}\int dk^2 e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} = \delta^2\left(\vec{r}-\vec{r}'\right)$$

we find:

$$H = \kappa\frac{1}{(2\pi)^4}\int d^2r\int d^2k\int d^2k'\nabla(e^{i\vec{k}\cdot\vec{r}}\phi_{\vec{k}})\nabla(e^{i\vec{k}'\cdot\vec{r}}\phi_{\vec{k}'}) = \kappa\frac{1}{(2\pi)^2}\int d^2k k^2\phi_{\vec{k}}\phi_{-\vec{k}}$$

from this we can easily find the correlation function:

$$\left\langle\phi_{\vec{k}}\phi_{\vec{k}'}\right\rangle = \delta(\vec{k}-\vec{k}')\kappa\frac{1}{k^2}.$$

Using the relation

$$F(|\vec{r}|) = \left|e^{i\vec{k}\cdot\vec{r}}-1\right|^2 = \left(\cos(kr)-1\right)^2+\sin^2(kr) = 1-2\cos(kr)+\cos^2(kr)+\sin^2(kr) = 2-2\cos(kr)$$

we notice that for $\vec{k}\cdot\vec{r}\ll 1$ the function $F(|\vec{r}|)$ is zero and for $kr\gg 1$ the cos terms is oscillating so that its integral will be paretically zero and we can ignore it. We therefore can approximate Eq. (10) by:

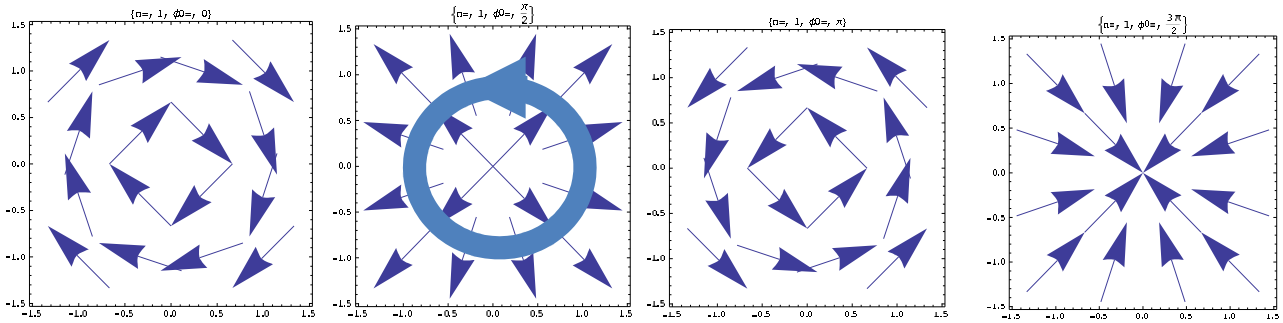
$$\left\langle (\phi(r)-\phi(0))^2 \right\rangle = \frac{2}{(2\pi)^2}\frac{T}{\kappa_0}\int_{1/r}^{1/a}\frac{2\pi k}{k^2}dk = \frac{T}{\pi\kappa_0}\log\left(\frac{r}{a}\right).$$

Here a is a short range (ultra violet) cutoff. Hence we have

$$\langle \psi^*(r)\psi(0)\rangle = |\psi_0|^2\left(\frac{1}{r}\right)^{\eta}, \qquad \eta = \frac{T}{2\pi\kappa_0}.$$

Thus we don't have long range order (only quasi long range order).

5.1 Vortexes in the XY model



$$\phi = -n\arctan(x/y) + \phi_0$$

$$S_x = \cos\phi; S_y = \sin\phi$$

$$\vec{v} = \vec{\nabla}\phi$$

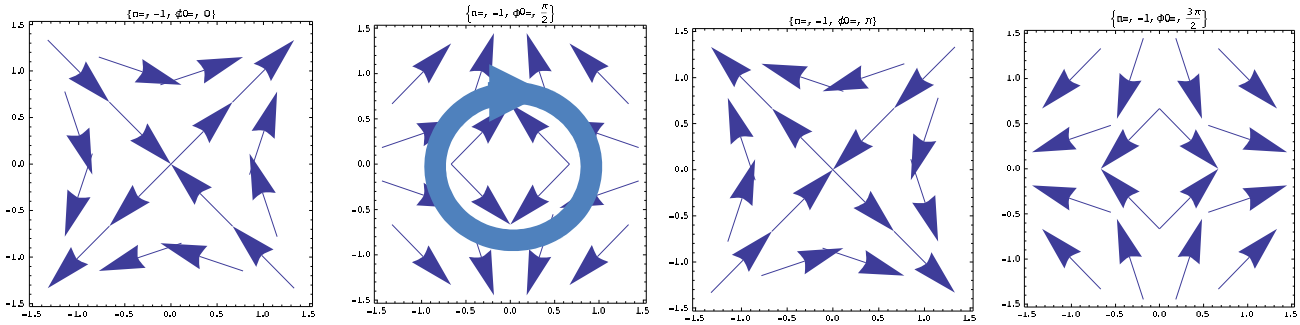


Figure 10: vortex and anti vortex in the spin configuration $\psi = \psi_0 e^{i\phi} = S_x - iS_y$ denoted by arrows in the spin direction and in the velocity $\nabla\phi$ dented by bold lines

For $\psi \neq 0$, we define a velocity field

$$\nabla\phi(r) = v(r)$$

From the fact that $e^{i\phi}$ is single valued, we have a condition for an integral around the core of a vortex

$$\oint v(r)dr = 2\pi n$$

from which we define the vortex charge

$$N = \frac{1}{2\pi}\oint v(r)dr$$

and integrating over a number of vortices

$$\oint v \cdot dr = \sum_i 2\pi N_i \equiv 2\pi N_c$$

We can define a vortex charge density

$$N(r) = \sum_i N_i \delta(r - R_i)$$

$$N_c = \int ds N(r) = \frac{1}{2\pi}\oint v(r)dr = \frac{1}{2\pi}\int \nabla \times v ds$$

$$N(r) = \frac{1}{2\pi}\nabla \times v$$

which gives the following continuity equation

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\sum_i N_i \delta(r - R_i) = \sum_i N_i \frac{\partial R_i}{\partial t}\frac{\partial}{\partial R_i}\delta(r - R_i) = -\sum_i N_i \frac{\partial R_i}{\partial t}\frac{\partial}{\partial r}\delta(r - R_i)$$

$$= -\frac{\partial}{\partial r}\left(\sum_i N_i \frac{\partial R_i}{\partial t}\delta(r - R_i)\right) \equiv -\frac{\partial}{\partial r}J_v$$

The energy of a single vortex of charge 1 in a system of size R is

$$E = E_{core} + \int_\xi^R 2\pi r dr \kappa_0 \frac{(\nabla\phi)^2}{2} = E_{core} + \pi\kappa_0 \int_\xi^R \frac{dr}{r} = E_{core} + \pi\kappa_0 \log\left(\frac{R}{\xi}\right)$$

This diverges unless we add another vortex with opposite charge at a distance $x_0 \gg \xi$, which gives

$$E_2 = 2\pi\kappa_0 \log\left(\frac{x_0}{\xi}\right) + const$$

where $const \approx 2E_{core}$.

5.2 Kosterlitz-Thouless Argument

The probability of creating a single vortex is

$$p_1 \sim e^{-H_1/T} = \exp\left(\frac{-\pi\kappa_0}{T}\log\frac{R}{\xi} + \frac{E_c}{T}\right) = e^{-E_c/T}\left(\frac{R}{\xi}\right)^{-\pi\kappa_0/T}$$

.

We have $(R/\xi)^2$ possible locations to place the core, giving

$$P = \left(\frac{R}{\xi}\right)^2 e^{-E_c/T}\left(\frac{R}{\xi}\right)^{-\pi\kappa_0/T} = e^{-E_c/T}\left(\frac{R}{\xi}\right)^{2-\pi\kappa_0/T}$$

.

Thus if

$$2 - \frac{\pi\kappa_0}{T} > 0 \Rightarrow T > \frac{\pi\kappa_0}{2} \equiv T_{KT},$$

it becomes preferable to crate vortexes. Below T_{KT} they will not be created. We saw

$$\langle\psi^*(r)\psi(0)\rangle = |\psi_0|^2\left(\frac{1}{r}\right)^{\eta}, \eta = \frac{T}{2\pi\kappa_0}$$

hence if $T < T_c \Rightarrow \eta < 1/4$.

The argument that we gave here is equivalent to the competition between the energy and the entropy of a single vortex. The free energy of the vortex is:

$$F = E - TS = \pi\kappa_0\log\left(\frac{R}{\xi}\right) - 2T\log\left(\frac{R}{\xi}\right)$$

for positive F when the energy "wins" vortexes are not created. For a negative F when the entropy wins at they are created. We expect a phase transition at $T_c = \pi\kappa_0$. Below T_c we will have vorteces bounded in pairs and above T_c we will have free vortexes – plasma of vortexes.

5.3 Describing vortices as a Coulomb gas

We will show now how this vortex plasma is mapped on to a 2D Coulomb gas, with logarithmic interactions. In the absence of magnetic field the total vorticity is zero hence hence the vortexes charges obey

$$\sum N_i = 0$$

and we define a velocity field and divide it into a rotational and irrotational part

$$\vec{v} = \vec{v}_0 + \vec{v}_1, \quad \nabla\times\vec{v}_1 = 0, \quad \nabla\cdot\vec{v}_0 = 0$$

$$N(r) = \frac{\nabla\times\vec{v}}{2\pi} \Rightarrow \nabla\times\vec{v}_0 = 2\pi N(r)$$

We shall also assume that at the boundary of the system we have:$v_{0\perp} = 0$ where $v_{0\perp}$ is the component of $\vec{v}$ perpendicular to the boundary.

From which we have

$$\int \vec{v}_1\cdot\vec{v}_0 d^2r = \int(\nabla\cdot\phi)\cdot\vec{v}_0 d^2r = \int\nabla\cdot\left(\phi\vec{V}_0\right)d^2r = \oint\phi\vec{v}_{0\perp}dl = 0. \tag{11}$$

The Hamiltonian of the system is given now by:

$$H = H_{\text{vortex}} + H_{\text{sw}}$$

with

$$H_{\text{vortex}} = \frac{\kappa_0}{2} \int |v_0|^2 \, d^2 r + \text{core energies}$$

and

$$H_{\text{sw}} = \frac{\kappa_0}{2} \int |v_1|^2 \, d^2 r.$$

.

(Due to Eq. (11) there are no cross terms.)

We can define now an analog of the electric field in the form

$$\vec{E} = -2\pi\kappa_0 \vec{z} \times \vec{v}_0$$

doing so we obtain

$$\nabla \cdot \vec{v}_0 = 0 \Rightarrow \nabla \times \vec{E} = 0$$

$$\vec{v}_{0\perp} = 0 \Rightarrow \vec{E}_{\parallel} = 0$$

$$\nabla \times \vec{v}_0 = 2\pi N(r) \Rightarrow \nabla \cdot \vec{E} = \frac{4\pi N(r)}{\varepsilon_0}, \varepsilon_0 = \frac{1}{\pi\kappa_0}$$

$$H_{\text{vortex}} = \frac{1}{8\pi\varepsilon_0} \int |E|^2 \, d^2 r + core = \frac{1}{2\varepsilon_0} \sum_{ij} N_i N_j G(\overrightarrow{R}_i, \overrightarrow{R}_j) + \sum N_i^2 E_c$$

where the last equality is from previous lectures. The effective interaction is similar to the 2D Coulomb interaction.

$$G(\vec{R}_i, \vec{R}_j) = -2 \log \left(\frac{\vec{R}_i - \vec{R}_j}{\xi} \right)$$

.

5.3.1 RG Approach

We expect that when we apply coarse graining, the dielectric constant of the medium will change due to screening by intermediate vortex pairs which act as effective dipoles between any given vortex pair. We can define a new dielectric constant

$$\varepsilon_R = \varepsilon_0 + 4\pi\chi$$

with χ being the dielectric susceptibility due to the vortices.

We expect the following behavior of the dielectric constant:

$$\varepsilon_R = \begin{cases} \text{finite} & T < T_c \\ \text{"}\infty\text{"} & T > T_c \end{cases}$$

here T_c is the temperature at which there will be vortex proliferation (they will still exist at lower temperatures due to fluctuations despite the fact that they are not energetically favorable).

$$\varepsilon = \epsilon_\xi + 4\pi \int \alpha(r) 2\pi r p(r) dr.$$

With $\alpha(r)$ being the polarizaility of a pair at distance r, and $p(r)$ the probability density to have a pair of vortices at distance r. Using $E_2(r) = 2E_c + 2\pi\kappa_0 \log \frac{r}{\xi}$, which gives

$$p(r) \sim \frac{1}{\xi^4} e^{-E_2(r)/T} = \frac{1}{\xi^4} e^{-\frac{2E_c + 2\pi\kappa_0 \log \frac{r}{\xi}}{T}}$$

We assume that the gas is dilute enough that the probability that there will be more than one pair is negligible. Thus this treatment only holds for $T < T_c$ near T_c.

$$= \frac{1}{\xi^4} y_0^2 \left(\frac{\xi}{r}\right)^{2\pi\kappa_0/T}$$

where $y_0 = e^{-E_c/T}$ is the fugacity which we assume to be small. The $1/\xi^4$ is there from dimensional considerations since this probability distribution is integrated with a measure of dimension x^4.

