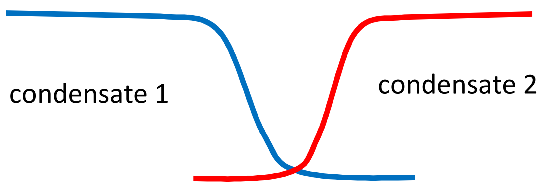
Exercise #3

Due date: 11/05/2014

1. **Josephson relation.** Consider the real-time action of two decoupled condensates

$$S_{GL} = \sum_{a=1,2} \int_0^t dt[-i\psi_a^+ \partial_t \psi_a + u(|\psi_a|^2 - \rho_0)^2]$$

which are close enough such there is a finite probability for a boson to tunnel from one condensate to the other (see figure).




This process is described by the following term $\delta S = -E_J \int dt\, \psi_1^+ \psi_2 + \mathrm{H.c.}$. (Note that for simplicity we have neglected the spatial modulations with in the condensates, which may have an important effect on the Josephson junction, as we will see in a future exercise).

   a. Write the action in terms of the complex fields $\psi_a$ in a polar representation, *i.e.,* $\psi_a = \sqrt{\rho_a}\, e^{i\theta_a}$ .
   b. Perform a canonical transformation to the sum and difference of phases $\theta_\pm = \frac{\theta_1 \pm \theta_2}{\sqrt{2}}$. What are the conjugate fields of $\theta_\pm$ ? For large $E_J$ the field $\theta_-$ is quenched, what is the physical consequence of that?
   c. Add to the action the potential term $eV\rho_-$, where $\rho_- = \rho_1 - \rho_2$, and obtain the classical equations of motion for the current $I = e\partial_t \rho_-$ and the phase $\theta_-$ ? These equations are known as the Josephson relation. How does the supercurrent $I$ depends on the voltage difference?

2. **Persistent currents in a superconducting ring.**
Consider the static GL free energy of a superconductor in a ring geometry

$$f_{GL} = a^2 \int_0^{2\pi} d\phi\, R\left[-\psi^+ \frac{(\nabla - ie^\star \boldsymbol{A})^2}{2m^\star} \psi + u(|\psi|^2 - \rho_0)^2\right].$$

Here $\phi$ is the angular coordinate along the ring, $R$ is the ring's radius, $a^2$ is the ring's cross section, $\psi$ is the cooper pair field and we have coupled the superconductor to a magnetic flux in the center of the ring, such that $\boldsymbol{A} = \frac{\Phi}{2\pi R}\, \hat{\phi}$ .

   a. Use the polar representation of the superconductor in terms of the density $\rho$ and the phase $\theta$ and find the physical constraint on the phase $\theta$ due to the periodic boundary conditions.

**b.** Neglecting fluctuations in $\rho$, find the free energy of the system as a function of the flux $\Phi$. (hint you must minimize the GL theory to determine the angular dependence of $\theta$). Plot the energy qualitatively as a function of $\Phi$.

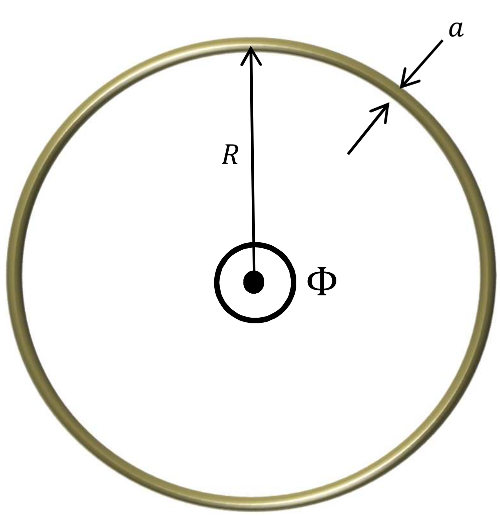
**c.** In order to obtain an expression for the canonical current in the system in the presence of the gauge field $\boldsymbol{A}$ let us add to the action the Maxwell term which describes the energy of the magnetic flux

$$F_{GL} = f_{GL} + \int d^d x \frac{1}{2} (\nabla \times \boldsymbol{A})^2$$

The equation of motion for $\boldsymbol{A}$ is given by the variation $\frac{\delta F_{GL}}{\delta \boldsymbol{A}} = 0$. Use the Maxwell equation $\boldsymbol{J} = \frac{1}{4\pi} \nabla \times \nabla \times \boldsymbol{A}$ to obtain the current density in terms of the parameters of $f_{GL}$.