To find the polarizability $\alpha(r)$ we note that the a dipole moment $\vec{P}$ in an external filed $\vec{E}$ has an energy V given by:

$$V = -\vec{P} \cdot \vec{E} = -|p||E| cos\theta = -q|r||E| \cos\theta.$$

Averaging over the configuration of the dipole moment we find:

$$\overline{P} = qr\overline{cos\theta} = qr \frac{\int e^{-V/kT} cos\theta d\theta}{\int e^{-V/kT} d\theta} = r \frac{\int e^{qrEcos\theta/kT} cos\theta d\theta}{\int e^{qrEcos\theta/kT} d\theta} \underbrace{=}_{a=\frac{qrE}{T}} \frac{d}{da} log\left(\int e^{a\cos\theta} d\theta\right) \underbrace{\approx}_{a\ll 1} \frac{1}{2} a$$

giving,

$$\overline{P} = \frac{r^2}{2T} E \equiv \alpha(r) E, \text{ with } \alpha(r) = \frac{r^2}{2T}.$$

Substituting in the expression for the dielectric constant we find:

$$\varepsilon = \varepsilon_\xi + 4\pi \int_\xi^\infty \frac{r^2}{2T} \frac{(2\pi r)}{\xi^4} y_0^2 \left(\frac{\xi}{r}\right)^{2\pi\kappa_0/T} dr = \varepsilon_\xi + \frac{4\pi^2}{T} y_\xi^2 \int_\xi^\infty \left(\frac{r}{\xi}\right)^{3-2/\varepsilon T} \frac{dr}{\xi}.$$

We now want to apply coarse graining

$$= \underbrace{\varepsilon_\xi + \frac{4\pi^2}{T} y_\xi^2 \int_\xi^{\xi(1+dl)} \left(\frac{r}{\xi}\right)^{3-2/\varepsilon T} d\left(\frac{r}{\xi}\right)}_{\equiv \varepsilon_{\xi'}} + \frac{4\pi^2}{T} y_\xi^2 \int_{\xi(1+dl)}^\infty \left(\frac{r}{\xi}\right)^{3-2/\varepsilon T} \frac{dr}{\xi}. \tag{12}$$

Defining $\xi' \equiv \xi\,(1 + dl)$ we have

$$= \varepsilon_{\xi'} + \frac{4\pi^2}{T} \underbrace{y_\xi^2 \left(\frac{\xi'}{\xi}\right)^{4-2/\varepsilon_\xi T}}_{\equiv y_{\xi'}^2} \int_{\xi'}^\infty \left(\frac{r}{\xi'}\right)^{3-2/\varepsilon T} d\frac{dr}{\xi'}$$

$$= \varepsilon_{\xi'} + \frac{4\pi^2}{T} y_{\xi'}^2 \int_{\xi'}^\infty \left(\frac{r}{\xi'}\right)^{3-2/\varepsilon T} d\frac{dr}{\xi'}$$

from the Eq. (12) that defines $\varepsilon_{\xi'}$, and using the fact that dl is small we have:

$$\varepsilon_{\xi'} = \varepsilon_\xi + y_\xi^2 \frac{4\pi^2}{T} dl$$

and hence

$$\frac{d\varepsilon}{dl} = \frac{4\pi^2}{T} y^2.$$

From the definition of $y_{\xi'}$ and ξ' we find

$$y_{\xi'}^2 = y_\xi^2 + y_\xi^2 \left(2 - \frac{2}{\varepsilon T}\right) dl$$

and

$$\xi' \equiv \xi\,(1 + dl) \Rightarrow \frac{d\xi}{\xi} = dl \Rightarrow \xi(l) = \xi_0 e^l$$

so that

$$\frac{dy^2}{dl} = y^2 \left(4 - \frac{2}{\varepsilon T}\right).$$

Defining

$$\kappa = \frac{1}{\varepsilon T \pi}$$

we have

$$\frac{dy}{dl} = (2 - \kappa\pi)\, y + O(y^3)$$

and

$$\frac{d\kappa^{-1}}{dl} = 4\pi y^2 + O(y^4).$$

For $\kappa\pi < 2$ the fugacity y increases while for $\kappa\pi > 2$ it decreases.
It is useful to use further the following definitions

$$x = \frac{2}{\kappa\pi} - 1 \Rightarrow \kappa = \frac{2}{\pi} \frac{1}{1+x} \approx \frac{2}{\pi}\,(1 - x) \;\text{ for } x \ll 1$$

$$d\kappa^{-1} = \frac{\pi}{2} dx \Rightarrow \frac{\pi}{2} \frac{dx}{dl} = 4\pi^2 y^2$$

giving

$$\frac{dy}{dl} = 2xy, \quad \frac{dx}{dl} = 8\pi^2 y^2$$

$$\frac{d2x}{dl} = (4\pi y)^2, \quad \frac{d\,(4\pi y)}{dl} = 2x\,(4\pi y)$$

with $\tilde{x} = 2x$ and $\tilde{y} = 4\pi y$ we have

$$\frac{d\tilde{x}}{dl} = \tilde{y}^2, \quad \frac{d\tilde{y}}{dl} = \tilde{x}\tilde{y} \tag{13}$$

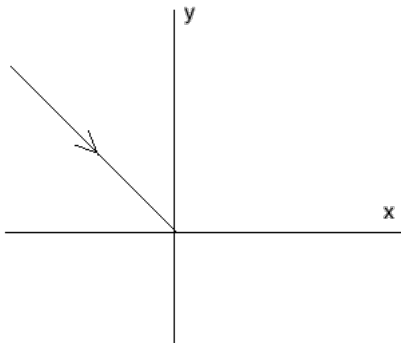
These are the KTB RG flow equations.

From them we can define a constant of the motion by multiplying the first equation by $\tilde{x}$, the second by $\tilde{y}$ and subtracting

$$\frac{d\tilde{x}^2 - \tilde{y}^2}{dl} = 0$$

$$\tilde{x}^2 - \tilde{y}^2 = const$$

We remove the tildes from now on. Along the line $x = -y$



$$\frac{dy}{dl} = -y^2$$

$$-\frac{dy}{y^2} = dl \Rightarrow \frac{1}{y_\xi} - \frac{1}{y_0} = l = \log\left(\frac{\xi}{\xi_0}\right)$$

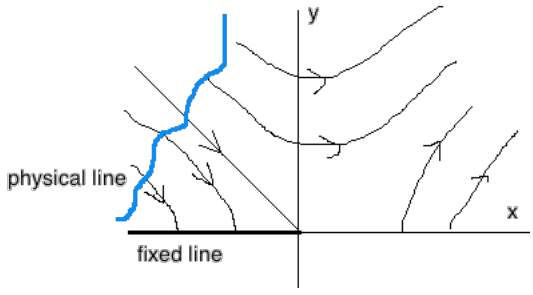
$$y_\xi = \frac{y_0}{1 + y_0 l} \approx \frac{1}{l}$$

$$y = -x = -\frac{1}{\log\left(\frac{\xi}{\xi_0}\right)}$$

at $T = T_C$ we have $x = 0 \Rightarrow \kappa = \frac{\kappa_R}{T_c} = \frac{2}{\pi}$, $\varepsilon_R = \frac{1}{2T_C}$ where ε_R is the dielectric constant at T_C.
The probability of the existence of a pair at a distance r from one another at T_C

$$P(r) = \frac{y^2}{r^4} = \frac{1}{r^4 log^2\left(r/\xi_0\right)}$$

We now look at other areas.



For $T < T_C$, near $y = 0$

$$x^2 - y^2 = \left(T - T_C\right) b^2$$

we can assume a linear dependence because $x^2 - y^2$ changes sign when we change the relative sizes of T and T_C

$$x = b\sqrt{T_C - T}$$

$$\frac{\kappa_R}{T_C} = \frac{2}{\pi} + b\sqrt{T_C - T}$$

This development breaks down in the area of positive x where y grows. This is because it was based on the assumption that the fugacity is small and the pair density is low.

For $T > T_C$

$$y^2 = x^2 + b^2 \left(T - T_C\right)$$

$$\frac{dx}{dl} = x^2 + b^2 \left(T - T_C\right)$$

$$\int_{\xi_0}^{\xi} dl = \int_{x_0}^{1} \frac{dx}{x^2 + b^2 \left(T - T_C\right)}$$

where $x = 1$ is some scale beyond which the RG treatment breaks down

$$l = \frac{1}{b\sqrt{T - T_C}} \tan^{-1}\left(\frac{x}{b\sqrt{T - T_C}}\right)\Bigg|_{x_0}^{1} \approx \frac{1}{b\sqrt{T - T_C}}$$

From this we can extract a length. We identify this temperature dependent length scale as the screening length

$$\xi_+ = r_{sc} = \xi_0 e^{1/b\sqrt{T - T_C}}$$

If we want to discover how many vortices we have, we can calculate the effective screening radius of a certain vortex concentration and compare to the one we found. According to Debye

$$\frac{4\pi n_{\text{free}}}{T\varepsilon} = 8\pi n_{\text{free}}$$

from which

$$n_{\text{free}} = \frac{1}{\xi_+^2}$$

5.3.2 The order of the KTB phase transition

Which thermodynamic quantity changes in this phase transition?

$$H_{NU} = \int \left[\alpha\psi^2 + \beta\psi^4 + J\left(\nabla\psi\right)^2\right] d^d r$$

and a saddle point approximation gives $\psi^2 = \frac{\alpha}{2\beta}$ and assuming the derivative terms are of the same scale as the potential terms

$$\frac{J}{\xi^2} \sim \alpha$$

$$\frac{1}{2}\int J\left(\nabla\psi\right)^2 d^dr \sim J\xi^{d-2}\left|\psi\right|^2 = \frac{J\xi^{d-2}\alpha}{2\beta} = \frac{J^2}{2\beta}\xi^{d-4}$$

for D=2

$$H_{NU} \propto \frac{1}{\xi^2}$$

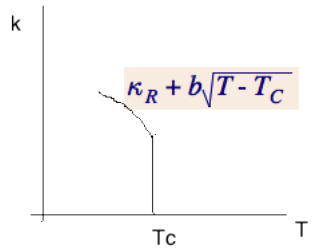
$$F_{singular} \sim \frac{1}{\xi^2}$$

thus

$$\frac{\partial^n F}{\partial T^n} = \frac{b}{\left(T-T_C\right)^{n+1/2}} e^{-2/b\sqrt{T-T_C}}$$

meaning the transition is continuous at any order.

The superfluid density however changes discontinuously.



We also saw that the 2 point function changes its behaviour

$$\langle\psi(r)\psi(0)\rangle = \begin{cases} T > T_C & e^{-r/\xi_+} \\ T < T_C & (1/r)^{T/2\pi\kappa} \end{cases}$$

5.4 Coulomb gas and the Sine-Gordon mapping (Momentum RG)

5.4.1 Mapping between Coulomb Gas and the Sine Gordon model

$$S_V = \frac{1}{T}\left[\frac{1}{2\varepsilon_0}\sum_{ij} N_i N_j G(\frac{\left|\overrightarrow{R}_i - \overrightarrow{R}_j\right|}{\xi}) + \sum N_i^2 E_c\right]$$

$$G(x) = -2log\left(x\right)$$

$$z = \sum_{N_i=0,\pm1} \exp\left(-S_V\right)$$

where we drop vortices with charge larger than 1. Using the general formula for a Gaussian integral

$$\int \frac{dx_1...dx_n}{\left(2\pi\right)^{n/2}} exp\left(-\frac{1}{2}x_i A_{ij} x_j + x_i J_i\right) = \left(detA\right)^{-1/2} exp\left(\frac{1}{2}J_i A_{ij}^{-1} J_j\right)$$

$$exp\left(-\frac{1}{2\varepsilon_0}N_iN_jV_{ij}\right)=det\left(\frac{V^{-1}}{2\varepsilon}\right)^{1/2}\frac{1}{(2\pi)^{1/2}}\int d\varphi_1..d\varphi_n exp\left(-\frac{1}{2}\varphi_iV_{ij}^{-1}\varphi_j+i\varphi_iN_i\right)$$

and adding the core energy term

$$z=N\int\mathcal{D}\varphi(x)exp\left(-\frac{1}{2}\sum_{i\neq j}\varphi(x_i)V^{-1}\left(x_i-y_i\right)\varphi(y_i)2\varepsilon_0T\right)\prod_i\sum_{N_i=0,\pm1,...}exp\left(i\varphi_iN_i+N_i^2E_c/T\right)$$

$$\approx N\int\mathcal{D}\varphi(x)exp\left(-\frac{1}{2}\sum_{i\neq j}\varphi(x_i)V^{-1}\left(x_i-y_i\right)\varphi(y_i)2\varepsilon_0T\right)\prod_i\left(1+2\left(cos\varphi_i\right)exp\left(-E_c/T\right)\right)$$

$$\approx N\int\mathcal{D}\varphi(x)exp\left(-\frac{1}{2}\sum_{i\neq j}\varphi(x_i)V^{-1}\left(x_i-y_i\right)\varphi(y_i)2\varepsilon_0T\right)exp\left(-y\sum2\left(cos\varphi_i\right)\right)$$

$V(q)=\frac{1}{q^2}\Rightarrow V^{-1}(q)=q^2\Rightarrow V(x)=\delta(x)\nabla^2$

$$=N\int\mathcal{D}\varphi(x)exp\left(-\frac{1}{2}\int\left(\nabla\varphi(x)\right)^2d^2x2\varepsilon_0T+\frac{2y}{a^2}\int d^2xcos\varphi(x)\right)$$

changing variables to $\sqrt{2\varepsilon_0T}\varphi,\beta=\frac{1}{\sqrt{2\varepsilon_0T}},M=2y_0=2e^{-E_c/T}$

$$=N\int\mathcal{D}\varphi(x)exp\left(-\frac{1}{2}\int\underbrace{(\nabla\varphi)^2d^2x}_{S_0}+\underbrace{M\int d^2x\cos(\beta\varphi)}_{S_1}\right)$$

5.4.2 (Momentum shell) Renormalization group of the sine Gordon Model

[Here we follow with some additional details chapter 10 of Ref. [6]]

We split a high momentum shell from the system ($b>1$)

$$\varphi_\Lambda(x)=\frac{1}{\sqrt{V}}\left(\underbrace{\sum_{K<\frac{\Lambda}{b}}e^{ikx}\varphi_K}_{\varphi_S(x)}+\underbrace{\sum_{\frac{\Lambda}{b}<K<\Lambda}e^{ikx}\varphi_K}_{\varphi_F(x)}\right)$$

and since S_0 is diagonal in momentum space,

$$z_\Lambda=\int\mathcal{D}\varphi_S\mathcal{D}\varphi_Fe^{-S_0[\varphi_S]-S_0[\varphi_F]-S_1[\varphi_F+\varphi_S]}$$

$$=\int\mathcal{D}\varphi_Fe^{-S_0[\varphi_F]}\int\mathcal{D}\varphi_Se^{-S_0[\varphi_S]}\frac{\int\mathcal{D}\varphi_Fe^{-S_0[\varphi_F]-S_1[\varphi_F+\varphi_S]}}{\int\mathcal{D}\varphi_Fe^{-S_0[\varphi_F]}}$$

$$=z_F\int\mathcal{D}\varphi_Se^{-S_0[\varphi_S]}\left\langle e^{-S_1[\varphi_F+\varphi_S]}\right\rangle_F$$

giving an effective action for the slow degrees of freedom:

$$S_{\text{eff}}(\varphi_S) = S_0(\varphi_S) - \log\left\langle e^{-S_1[\varphi_F+\varphi_S]}\right\rangle_F.$$

The basic idea here is to average over the fast degree of freedom a φ_f and obtain an effective action in terms of the slow variables φ_s. The parameters of the action S are renormalized and the RG procedure is established.

We assume the core energy is small, hence M is small. Thus using the cumulant expansion $log\left\langle 1 - S_1 + \frac{1}{2}S_1^2\right\rangle = \langle S_1\rangle - \frac{1}{2}\left\langle S_1^2\right\rangle + \frac{1}{2}\langle S_1\rangle^2$

$$S_{\text{eff}}(\varphi_S) = S_0(\varphi_S) - \langle S_1(\varphi_F+\varphi_S)\rangle + \frac{1}{2}\left\langle S_1^2(\varphi_F+\varphi_S)\right\rangle - \frac{1}{2}\langle S_1(\varphi_F+\varphi_S)\rangle^2$$

a First order term

a.1 Integrating out the fast variables – thinning the degrees of freedom The first order term is

$$\langle S_1(\varphi_F+\varphi_S)\rangle = \frac{M}{a^2}\int d^2x\left\langle cos\left(\beta(\varphi_S+\varphi_F)\right)\right\rangle_F$$

$$= \frac{1}{2}\frac{M}{a^2}\int d^2x\sum_{\sigma=\pm1} e^{i\beta\sigma\varphi_S}\left\langle e^{i\beta\sigma\varphi_F}\right\rangle_F$$

$$\left\langle e^{i\beta\sigma\varphi_F}\right\rangle_F = e^{-\frac{1}{2}\beta^2\left\langle\varphi_F^2\right\rangle} \equiv e^{-\frac{1}{2}\beta^2 G(0)} \equiv A(0)$$

$$G(0) = \langle\varphi_F(0)\varphi_F(0)\rangle = \frac{1}{(2\pi)^2}\int_{\Lambda-d\Lambda=\Lambda/b}^{\Lambda} d^2k\frac{1}{k^2} = \frac{1}{2\pi}\int\frac{dk}{k} = \frac{1}{2\pi}dlog\Lambda \equiv \frac{1}{2\pi}dl$$

$$A(0) = e^{-\frac{1}{2}\beta^2\left\langle\varphi_F^2\right\rangle} = 1 - \frac{\beta^2}{4\pi}dl$$

Thus to first order

$$S_{\text{eff}}(\varphi_S) = \frac{1}{2}\int(\nabla\varphi_S)^2 d^2x + M\left(1-\frac{\beta^2}{4\pi}dl\right)\int\frac{d^2x}{a^2}\cos(\beta\varphi_S)$$

a.2 Rescaling We apply

$$x' = x/b \Rightarrow q' = bq$$

where $b = \frac{1}{1-dl}$

which doesn't change S_0

$$S_{\text{eff}}(\varphi_S) = \frac{1}{2}\int(\nabla\varphi_S)^2 d^2x' + M\left(1+\left(2-\underbrace{\frac{\beta^2}{4\pi}}_{dim}\right)dl\right)\int\frac{d^2x'}{a^2}cos(\beta\varphi_S)$$

$$M' = M(1+(2-dim)\,dl)$$

$$M' - M = \frac{dM}{dl} = (2 - dim)\, M$$

$$M(L) = M_0 \left(\frac{L}{a}\right)^{2-dim}$$

M_0 represents the fugacity. We can see that for $dim < 2$ the coefficient M grows and our approximation breaks. This allows us to obtain a scale which is an effective screening length, since beyond it there are many vortices and our treatment breaks down.

$1 = M_0 \left(\frac{\xi_+}{a}\right)^{2-dim} \Rightarrow \xi_+ = \left(\frac{1}{M_0}\right)^{1/(2-dim)} a.$

It is useful to remember that an operator with a large dimension is irrelevant in the renormalization group sense

a.3 Field rescaling We neglected the fact that φ_S also depends on x when we changed variables to x'. We can show that to first order this is trivial

$$\varphi(x) = \overline{\zeta}\varphi_S\left(x'\right)$$

$$\varphi_q = \zeta\varphi_{q'}$$

we choose $\overline{\zeta} = 1$ to keep S_0 invariant. This means

$$\varphi_q = \int d^2x e^{iqx}\varphi_S(x) = b^2 \int d^2x'\varphi_S(x')e^{iq'x'} = b^2\varphi_q$$

giving

$$\zeta = b^2$$

b Second order term

b.1 Integrating out the fast variables – thinning the degrees of freedom We now consider the $O(M^2)$ terms in the cumulant expansion of $S_{\text{eff}}(\varphi_S)$:

$$2O(M^2) = \left\langle S_1^2\left(\varphi_F + \varphi_S\right)\right\rangle - \left\langle S_1\left(\varphi_F + \varphi_S\right)\right\rangle^2$$

$$= M^2 \int d^2x_a d^2x_b \left\langle cos\left(\beta\left(\varphi_F^a + \varphi_S^a\right)\right) cos\left(\beta\left(\varphi_F^b + \varphi_S^b\right)\right)\right\rangle - \left\langle cos\left(\beta\left(\varphi_F^a + \varphi_S^a\right)\right)\right\rangle \left\langle cos\left(\beta\left(\varphi_F^b + \varphi_S^b\right)\right)\right\rangle$$

$$= \frac{M^2}{4} \int d^2x_a d^2x_b \sum_{\sigma_a,\sigma_b=\pm 1} \left(\left\langle e^{i\sigma_a\beta\varphi_F^a} e^{i\sigma_b\beta\varphi_F^b}\right\rangle - \left\langle e^{i\sigma_a\beta\varphi_F^a}\right\rangle \left\langle e^{i\sigma_b\beta\varphi_F^b}\right\rangle\right) e^{i\sigma_a\beta\varphi_S^a} e^{i\sigma_b\beta\varphi_S^b}.$$

The sum over $\sigma_a, \sigma_b = \pm$ gives four terms. The one with $\sigma_a = \sigma_b = 1$ (whose integrant we denote by Y_1) is given by

$$\frac{M^2}{4} \int d^2x_a d^2x_b \left(\left\langle e^{i\beta\varphi_F^a} e^{i\beta\varphi_F^b}\right\rangle - \left\langle e^{i\beta\varphi_F^a}\right\rangle \left\langle e^{i\beta\varphi_F^b}\right\rangle\right) e^{i\beta\varphi_S^a} e^{i\beta\varphi_S^b} \equiv \frac{M^2}{4} \int d^2x_a d^2x_b Y_1 e^{i\beta\varphi_S^a} e^{i\beta\varphi_S^b}.$$

Since the free part of the action is quadratic we have $\left\langle e^{\pm i\beta\varphi_F^a}\right\rangle = 1 \pm \left\langle i\beta\varphi_F^a\right\rangle - \frac{\beta^2}{2}\left\langle\varphi_F^{a2}\right\rangle + \cdots = 1 - \frac{\beta^2}{2}\left\langle\varphi_F^{a2}\right\rangle = e^{-\frac{\beta^2}{2}\left\langle\varphi_F^{a2}\right\rangle}$.

(An alternative way to prove this identity is to complete to a square and performing the Gaussian integration.) Using this identity we find:

$$Y_1 = e^{-\frac{\beta^2}{2}\left\langle\left(\varphi_F^a+\varphi_F^b\right)^2\right\rangle} - e^{-\frac{\beta^2}{2}\left\langle\varphi_F^{a2}\right\rangle}e^{-\frac{\beta^2}{2}\left\langle\varphi_F^{b2}\right\rangle} = e^{-\beta^2\left\langle\varphi_F^{a2}\right\rangle}\left(e^{-\beta^2\left\langle\varphi_F^a\varphi_F^b\right\rangle}-1\right)$$

$$\equiv e^{-\beta^2 G(0)}\left(e^{-\beta^2 G(x_a-x_b)}-1\right) \equiv A^2\left(0\right)\left(A^2\left(x_a-x_b\right)-1\right).$$

Averaging over the free field action S_0 give the following expression for the function G:

$$G(x_a-x_b) = \int_{\Lambda/b}^{\Lambda} \frac{e^{ip(x_a-x_b)}}{\left(2\pi\right)^2 p^2} d^2p.$$

A similar term $Y_2 = Y_1$ when taking $\sigma_a = \sigma_b = -1$ and two other terms where the sign in the first exponent is flipped, giving

$$O(M^2) = \frac{M^2}{4}\int d^2x_a d^2x_b \left[A^2\left(0\right)\left(A^2\left(x_a-x_b\right)-1\right)\cos\left(\beta\left(\varphi_S^a+\varphi_S^b\right)\right) + A^2\left(0\right)\left(\frac{1}{A^2\left(x_a-x_b\right)}-1\right)\cos\left(\beta\left(\varphi_S^a-\varphi_S^b\right)\right)\right].$$

For $x \gg \frac{b}{\Lambda}$ the oscillations in $G(x) \propto \int_{\Lambda/b}^{\Lambda} \frac{e^{ipx}}{p^2} dp$ lead to $G(x) \approx 0 \Rightarrow A^2(x) \approx 1$, and thus $A^2(x) \neq 1$ only for small $x < b/\Lambda$.

Hence x_a is very close to x_b. Setting $x_a = x_b$ the first term contains $\cos\left(2\beta\varphi_S^a\right)$ which is a vortex of charge 2 made by two overlapping charge 1 vortices - even though we didn't include such vortices in our partition function. In such situations we say that the RG procedure *g*enerated a new term. For parameters near the KTB transition (large

The second will act like a derivative.

Defining $z = \frac{x_a+x_b}{2}, \xi = x_a - x_b$, around $x_a = x_b$

$$O(M^2) = \frac{M^2}{2}\int d^2z d^2\xi \left[A^2\left(0\right)\left(A^2\left(\xi\right)-1\right)\cos\left(2\beta\varphi_S\left(z\right)\right) + A^2\left(0\right)\left(\frac{1}{A^2\left(\xi\right)}-1\right)\cos\left(\xi\beta\partial_z\varphi_S\left(z\right)\right)\right].$$

and since ξ is very small we can use the approximation $\cos\left(\xi\beta\partial_z\varphi_S\left(z\right)\right) \approx 1 - \frac{1}{2}\left(\xi\beta\partial_z\varphi_S\left(z\right)\right)^2$ to find:

$$O(M^2) = \frac{M^2}{2}\int d^2z \left[a_1\cos\left(2\beta\varphi_S\left(z\right)\right) + a_3 - \frac{a_2\beta^2}{2}\left(\nabla\varphi_S\right)^2\right],$$

where

$$a_1 = \int d^2\xi A^2\left(0\right)\left(A^2\left(\xi\right)-1\right), a_2 = \xi^2\int d^2\xi A^2\left(0\right)\left(\frac{1}{A^2\left(\xi\right)}-1\right), a_3 = \int d^2\xi A^2\left(0\right)\left(\frac{1}{A^2\left(\xi\right)}-1\right).$$

In addition

$$G(\xi) = \frac{1}{\left(2\pi\right)^2}\int_{\Lambda/b}^{\Lambda}\frac{e^{ip\xi cos\theta}}{p^2} p dp d\theta = \frac{1}{2\pi}J_0\left(\xi\Lambda\right)\frac{d\Lambda}{\Lambda}$$

where J_0 is a Bessel function. It features oscillations which are due to the fact that we used a sharp cutoff. Developing the exponent in A,

$$a_2 \approx \beta^2\frac{1}{2\pi}\int d^2\xi J_0\left(\xi\Lambda\right)\underbrace{\frac{d\Lambda}{\Lambda}}_{dl} \equiv \beta^2 dl C$$

where C is a number. Carefully examine the last integral we see that it does not converge, it can be shown (we do not show it here) that when the sharp cutoff is replaced by a soft one, the expression for C converges. This gives

$$O(M^2) = \frac{1}{2} M^2 \beta^4 (2C)\, dl\, (\nabla \varphi_S)^2 .$$

Gathering terms of order M also we have:

$$S_{\text{eff}} = \frac{1}{2} \int d^2x \left[\left(1 + M^2 \beta^4 2Cdl\right) (\nabla \varphi_S)^2 + M \left(1 - \frac{\beta^2}{4\pi} dl\right) \cos(\beta \varphi_S) + a_1 M^2 \cos(2\beta \varphi_S) + a_3 \right]$$

where the term $\propto a_2$ is irrelevent in the RG sense, and the term a_3 is a normalization constant. The normalization constant is not important as it will drop off in all the Gaussian averaging procedures.

b.2 Rescaling Rescaling the space x as in sub paragraph a.2 we obtain the equation:

$$M' = M \left(1 - \left(\underbrace{2}_{\text{from rescaling of } x} - \frac{\beta^2}{4\pi} \right) dl \right)$$

b.3 Field Rescaling Unlike the first order case the field rescaling here is not trivial. Defining

$$\varphi_{\text{new}} = \sqrt{1 + M^2 \beta^4 2Cdl}\, \varphi_S \approx \left(1 + M^2 \beta^4 Cdl\right) \varphi_S$$

$$\beta' = \beta \left(1 - M^2 \beta^4 A dl\right)$$

So that eventually

$$S_{\text{eff}} == N' \int \mathcal{D}\varphi(x) exp \left(-\frac{1}{2} \int \underbrace{(\nabla \varphi)^2 d^2x}_{S_0} + \underbrace{M' \int d^2x \cos(\beta' \varphi)}_{S_1} \right)$$

To complete the RG equation near the KTB transition we define $\frac{\beta^2}{4\pi} = 2 + x, y = \sqrt{2}M$ and get the RG equations as in the real space approach in Eq. (13)

$$\frac{dy}{dl} = -xy, \frac{dx}{dl} = -y^2. \tag{14}$$

The sign change compared to the former renormalization is because now we go to smaller momenta vs. longer lengths.

6 Introduction to Graphene

This chapter should be taught at the beginning of the course.

7 The Quantum Hall Effect – This chapter is still under construction

A nice review of the quantum Hall Effect is given in [5] it includes discussions of Graphine as well.

7.1 Classical Hall Effect

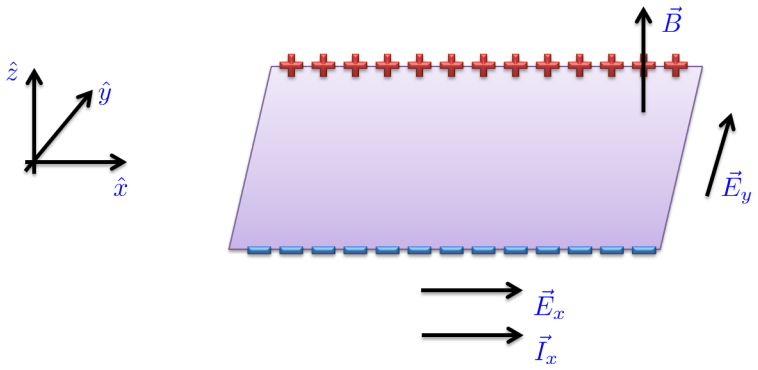


Figure 11: The Hall effect in classical system, current is driven in the x direction and in the presence of magnetic field the Lorentz force pushes electrons to the boundaries of the sample. This process continues until the electric field due to the electrons accumulated at the boundaries cancels the Lorentz force

The classic Hall experiment is depicted in Fig. 11. In response to the electric Field E_x current J_x is flowing in the wire. In addition due to the magnetic field B in the $\hat{z}$ direction a Lorentz force deflect the electrons. The deflected electrons are accumulated in the sample boundary and build up an electric field E_y that cancels the Lorentz force. Working in the convention $e = -|e|$ we find:

$$\frac{e}{c}\vec{v}\times\vec{B} = -eE_y.$$

Defining j in terms of the carrier density n as $j = env$ we find:

$$B\frac{env}{c}\times\hat{z} = -enE_y \quad \Rightarrow \quad \vec{E} = -\frac{B}{enc}\vec{J}\times\hat{z},$$

which gives

$$\frac{E_y}{J_x} \equiv R_B = \frac{-B}{enc}.$$

Hence, measuring the Hall voltage allows measuring the carrier density n and the sign of their charge. Notice that in the absence of impurities the current flows in a direction that is perpendicular to both the the electric and the magnetic field $\vec{J} \propto \vec{E}\times\vec{B}$.

If we add impurities, quantify their strength by the mean free collision time τ between them the EOM become:

$$m\dot{\vec{v}} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B} - \frac{m\vec{v}}{\tau} \quad \Rightarrow \quad \vec{J} = \frac{ne^2}{m}\vec{E} + \frac{e}{cm}\vec{J}\times\vec{B} - \vec{J}/\tau.$$

We solve the equation in the frequency space, with (i) $\dot{\vec{J}} = i\omega\vec{J}$ and (ii) the matrix relation between $\vec{J}$ and $\vec{E}$

$$\vec{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \hat{\rho}\vec{J} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix}\begin{pmatrix} J_x \\ J_y \end{pmatrix}$$

we find:

$$\rho = \begin{pmatrix} \frac{m}{ne^2\tau}(1+i\omega\tau) & \frac{B}{nec} \\ -\frac{B}{nec} & \frac{m}{ne^2\tau}(1+i\omega\tau) \end{pmatrix}.$$

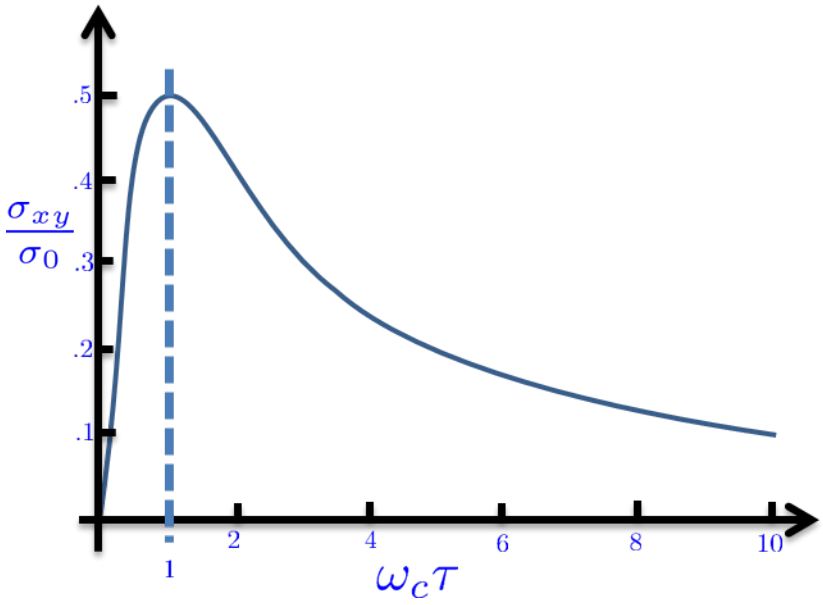


Figure 12: the Hall cobductance σ_{xy} as a function of magnetic

Using the matrix relation demand careful examination of the boundary conditions, for example assuming that we apply electric field $\vec{E} = E\hat{x}$ in the presence of magnetic filed, $\vec{B} = B\hat{z}$ because current is not flowing in the y direction we will obtain $E_y = \rho_{xy} J_x$.

We can rearrange $\hat{\rho}$ in the form:

$$\hat{\rho} = \frac{1}{\sigma_0} \begin{pmatrix} 1 + i\omega\tau & \omega_c\tau \\ -\omega_c\tau & 1 + i\omega\tau \end{pmatrix},$$

where $\sigma_0 = \frac{ne^2\tau}{m}, \omega_c = \frac{eB}{mc}$. Inversion of the resistivity matrix to the conductivity we have:

$$\hat{\sigma}(\omega = 0) = \frac{\sigma_0}{\left(1 + (\omega_c\tau)^2\right)} \begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix} \underset{\omega_c\tau \gg 1}{\longrightarrow} \begin{pmatrix} \frac{\sigma_0}{(\omega_c\tau)^2} & -\frac{nec}{B} \\ \frac{nec}{B} & \frac{\sigma_0}{(\omega_c\tau)^2} \end{pmatrix}. \tag{15}$$

We note that for large $\omega_c\tau$ (large B) both the resistivity (ρ_{xx} and ρ_{yy}) and conductivity (σ_{xx} and σ_{yy}) on the diagonal are negligible (with $\omega = 0$) with respect to the Hall terms. This "wired"situation that both resistance and conductance are small is special to the motion of electrons in strong magnetic field. To understand qualitatively the origin of this behavior we note that the terms on the diagonal can be written as:

$$\frac{e^2 n}{m}\left[\tau \frac{1}{(\omega_c\tau)^2}\right]$$

giving

$$\tau_{\text{eff}} = \frac{1}{\omega_c^2 \tau}$$

From dimensional considerations we can define a diffusion constant. $[D] = [lv] = v^2\tau = l^2/\tau$ with l typical step length in a random motion and τ the typical time between steps. And hence (since the electrons move typically in the Fermi velocity) we have

$$D = v_F^2 \tau_{\text{eff}} = \frac{(v_F \omega_c)^2}{\tau} = \frac{R_c^2}{\tau}.$$

In strong magnetic field the typical step is R_c the cyclotron radius and the typical time between "hoping between circles" is τ. We note the surprising fact that increasing the impurities and hence decreasing τ will actually increase the conductivity. This can be explained by the fact that without impurities the electrons will simply perform circular orbits. The scattering from the impurities will cause them to move in the direction of the external field.

7.2 Quantum Hall effect

The (Integer) Quantum Hall effect was discovered in 1980 by Klaus von Klitzing who measured the Hall resistance in a two-dimensional electron gas, realized with a silicon metal-oxide-semiconductor field-effect transistor. The Hall conductance was quantized of $e^2/h\nu$ with ν being an integer. Later, in 1982 Dan C. Tsui, Horst L. Stormer, and Art C. Gossard observed conductance quantization at a fractional values $\nu = 1/3$ more fraction were measured later on $\nu = 2/5, 3/7, 2/3, 3/5, 1/5, 2/9, 3/13, 5/2, 12/5 \ldots$. The fractional values are observed originally and in most experiments later on heterostructure of Aluminium Gallium Arsenide/ Gallium Arsenide. In the boundary between the two materials a clean two dimensional electron gas is formed. Recently quantum and fractional Hall effect where also observed in Graphene.

[2]

Two Nobel prises were awarded for the discoveries and the explanation of the Integer and fractional quantum Hall effect. The purpose of this chapter is to highlight the experimental finding and to show how we use topological arguments to explain the novel state of matter that give integer values for the Hall conductance.

The discussion here tries to exemplify the dialogue between theory and experiment. Experimental results give hints and clues for the state of the system then clever arguments lead to additional conclusions about the nature of the system, those are drawn even without having a microscopic theory.

7.2.1 Experimental observations

Fig. 13 depicts typical experimental observation of the quantum Hall effect, the main effects are:

1. **Quantized values of the Hall resistance**

   The Hall resistance show plateaus at

   $$\rho_{xy} = \frac{h}{e^2 \nu}, \nu \in N$$

   With a stronger field there are also plateaus at fractional values

   $$\nu = \frac{1}{3}, \frac{2}{5}, \frac{3}{4}, ...$$

2. **Effect of disorder** It was found experimentally that for cleaner samples we have more plateaus however that plateaus are narrower.

3. **Vanishing longitudinal resistance** R_{xx}

   As we can see in the Fig. 13 at small magnetic field there are small oscillations in the resistance these are known as the Shubnikov-deHaas oscillations, similar oscillations in the magnetization are deHaas van Alphen oscillations). At stronger magnetic field when plateaus are developed in the Hall resistance R_{xy} the longitudinal resistance R_{xx} is close to zero. Notice that due to the strong magnetic field, not surprisingly, the longitudinal conductance is also very small. The temperature dependance of the resistance is not shown in Fig. 13, but it was found that it increases exponentially with the temperature, i.e., $\sigma \sim e^{-T_0/T}$.

[2] Notice that in a square 2D sample of thickness d and linear dimension $L_x = L_y = L$ the resistance $R_{xx} = \rho \frac{L_x}{dL_y} = \frac{\rho}{d}$.

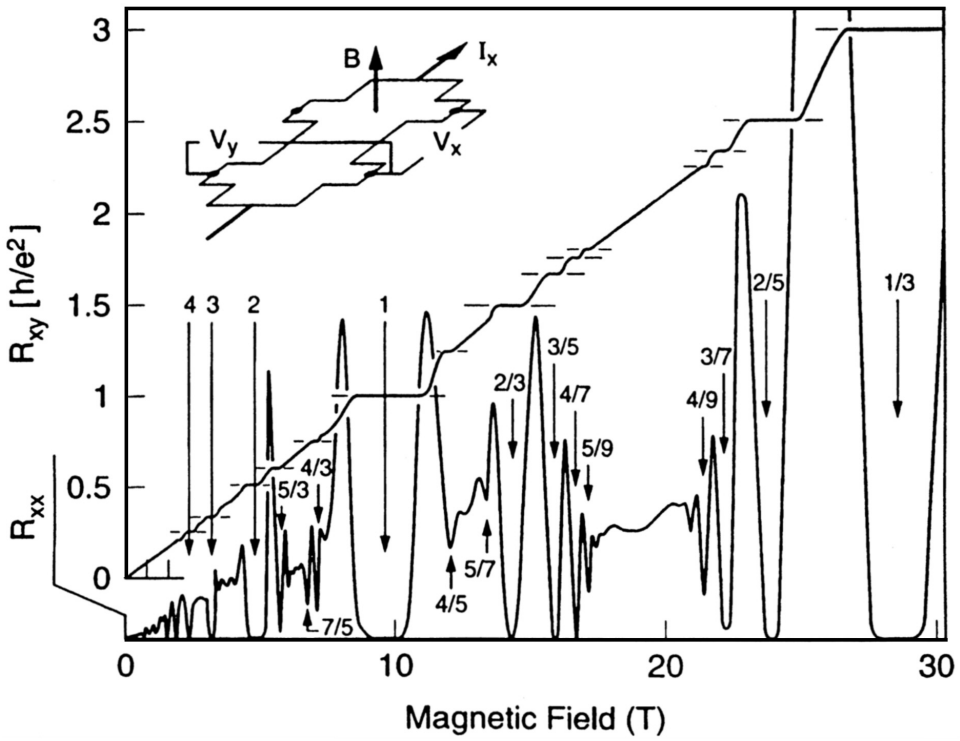


Figure 13: Here we plot the Hall resistance R_{xy} and the resistance R_{xx} In the classical Hall effect R_{xx} is constant and R_{xy} depends linearly on the Magnetic field. As we see the experimental result are quite different, R_{xx} fluctuates with the magnetic field and is zero at filling integer and fractional filling factor.

7.3 Basic explanation of the quantization effect

The Hamiltonian describing electrons moving in three dimensions in the presence of magnetic field is given in Landau's gauge by:

$$\hat{H} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2m}\left(\hbar k_x - eBy\right)^2 + eE_x x + U(x,y) + \frac{k_z^2}{2m} - \mu. \tag{16}$$

In finite width samples (in the z direction) electrons can not move in the z direction hence $k_z = \pi n_z / L_z$ is quantized. We will assume henceforth that only the first sub-band with $n_z = 1$ is occupied and absorb the energy $k_z^2/2m$ in the definition of μ . The potential $U(x,y)$ describes the effect of disorder – scattering of the electrons due to impurities. In principle, we should include also interaction effects between the electrons. Microscopically, these interactions are responsible for the formation of the fractional states.

The motion of the electrons in a strong electric field lead to the formation of Landau's levels. Taking periodic boundary conditions in the x direction, (having a Corbino geometry) we obtain a set of of Landau levels as follows. First choose a convenient gauge $\psi(x,y) = e^{ikx}\phi(y)$ the Schrodinger equation become:

$$\hat{H}\psi(x,y) = e^{ik_x x}\left[\frac{1}{2m}\left(\hbar k_x - eyB\right)^2 + \frac{1}{2m}\partial_y^2 - \mu\right]\phi(y) \tag{17}$$

with the periodic boundary condition in the x direction we have $k_x = \frac{2\pi}{L}n_x$ with n_x an integer. Defining now:

$$y_{n_x} = \frac{2\pi\hbar}{Be}n_x = \frac{l_B^2}{L_x}n_x; \quad Bl_B^2 \equiv \frac{h}{e} = \Phi_0$$

we obtain

$$H\psi(x,y) = e^{ik_x x}\left[\frac{\hbar\omega_c}{2}(y-y_{n_x})^2 + \frac{1}{2m}\partial_y^2 - \mu\right]\phi(y) \tag{18}$$

hence the Landau levels with energy $E = \hbar\omega_c(n_{\text{Lan}} + 1/2)$ are centered around y_{n_x}.

From the requirement $0 < y_{n_x} < L_y$ we can find the degeneracy of each level:

$$0 < n_x < \frac{L_x L_y}{l_B^2} = \frac{L_x L_y B}{\Phi_0} = \frac{\Phi}{\Phi_0}.$$

Denoting N the total number of particle we can define also the filling factor:

$$\nu = \frac{\text{number of electrons}}{\text{degeneracy of LL}} = \frac{N}{\Phi/\Phi_0} = \frac{\Phi_0}{\Phi/N} = \frac{\text{flux quantum}}{\text{flux per electron}} = \frac{N\Phi_0}{\Phi} = \frac{N/(LxLy)\Phi_0}{\Phi/(L_xLy)} = \frac{n\Phi_0}{B}.$$

For example for $\nu = 2$ we have exactly one flux quantum for two electrons, and the spin up and spin down levels of the first Landau's level (with $n_{\text{Lan}} = 0$) are filled. For $\nu = 1/3$ we have 3 quantum fluxes per electron.

Semiclassically, we expect that near the edge of the sample the parabola will be effectively narrower (as one side is blocked by the infinite potential of the sample edge) so that the energies of the Landau levels are higher near the edge.

Similarly we expect that smooth disorder (smooth on the scale of l_B) will tend to change the Landau level energies in the bulk of the sample. It can be shown (we do not explain it in details here) that states with energy at the center of the Landau level are delocalized through out the sample while those away from the center are localized. These qualitative explanations are summarized in the Fig. 15.

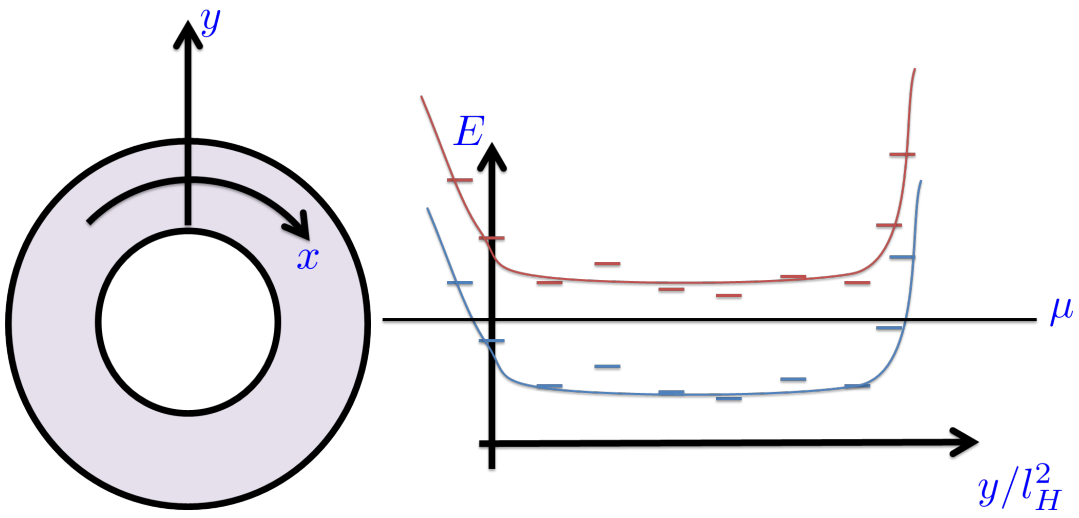


Figure 14: (a) Corbino geometry. (b) The $n_{\text{Lan}} = 0, 1$ Landau levels, in the presence of disorder the level is broaden, at strong magnetic filed away from the Landau level center the state are localized. (Zeeman effect that split the level further is not shown here). Near the edges of the sample we have extended states.

Having this rough picture in mind we can understand qualitatively the integer quantum Hall effect.

1. The first and most crucial observation of plateaus: As we increase the external magnetic field or decrease the electron density by reducing the chemical potential the filling factor ν crosses integer values. Since the states between the Landau levels are localized and can not conduct we observe conduction plateaus.

   The fractional case is much more delicate and goes beyond the scope of this course.

2. Effect of disorder: An increase in the disorder will make the plateaus wider as more states between the Landau level are localized and do not contribute to the conductance as we change the chemical potential.

Interestingly from the third experimental observation, i.e., the exponential temperature dependance of the resistances we can conclude that there us a gap in the system. This will lead us later on the a set of conclusions.

3. In the classical limit we saw that due to the matrix structure of the resistance both the conductance and the resistance are small compared to the off diagonal terms. [3]

   Experimentally it is observed that both

   $$\frac{\rho_{xx}}{h/e^2} \sim e^{-T_0/T}$$

   and we expect that also the conductance will have similar exponential dependance on the temperature.

   As we know the dissipative conductance is related to the rate W of absorbing external electro-magnetic field $W \sim \vec{J}\vec{E} \sim \sigma E^2$. To calculate this rate we can use the standard Fermi golden rule:

   $$W \sim \sum_{\text{initial states}} \frac{e^{-\varepsilon_i/t}}{z} \sum_{\text{final states}} \hbar\omega \left|\langle f|H'|i\rangle\right|^2 \left(\delta\left(\varepsilon_f - \varepsilon_i - \omega\right) + \delta\left(\varepsilon_f - \varepsilon_i + \omega\right)\right),$$

   with H' presenting the interaction of the electrons in our sample with the external EM field. ϵ_i the energy of the initial state and ϵ_f the energy of the final states. Since we are interested in the limit of $\omega = 0$ we have to require $\varepsilon_f \sim \varepsilon_i$. If there is a gap in the system then the factor $\propto e^{-\varepsilon_i/T}$ will yeild an exponential dependance in the temperature.

7.4 Additional conclusions, based on the existence of a gap in the system

7.4.1 Edge states

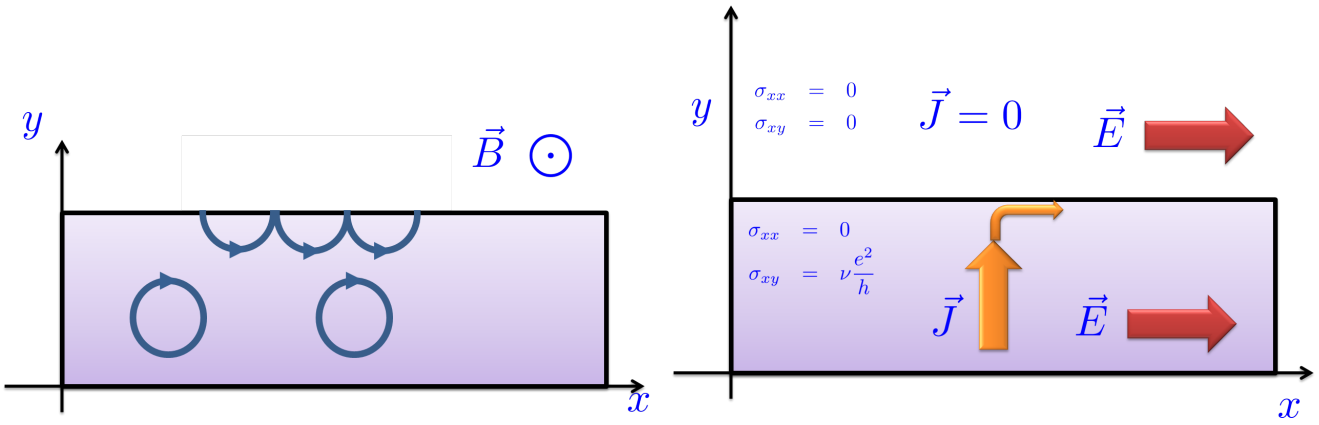


Figure 15: (a) Semiclassically: in the bulk electrons move in circles and near the edge in skipping orbits. (b) When we have a gap we find $\sigma_{xx} = 0$, then on the boundary separating $\sigma_{xy} \neq 0$ and $\sigma_{xy} = 0$ we must have an edge state.

Due to the exponential dependance on the temperature we expect that at zero temperature $\sigma_{xx} \approx 0$ so that the electrons move perpendicularly both to the electric field and to the magnetic field. We saw that

$$\sigma_{xy} = \frac{\nu e^2}{h}, \sigma_{xx} = 0$$

and since the current is perpendicular to both the magnetic and electric field, if we look at an edge parallel to the electric field we have a current towards the edge. As we reach the edge, since charge can not be accumulated on the edge, we must have a current parallel to the edge.

[3] Notice that in thin sample the resistance is $R_{xx} = \rho_x x L_x/(L_y d)$ so that for a square sample with $L_x = L_y$ the resistance R is equal to the resistivity ρ/d. Similarly the conductance G is equal to the conductivity σ times the thickness d. We therefore usually do not distinguish between conductivity and conductance (per square) or resistivity and resistance per square when discussing 2D systems.

This can be explained semiclassically in terms of skipping orbits performed by electrons near the edge. As we already so the motion of the electrons is perpendicular to both E and B the edge its self exsert an electric field on the electrons perpendicular to the edge so that electron will move along the edge.

7.4.2 Fractional charges (Laughlin)

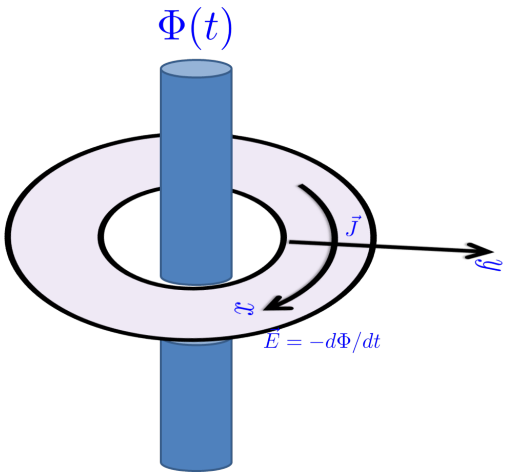


Figure 16: A change of the flux by ϕ_0 does not change the spectrum but push charges from edge to edge.

Noting $\sigma_{xy} = \nu\frac{e^2}{h}$ we will show now (following an argument by Laughlin) that there are fractional charges in the system. To do so we will use to important concepts:

1. **The adiabatic principle**: Assume that the system has a gap E_g for excitations above the (non degenerate) ground state $|\Psi(R(t))\rangle$. When we change the parameters $R(t)$ controlling the system at a slow rate Γ such that $E_g \gg \hbar\Gamma$ then the system will follow the ground state $|\Psi(R(t))\rangle$ (and will not be excited to a different state).
2. **Gauge invariance:** Byers and Yang (1961): showed using gauge invariance, that spectra of rings threaded by flux are periodic in the flux with period of the flux quantum $\phi_0 = h/e$. The Byers and Yang theorem is valid also in the presence of disorder and electron electron interaction.

Assuming we have a solenoid passing through the core of the system and we slowly (compared to the gap in the system) increase the flux. This generates an electric field in the $\hat{x}$ direction. (See Fig. 16) As there is a constant magnetic filed perpendicular to the sample a Hall current will flow perpendicular to $\vec{B}$ and $\vec{E}$, which will cause electrons to flow outward towards the edge.

From gauge invariance, after changing the flux by a flux quantum ϕ_0 we must return to the original spectrum. Hence the total charge the transferred between the edges is:

$$\Delta Q = \int I_y dt = \int J 2\pi r dt = \sigma_{xy} \int 2\pi r E dt = \sigma_{xy} \int \frac{\partial \Phi}{\partial t} dt = \sigma_{xy} \Delta\Phi$$

Using $\delta\Phi = \phi_0 = h/e$ we obtain: $\Delta Q = \nu\frac{e^2}{h}\frac{h}{e} = \nu e$

i.e. a fractional charge was transferred from edge to edge.

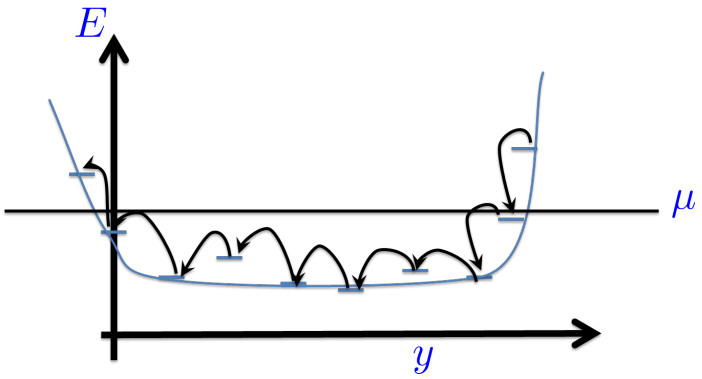


Figure 17: A change of the flux adiabatically move the Landau states. At ϕ_0 each state is replacing its neighbor position.

7.4.3 Quantization of Hall conductance in the integer case

In the presence of the flux the boundary condition on k_x after Eq. (17) are modified so that $e^{ik_xL} = e^{i2\pi\Phi(t)/\phi_0}$ which gives:

$$k_x(t) = \frac{2\pi}{L}\left(n + \frac{\Phi(t)}{\phi_0}\right).$$

Pictorially, the parabolas of the Landau levels are adiabatically moving in the y direction until for $\Phi(t) = \phi$ each parabola substitution the position of its neighbor. For the $\nu = 1$ case each state was filled exactly by 1 electron so that in this process 1 electron was transected from edge to edge. So that we have:

$$e = \Delta Q = \int I_y dt = \int J 2\pi r dt = \sigma_{xy}\int 2\pi r E dt = \sigma_{xy}\int \frac{\partial\Phi}{\partial t}dt = \sigma_{xy}\Delta\Phi$$

hence for a change of one flux quantum

$$\sigma_{xy} = \frac{e}{\Phi_0} = \frac{e^2}{h}$$

7.4.4 Quantization of the Hall conductance and Chern numbers using Linear response

In this subsection we are following an argument by Thouless that lead to quantization of the Hall conductance. This argument is very similar to Laughlin arguments mentioned above and also related to works by Berry (The Berry phase) and Avron. It is based on a formalism that is called "The Linear Response" formalism.

in the y direction, see Fig. 20

In the presence of electric field the Hamiltonian of Eq. (16) is modified by adding a term

$\delta H = e\mathcal{E}x$

where $\mathcal{E}$ is the applied electric field in the x direction. For $\mathcal{E} = 0$ we must have

$$J_y = \left\langle \Psi_0 | \hat{J}_y | \Psi_0 \right\rangle = 0.$$

Here $|\Psi_0\rangle$ is the many body ground state. We now assume that $\mathcal{E}$ is small and calculate the change in the current using perturbation theory.

The first order perturbation theory (in the operator $\mathcal{E}\hat{x}$) for the ground state gives

$$\delta|\Psi_0\rangle = \sum_{m\neq 0}\frac{\hat{x}_{0m}}{E_0 - E_m}|m\rangle.$$

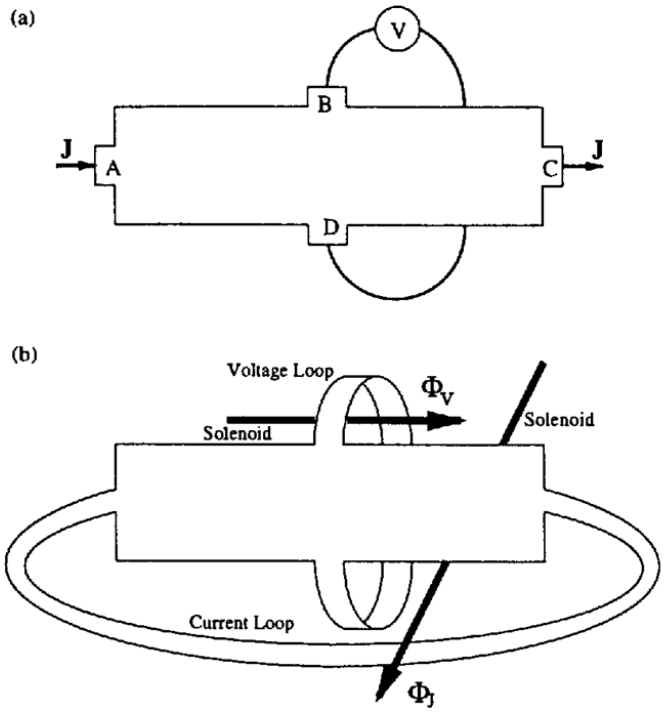


Fig. 14. The Hall bar, with current and voltage leads, shown in (a) can be replaced by the arrangement shown in (b), where the voltage is supplied by changing flux Φ_V through one loop, and the current is monitored by observing changes of the flux Φ_J through the other loop.

Figure 18: This figure is taken from Thouless []

Writing it in a slightly different and compact form, that will simplify the notation later we have:

$$\delta |\Psi_0\rangle = e\mathcal{E}\frac{\mathcal{P}}{E-\hat{H}}\hat{x}\,|\Psi_0\rangle \left[= e\mathcal{E}\sum_m \frac{\mathcal{P}}{E-\hat{H}}\,|m\rangle\langle m|\,\hat{x}\,|\Psi_0\rangle = \sum_{m\neq 0}\frac{e\mathcal{E}}{E_0-E_m}x_{m0}\right]$$

where $\mathcal{P}$ is a projection operator that projects out the ground state and $E = E_0$.

Similarly we have for the bra part of the ground state:

$$\delta\langle\Psi_0| = e\mathcal{E}\langle\Psi_0|\,\hat{x}\frac{\mathcal{P}}{E-\hat{H}}.$$

Combining the expressions we find:

$$\begin{aligned}\langle\Psi|\,\hat{J}_y\,|\Psi\rangle &= (\langle\Psi_0| + \delta\langle\Psi_0|)\,\hat{J}_y\,(|\Psi_0\rangle + \delta|\Psi_0\rangle)\\ &= e\mathcal{E}\left(\left\langle\Psi_0|\hat{J}_y\frac{\mathcal{P}}{E-\hat{H}}\hat{x}|\Psi_0\right\rangle + \left\langle\Psi_0|\hat{x}\frac{\mathcal{P}}{E-\hat{H}}\hat{J}_y|\Psi_0\right\rangle\right). \qquad (19)\end{aligned}$$

In addition we have the relation

$$\hat{J}_x = e\frac{\partial \hat{x}}{\partial t} = \frac{ei}{\hbar}\left[\hat{H}, \hat{x}\right] \Rightarrow$$

$$-i\frac{\hbar}{e}J_x \left|\Psi_0\right\rangle = \left[\hat{H}, \hat{x}\right]\left|\Psi_0\right\rangle = \left(H - E\right)\hat{x}\left|\Psi_0\right\rangle \Rightarrow$$

$$\hat{x}\left|\Psi_0\right\rangle = \frac{i\hbar}{e}\frac{\mathcal{P}}{E - \hat{H}}J_x\left|\Psi_0\right\rangle. \tag{20}$$

Substituting Eq. (20) (and a similar expression for the bra term.) in Eq. (19) we obtain

$$J_y = -i\hbar\mathcal{E}\left\langle \Psi_0 \left| \hat{J}_y \frac{\mathcal{P}}{\left(E - \hat{H}\right)^2}\hat{J}_x - \hat{J}_x \frac{\mathcal{P}}{\left(E - \hat{H}\right)^2}\hat{J}_y \right| \Psi_0 \right\rangle \tag{21}$$

We can continue and express the current operators in a different form. Closing the two leads in each direction to form a loop which allows to pass a flux through it, using the relation

$$\hat{J} = \frac{\delta H}{\delta A}, \tag{22}$$

and defining Φ_v as the flux through the loop in the plane perpendicular to $\hat{y}$ and Φ_J as the flux in the loop in the plane perpendicular to $\hat{x}$ (see Fig. 20) we have

$$\hat{J}_x = \frac{\partial H}{\partial \Phi_v}, \quad \hat{J}_y = \frac{\partial H}{\partial \Phi_J}.$$

Substituting in Eq. (21) we have:

$$\sigma_H\left(\Phi_v, \Phi_J\right) = i\hbar\left\langle \Psi_0 \left| \frac{\partial H}{\partial \Phi_J}\frac{\mathcal{P}}{\left(E - \hat{H}\right)^2}\frac{\partial H}{\partial \Phi_v} - \frac{\partial H}{\partial \Phi_v}\frac{\mathcal{P}}{\left(E - \hat{H}\right)^2}\frac{\partial H}{\partial \Phi_J} \right| \Psi_0 \right\rangle.$$

We can further simplified the expression by noting that writing

$$H = H_0 + \frac{\partial H}{\partial \Phi_v}\delta\Phi_v$$

we have

$$\delta\left|\Psi\right\rangle = \delta\Phi_v \frac{\mathcal{P}}{E - H}\frac{\partial H}{\partial \Phi_v}\left|\Psi_0\right\rangle$$

giving

$$\left|\frac{\partial H}{\partial \Phi_v}\right\rangle = \frac{\mathcal{P}}{E - H}\frac{\partial H}{\partial \Phi_v}\left|\psi_0\right\rangle$$

and hence

$$\sigma_H = i\hbar\left(\left\langle \frac{\partial \psi_0}{\partial \Phi_v}\middle|\middle|\frac{\partial \psi_0}{\partial \Phi_J}\right\rangle - \left\langle \frac{\partial \psi_0}{\partial \Phi_J}\middle|\middle|\frac{\partial \psi_0}{\partial \Phi_v}\right\rangle\right).$$

Since the result cannot depend on the flux we used to perturb the system and create an electric field (as long as there is a gap in the spectrum), we can integrate over it and find:

$$\sigma_H = i\hbar\left(\frac{e}{h}\right)^2 \int_0^{h/c} d\Phi_J \int_0^{h/c} d\Phi_v \left(\left\langle \frac{\partial \psi_0}{\partial \Phi_v}\middle|\middle|\frac{\partial \psi_0}{\partial \Phi_J}\right\rangle - \left\langle \frac{\partial \psi_0}{\partial \Phi_J}\middle|\middle|\frac{\partial \psi_0}{\partial \Phi_v}\right\rangle\right) \equiv i\hbar\left(\frac{e}{h}\right)^2 \int_0^{h/c} d\Phi_J \int_0^{h/c} d\Phi_v B.$$

Due to the periodicity in $\phi_0 = h/e$ (due to the gauge invariance) both in Φ_V and Φ_J the space of the fluxes forms a torus. We can define a Berry curvature B and Berry connection A which obey

$$\nabla \times A = B$$

$$\vec{A} = \frac{1}{2}\left(\left\langle \frac{\partial \Psi_0}{\partial \Phi_J}\middle|\middle|\Psi_0\right\rangle - \left\langle \psi_0\middle|\middle|\frac{\partial \Psi_0}{\partial \Phi_J}\right\rangle, \left\langle \frac{\partial \psi_0}{\partial \Phi_v}\middle|\middle|\psi_0\right\rangle - \left\langle \psi_0\middle|\middle|\frac{\partial \psi_0}{\partial \Phi_v}\right\rangle\right).$$

Notice that $\vec{A}$ is in fact given by the gradient of the argument of the wave function indeed, if we write $|\Psi_{0,\Phi_J,\Phi_V}\rangle = \Psi_{0,\Phi_J,\Phi_V}(\{x_i\} = r_{\Phi_J,\Phi_V}\{x_i\}e^{i\eta_{\Phi_J,\Phi_V}\{x_i\}}$, with the set $\{x_i\}$ being the set of the location of all the particles we have (suppressing the arguments in the middle terms):

$$i\,\mathrm{Im}\left(\langle\partial\Psi_0|\,|\partial_\Phi\Psi_0\rangle\right) = \frac{1}{2}i\int\prod_i dx_i\,\mathrm{Im}\left(r'e^{i\eta}re^{-i\eta} + i\eta' r^2 e^{i\eta}e^{-i\eta} + cc\right) = i\int\prod_i dx_i\frac{\partial\eta}{\partial\Phi} = i\frac{\partial\langle\eta\rangle}{\partial\Phi} \equiv i\frac{\partial\eta}{\partial\Phi}$$

hence

$$\vec{A} = \left(\frac{\partial\eta}{\partial\Phi_J}, \frac{\partial\eta}{\partial\Phi_v}\right).$$

We now have to perform the integral. We use Stokes' theorem to convert the integrals to ones over A.

$$\int B dv = \int \nabla \times A dv = \oint A ds$$

Since the integral is on a torus we have to be a bit careful.

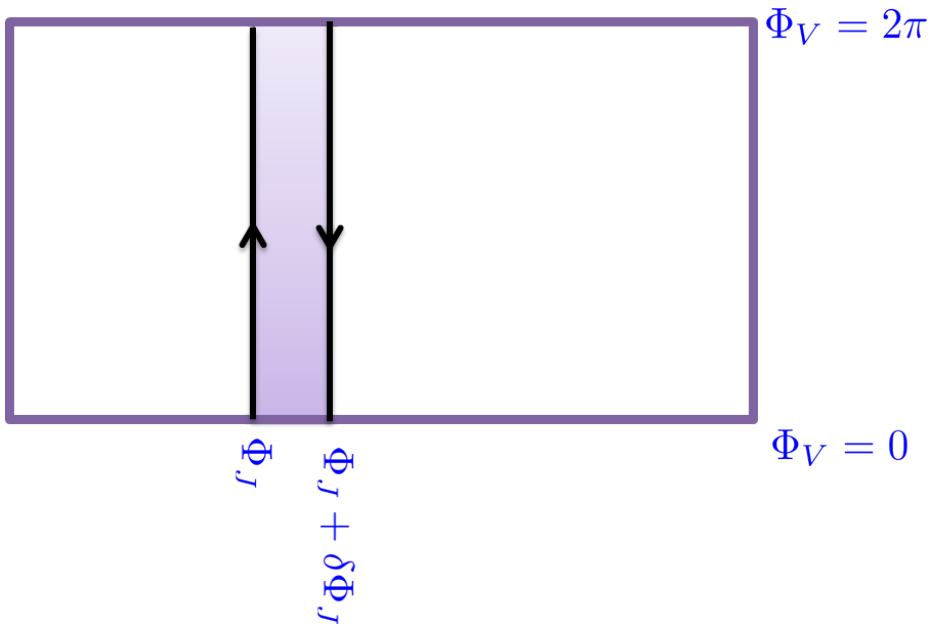


Figure 19: The integration contour for the calculation of the Chern number

We first integrate on loops of Φ_V at different values of Φ_J. This integral will give the flux through that slice.

$$\oint A\left(\Phi_J + \delta\Phi_J\right)d\Phi_v - \oint A\left(\Phi_J\right)d\Phi_v = \eta\left(\Phi_J + \delta\Phi_J\right) - \eta\left(\Phi_J\right) = \frac{\partial\eta}{\partial\Phi_J}\delta\Phi_J.$$

And now it remain to integrate on the loop of Φ_J:

$$\oint \frac{\partial \eta}{\partial \Phi_J} d\Phi_J = \eta(2\pi) - \eta(0) = 2\pi n$$

The last equal sign is correct because η is a phase on a loop and mast return to its value mod 2π after the rotation so that the wave function is unique.

Substituting in the expression for the conductance we find finally

$$\sigma_H = \frac{e^2}{h} n$$

with n defined as the Chern number, which we have shown is an integer.

8 Graphene – Quantum Hall effect without constant magnetic field

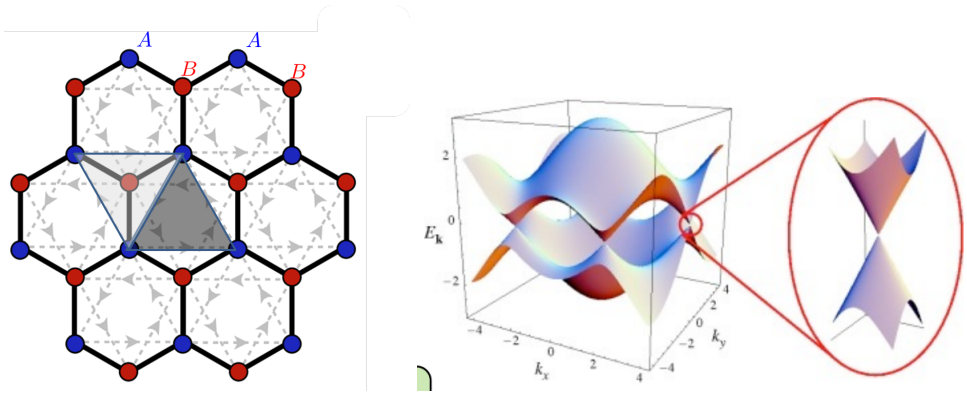


Figure 20: Haldane's Hamiltonian of Graphene with next nearest neighbor interactions finite flux in the shaded triangles and opposite flux in the complementary triangles

In this chapter we will show that a non zero Chern number leading to $\sigma_{xy} \neq 0$ and to the presence of edge modes can be obtain even when the average magnetic field is zero.

8.1 Preliminary notations

In Graphene the vectors connecting a site to its neighbors are

$$\delta_1 = \frac{a}{2}\left(1, \sqrt{3}\right), \delta_2 = \frac{a}{2}\left(1, -\sqrt{3}\right), \delta_3 = a(-1, 0).$$

At points

$$K = \frac{2\pi}{3a}\left(1, \frac{1}{\sqrt{3}}\right), K' = \frac{2\pi}{3a}\left(1, -\frac{1}{\sqrt{3}}\right)$$

the two bands of the Graphene touch each other.

Following Haldane, PRL 61, 2015 (1988) we define a hopping Hamiltonian which includes next nearest neighbors interaction:

$$H = t_1 \sum_{\langle ij \rangle} c_i^\dagger c_j + t_2 \sum_{\langle\langle ij \rangle\rangle} e^{-i\nu_{ij}\phi} c_i^\dagger c_j + M \sum_i \varepsilon_i c_i^\dagger c_i \tag{23}$$

with ε_i = the on site energy . In Graphene all ϵ_i are identical, since they are all carbon atoms, but in other examples, like that of Boron Nitride where the A and B atoms are different the situation is modified. The gist of Haldane idea is to study situations with an alternating magnetic field whose average is zero. For example a situation for which triangles around the B atoms circling positive flux and the complementary triangles negative flux.

Formally, we can include the alternating magnetic field by adding phases to the second nearest neighbors hopping matrix elements. For example we can choose ν_{ij} in Eq. (23) to be:

$$\nu_{ij} = \text{sign } \left(\delta_i \times \delta_j\right)$$

where δ_i, δ_j are the vectors between sites i and j and the site between them. This ensures that a loop of 3 sites which contains a site gives a positive sign for ν while one that doesn't gives a negative sign. Any other choice of phases consistent with that condition is possible.

In Fourier space, choosing $\varepsilon_A = 1, \varepsilon_B = -1$ for the two sublattices we have:

$$H = \sum c_k^\dagger h(k) c_k \text{with } h(k) = \varepsilon(k)\mathbb{I} + d_i(k) \cdot \sigma_i \tag{24}$$

$$\varepsilon(k) = 2t_2 \cos\phi \sum_i \cos\left(k \cdot \overline{\delta}_i\right)$$

$$d_1 = t_1 \sum_i \cos\left(k \cdot \delta_i\right), \quad d_2 = t_1 \sum_i \sin\left(k \cdot \delta_i\right), \quad d_3 = M - 2t_2 \sin\phi \left(\sum_i \sin\left(k \cdot \overline{\delta}_i\right) \right)$$

where $\overline{\delta}_1 = \delta_2 - \delta_3, \overline{\delta}_2 = \delta_3 - \delta_1, \overline{\delta}_3 = \delta_1 - \delta_2$ are the vectors connecting the next nearest neighbors.

Few symmetry properties of the Hamiltonian are noticeable. To study them let us first analyze the consequences that time-reversal has for Bloch Hamiltonians.

Time reversal symmetry
For spinless particles, T leaves the on-site creation operators unchanged, (unlike the case for spinful particles). We have:

$$Tc_jT^{-1} = c_j$$

where we can add any orbital indices to the creation operators as long as the index is not spin.
Using these rules we then have in the Fourier space:

$$Tc_jT^{-1} = \frac{1}{\sqrt{N}}\sum_k e^{-ikR_j}Tc_kT^{-1} = \frac{1}{\sqrt{N}}\sum_k e^{-ikR_j}Tc_kT^{-1}$$

In the above, the Fourrier exponent was complex conjugated because of the action of the T operator on complex c numbers is to complex conjugate them. Hence the action of the time-reversal operator on the annihilation operator of an electron at momentum k just flips the sign of the momentum. In order to fulfill the condition $Tc_jT^{-1} = c_j$ we must have

$$Tc_kT^{-1} = c_{-k}$$

this forms the rules for time reversal transformation of spin less particles in the momentum space.
We are now ready to explore the condition for time reversal of a Bloch Hamiltonian $h(k)$. By definition in clean systems we have:

$$H = \sum_{ij} c_j^\dagger h_{i-j} c_i = \sum_k c_k^\dagger h(k) c_k$$

The time reversal operation will be then:

$$THT^{-1} = \sum_k c_{-k}^\dagger Th(k)T^{-1}c_{-k} = \sum_k c_{-k}^\dagger h*(k)c_{-k},$$

we therefore conclude that THT^{-1} will be identical to H, i.e. will be time reversal invariant when $h*(k) = h(-k)$.
Coming back to Haldane's Hamiltonian in Eq, (24) we note that for the special case of $\phi = 0, \pi$ time reversal symmetry is preserved.
Indeed at this point we have

$$\varepsilon(k)^* = \varepsilon(k) = \varepsilon(-k)$$

$$d_i(k)^* = d_i(k) = d_i(-k).$$

hence time reversal symmetry is preserved at these values of the flux.
In addition $h(k)$ unchanged also under the inversion symmetry I

$$I : \sigma_x h(-k)\sigma_x = h(k)$$

In the discussion of the conductivity using the fluxes we use Eq. (25) for the definition of current when the momentum k is a good quantum number the current can also be defined as:

$$\hat{J} = \frac{e}{\hbar}\frac{\delta H}{\delta k}, \tag{25}$$

The integration over the periodic fluxes is than substituted by an integration over a full Brillouin zone and we can also write the conductivity as

$$\sigma_{xy} = \frac{e^2}{h}\int_{\text{Full Brillouin Zone}} dk_x dk_y F_{yx}(k) \tag{26}$$

$$F_{xy} = \frac{\partial A}{\partial k_x} - \frac{\partial A}{\partial k_y} \tag{27}$$

$$A_j = -i\sum_{\text{Full bands}} \left\langle \alpha, \vec{k} \middle| \frac{\partial}{\partial k_j} \middle| \alpha, \vec{k} \right\rangle \tag{28}$$

where α is a band index. The Chern number is then given by:

$$\mathrm{Ch} = \frac{1}{2\pi} \int\limits_{FBZ} dk_x dk_y F_{yx}(k) = n \tag{29}$$

Generally when $h(k)$ is a 2×2 matrix (as in Eq. (24)) its diagonalization gives:

$$E_{\pm} = \varepsilon \pm d(k) = \varepsilon \pm \sqrt{d_1^2 + d_2^2 + d_3^2}$$

$$\psi_+ = \frac{1}{\sqrt{d\,(d+d_3)}} \begin{pmatrix} d_3 + d \\ d_1 - i d_2 \end{pmatrix}, \psi_- = \frac{1}{\sqrt{2d\,(d+d_3)}} \begin{pmatrix} d_3 - d \\ d_1 - i d_2 \end{pmatrix}$$

hence:

$$A_{ij} = i \left\langle \psi_- | \partial_{k_j} | \psi_- \right\rangle = -\frac{1}{2d\,(d+d_3)} \left(d_2 \partial_j d_1 - d_1 \partial_j d_2 \right) \tag{30}$$

$$F_{ij} = \frac{1}{2} \varepsilon_{abc} \hat{d}_a \partial_i \hat{d}_b \partial_j \hat{d}_c \tag{31}$$

8.2 Chern number in Haldane's model and Obstruction of Stokes' theorem

8.2.1 Obstruction of Stokes' theorem

There is a nice topological interpretation for the integral of the k space. Due to the periodic boundary conditions the k variables span a two dimensional torus T^2. On the other hand the unit vector $\hat{d}(k_x, ky)$ is a point in the two dimensional sphere S^2.

The expression for the Chern number in Eq, (29) forms a mapping from the torus (k) space to the sphere (d) space. i.e., $T^2 \mapsto S^2$.

It appears that the Berry curvature F is simply the Jacobian of this transformation, indeed according to the result of Eq. (31) we have:

$$F_{ij} = \frac{1}{2} \begin{vmatrix} \hat{d}_1 & \hat{d}_2 & \hat{d}_3 \\ \partial_{k_x} \hat{d}_1 & \partial_{k_x} \hat{d}_2 & \partial_{k_x} \hat{d}_3 \\ \partial_{k_y} \hat{d}_1 & \partial_{k_y} \hat{d}_2 & \partial_{k_y} \hat{d}_3 \end{vmatrix} = \sin\theta \begin{vmatrix} \frac{\partial\theta}{\partial k_x} & \frac{\partial\phi}{\partial k_x} \\ \frac{\partial\theta}{\partial k_y} & \frac{\partial\phi}{\partial k_y} \end{vmatrix}$$

in the last equation we used the standard spherical coordinate system

$$d_1 = \sin\theta\cos\phi, d_2 = \sin\theta\sin\phi, d_3 = \cos\theta$$

The expression for the Chern number is now:

$$\mathrm{Ch} = \frac{1}{4\pi} \int_{\partial} \sin\theta d\theta d\phi,$$

but we have to determine the boundaries ∂ of integration.

To do so we neglect the ε which will not change the eigen vectors then the Hamiltonian is:

$$h(\vec{k}) = \varepsilon(\vec{k}k) + \sigma \cdot d(\vec{k}) \to \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

and the lower energy solution is:

$$\psi_I = \frac{1}{\sqrt{2d\left(d+d_3\right)}}\begin{pmatrix} d_3 - d \\ d_1 - id_2 \end{pmatrix} \to \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

which isn't defined at $\theta = 0$. On the other hand we can multiply by a phase and have

$$\psi_{II} = \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}$$

which isn't defined at $\theta = \pi$. Hence we can use each ψ only in the half sphere where it is well defined. We can then use Stokes' theorem to obtain that the integral over the sphere can be converted to two integrals over the respective Berry connections of the two half spheres on the equator, giving:

$$\frac{1}{4\pi}\int \cos\theta d\phi = \frac{1}{4\pi}\int \left(A_I - A_{II}\right) dl$$

where $A_j = i\left\langle\psi_j|\nabla|\psi_j\right\rangle$ is the Berry connection for the region where ψ_j is defined.

We can easily calculate A_I and A_{II} on the equator and and find

$$A_I = \frac{i}{2}\left(e^{i\phi},1\right)\partial_\phi\begin{pmatrix} -e^{i\phi} \\ 1 \end{pmatrix} = 1$$

and similarly

$$A_{II} = -1$$

giving

$$\mathrm{Ch} = \frac{1}{4\pi}\int 2d\phi = 1$$

The conclusion of these procedure, known as the obstruction of stokes theorem, is the following: if the vector d of the solution reaches both the north and the south poles of S^2 the Chern number is 1. If on the other hand d span only the north hemisphere then we can do the whole integration with one of the wave function shrink the loop of integration to a point and the Chern number is zero.

8.2.2 Chern number in Haldane's model

Coming back to Haldane's model in Eq. (24) we find the following results

for $\phi = 0$, $d_3(k) = M = const$ so clearly as we change k we will not get into the south pole and the Chern number is zero.

A more careful analysis of the dependance of $\vec{d}(k)$ on k (we will discuss that in the next section) shows that it is sufficient to discuss the behavior of d_3 near the special points K and K'. At these points:

$$d_3(K) = M - 3\sqrt{3}t_2\sin\phi, \quad d_3(K') = M + 3\sqrt{3}t_2\sin\phi$$

and we can use these values to see whether d reaches the south pole, which means we have non-zero Chern number. This gives the phase diagram

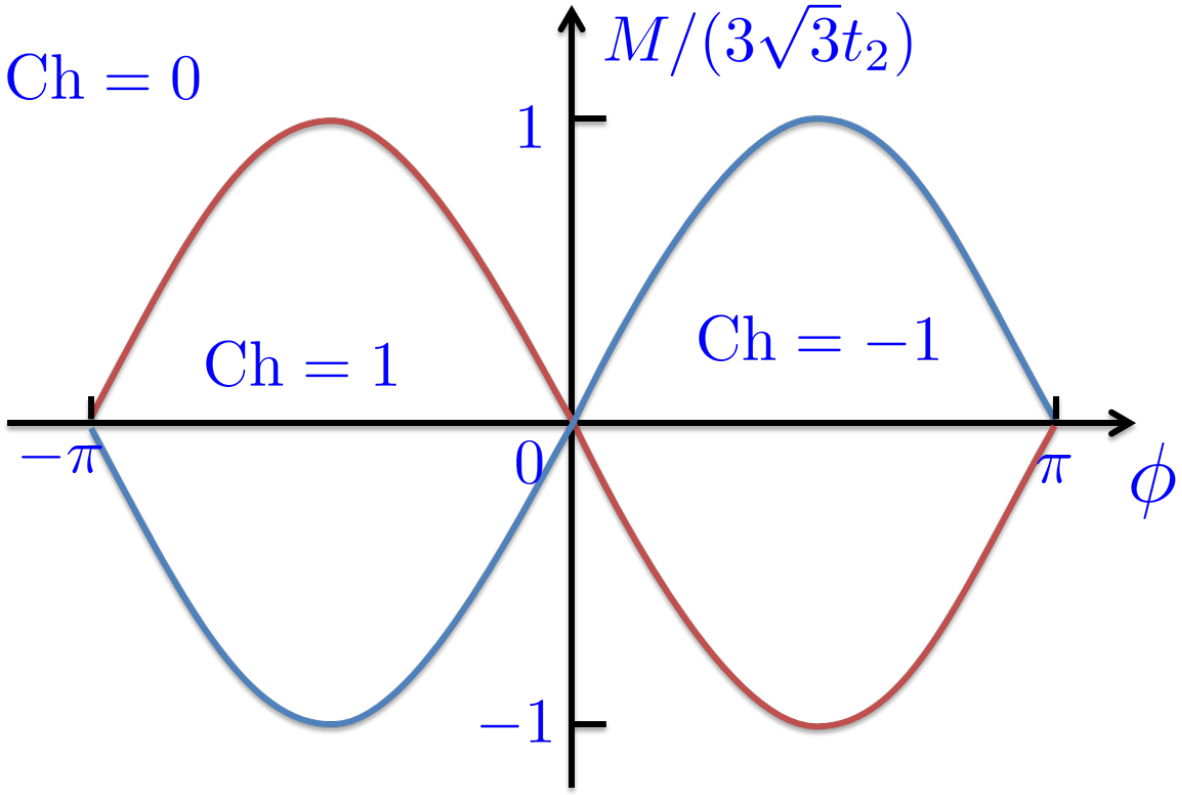


Figure 21: The Chern number for Haldane model

8.2.3 Chern number near a Dirac point

Developing the Hamiltonian around K, K' for the Haldane model in Eq. (24) we have:

$$\begin{aligned} H(k) &= -3t_2\cos(\phi) + \frac{3}{2}t_1\left(k_x\sigma_x + k_y\sigma_y\right) + \left(M \mp 3\sqrt{3}t_2\sin\phi\right)\sigma_z \\ &+ \frac{9}{4}t_2\cos(\phi)\left(k_x^2 + k_y^2\right) + \frac{9}{4}\sqrt{3}\sin\phi t_2\left(k_x^2 + k_y^2\right)\sigma_z \end{aligned} \tag{32}$$

where the $\mp$ is for K, K' respectively.

The linear part (divided by $3/2t_2$ with the relation $\left(\left(M \mp 3\sqrt{3}t_2\sin\phi\right)/(3/2t_2) = m_\pm\right)$ gives

$$\begin{pmatrix} m_\pm & k_x - ik_y \\ k_x + ik_y & -m_\pm \end{pmatrix} = k_x\sigma_1 + k_y\sigma_2 + m_\pm\sigma_3$$

the Berry curvature near the massive Dirac point is given by

$$F_{xy} = \frac{m}{2\left(k^2 + m^2\right)^{3/2}}$$

So that we can easily calculate the Chern number:

$$\mathrm{Ch} = \frac{1}{2\pi}\int \frac{m}{2\left(k^2 + m^2\right)^{3/2}} 2\pi k dk = \frac{m^3}{\left(m^2\right)^{3/2}}\int_0^\infty \frac{dy}{(1+y)^{3/2}} = \frac{\mathrm{sign}(m)}{2}$$

The Chern number depends on the sign of m, and in the phase diagram the lines in fig 21represent the values of M and ϕ where m is 0 and changes sign across them.

Notice that the Chern number that we got from the the analysis of the massive dirac point is half integer. We prove however that it has to be a n integer. That occurs because we did not treat properly the other parts of the spectrum. The contribution of the other parts of the spectrum that must convert the Chern number to be an integer is called "the contribution of spectator fermions". The change of the mass, however, at the dirac points give correctly the change of the Chern number. it jumps by one when the mass change its sign.

In order to determine the contribution of the spectator fermions (that in principle could be positive or negative) we examine the system in simple cases. In the so cold atomic limit when $M \to \infty$ we expect that system will not conduct at all, as there is a large gap and the electrons are localized on the atomic level. Thus for large M we expect that Ch $= 0$. The changes in the value of m gives the phase diagram of Fig. 21.

8.2.4 Explicit solution of edge mode in a massive Dirac spectrum

We show that a massive dirac spectrum may have a Chern number that is not zero. From our studies of the Hall effect we expect that when the Chern number is not zero the system will have edge modes. Unlike the Hall effect the simple interpretation in terms of skipping orbits on the edges does not exist here.

In this subsection we will construct an explicit solution of the edge modes: In order to do so we assume that we have a semi infinite system for positive x and write H as:

$$H = H_x + H_y,$$

examine Eq. (32) we find

$$H_x = \left(C - Dk_x^2\right) + \left(m - Bk_x^2\right)\sigma_z + Ak_x\sigma_x$$

$$H_y = -Dk_y^2 - Bk_y^2\sigma_z + Ak_y\sigma_y.$$

We want to solve this Hamiltonian on the edge of the model. We notice that the Dirac cone have a symmetry under rotations around the x axis which allows us to set initially

$$k_y = 0, H_y = 0$$

to assume that there is a solution at $E = 0$ and then treat H_y perturbatively. Then the Hamiltonian is:

$$\begin{pmatrix} C + D\partial_x^2 + m + B\partial_x^2 & iA\partial_x \\ -iA\partial_x & C + D\partial_x^2 - m - B\partial_x^2 \end{pmatrix}\psi_0 = E\psi_0.$$

We use an anzats

$$\psi_0 = \overline{a}e^{\lambda x}$$

with $\overline{a}$ being a two component vector. This yields

$$\left(m + B\lambda^2\right)\sigma_z\overline{a} = -i\sigma_x\overline{a}A\lambda,$$

multiplying by $i\sigma_x$ we find

$$\left(m + B\lambda^2\right)\sigma_y\overline{a} = \overline{a}A\lambda.$$

The eigenvalues of σ_y are

$$a_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\pi/4} \\ \mp e^{-i\pi/4} \end{pmatrix}, \sigma_y a_\pm = \pm a_\pm.$$

Setting these in the equation we find

$$\left(m + B\lambda^2\right) a_\pm = \pm A\lambda a_\pm.$$

We have 2 quadratic equations with 4 solutions. 2 of them are

$$\lambda_{1,2} = \frac{1}{2B}\left(A \pm \sqrt{A^2 - 4MB}\right)$$

(for the other two take $A \to -A$) thus the eigenvectors are

$$\psi_0 = \left(ae^{\lambda_1 x} + be^{\lambda_2 x}\right) a_+ + \left(ce^{-\lambda_1 x} + de^{-\lambda_2 x}\right) a_-$$

We impose a boundary condition $\psi(0) = 0 \Rightarrow b = -a, d = -c$. The normalizability condition of the wave function will prevent ψ from exploding at infinity.

If $Re\left(\lambda_1\right) > 0$ and $Re\left(\lambda_2\right) < 0$ or vice versa the normalization can't be achieved without setting all coefficients to 0, hence in this situation there is no solution.

However, if $Re\left(\lambda_1\right) < 0$ and

$$\text{sign}(Re(\lambda_1)) = \text{sign}(Re(\lambda_2)),$$

which is satisfied if

$$\sqrt{A^2 - 4MB} < |A|$$

Then we have two options:

$$A/B < 0 \Rightarrow Re(\lambda_1, \lambda_2) < 0$$

$$A/B > 0 \Rightarrow Re(\lambda_1, \lambda_2) > 0$$

Giving solutions

$$\psi_0 = a\left(e^{\lambda_1 x} - e^{\lambda_2 x}\right) a_+, A/B < 0, Re(\lambda_1, \lambda_2) < 0$$

$$, c\left(e^{-\lambda_1 x} - e^{-\lambda_2 x}\right) a_-, A/B > 0, Re(\lambda_1, \lambda_2) > 0$$

We found that we have a 1D edge state with a linear dispersion relation if we perturb around it in the y direction.

$$\Delta E = \langle\psi_0|H_y|\psi_0\rangle \to Ak_y$$

8.3 Topological insulators and Spin-orbit coupling, Kane and Mele Model (2005))

Given some magnetic field induced by movement in an electric field

$$\overrightarrow{B} = -\frac{\overrightarrow{v}}{c} \times \overrightarrow{E}$$

We add a spin-orbit interaction (?sign)

$$H_{so} = -\frac{1}{2} g\mu_B \overrightarrow{\sigma} \cdot \overrightarrow{B} \Rightarrow \frac{\hbar e}{2m^2c^2} \overrightarrow{\sigma} \cdot \overrightarrow{p} \times \overrightarrow{E} \frac{1}{2}$$

where the $\frac{1}{2}$ factor comes from an exact development of this term from the Dirac equation. In the case of the atom with a radial field with term will be proportional to $\sigma \cdot L$.

In matter SO coupling can be a significant effect due to the strong fields in the locality of ions, as opposed to freely propagating fermions. We can write

$$H_{so} = \frac{1}{2m}\left\{\frac{e}{c}\overrightarrow{p}\cdot\left(\frac{\hbar}{4mc}\overrightarrow{\sigma}\times\overrightarrow{E}\right)+\frac{e}{c}\left(\frac{\hbar}{4mc}\overrightarrow{\sigma}\times\overrightarrow{E}\right)\cdot\overrightarrow{p}\right\}$$

and identify $\overrightarrow{A}_{so} \equiv \frac{\hbar}{4mc}\overrightarrow{\sigma}\times\overrightarrow{E}$.

The AB flux from a field passing through a ring of radius R is

$$A_\theta = \frac{\Phi}{2\pi R}$$

If we want an electric field which decays in the same manner we can generate it with a charged wire. This will affect different spin states in the opposite direction.

Where can such electric field arise? It is natural to assume that there will be an abundance of positive charge around the ions giving an effective negative charge between them

(drawing of graphene with charge distribution)

This effect can be introduced by adding the following term to the Hamiltonian

$$i\lambda_R\sum_{ij}c_i^\dagger\left(\vec{s}\cdot(\vec{d_i}\times\vec{d_j})\right)c_j$$

With $\vec{d}_{i(j)}$ are in-plane vectors connecting the next nearest neighbor i,j through a common neighbor. Each band produces two edge modes for the two spin states. These are protected from scattering off each other from time reversal symmetry (there is no magnetic field in the problem). However, such states from different bands can scatter off each other which usually destroys the conductance which can be produced by them.