**d.** Use this expression to obtain the current through some cross section of the ring as a function of $\Phi$. What is the unit of the periodicity in $\Phi$?

**e.** Bonus: What happens if the ring has one narrow area which forms a Josphson junction (see question 1)?

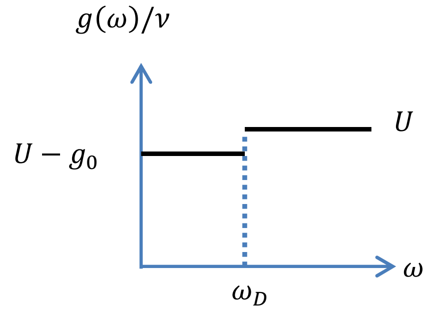



2. **Anderson-Morel gap mechanism -** The objective of this question is to resolve one of the biggest questions in conventional superconductivity. Namely, *how do two electrons which repel one another very strongly on the microscopic scale end up forming a bound-state at low energies?* To do so we will use the gap equation obtained in the tutorial (read the supplement to this exercise sheet in the course's homepage)

$$\Delta(\omega) = - \int_0^{\mu} dz \int_0^{\infty} d\xi \frac{\Gamma(\omega - z)\Delta(z)}{z^2 + \xi^2 + |\Delta_0|^2}$$

where

$$\Gamma(\omega) = \begin{cases} \nu(U - g_0) & , \ \omega < \omega_D \\ \nu\, U & , \ \omega > \omega_D \end{cases}$$




is a frequency dependent (dimensionless) interaction (it is dimensionless because we have absorbed a density of states $\nu$ when transforming from an integration over *k* to integration over $\xi$ ). $U$ is the repulsive Coulomb interaction. $g_0$ is the attractive phonon-mediated interaction that appears only for $\omega < \omega_D$ due to retardation effects (here we have chosen a notation where all couplings are positive). $\xi$ is the fermion dispersion with constant density of states $\nu$.

We will seek a solution for the gap which has the same form, i.e.,

$$\Delta(\omega) = \begin{cases} \Delta_0, & \omega < \omega_D \\ -\Delta_1, & \omega > \omega_D \end{cases}$$

where $\Delta_1 > 0$ and $\Delta_2 > 0$ are to be determined from the gap equation.

**a.** Obtain the following equations for $\Delta_0$ and $\Delta_1$

$$\Delta_0 = (g_0 - U)\Delta_0 \log\frac{\omega_D}{\Delta_0} + U\Delta_1 \log\frac{\mu}{\omega_D}$$

$$\Delta_1 = -U\Delta_0 \log\frac{\omega_D}{\Delta_0} + U\Delta_1 \log\frac{\mu}{\omega_D}$$

The first (second) equation is obtained by taking $\omega < \omega_D$ ($\omega > \omega_D$).

**b.** Solve the equations. Show that

$$\Delta_0 = \omega_D e^{-\frac{1}{g^*}}$$

what is $g^*$? Explain your result physically, how does $g_e$ effect the value of $\Delta_0$? Compare to section d in question 1. The realistic regime is where $\gg g_0$ , is there a solution in this regime? Discuss the limits of superconductivity

**c.** Now let us obtain the same result from the point view of RG. Use the RG equation

$$\frac{d\tilde{g}}{d\log D} = \nu\, \tilde{g}^2$$

to determine the gap. First, write this equation in a dimensionless form using $g = N(0)\tilde{g}$. Next, Integrate the equation from $\mu$ down to $\omega_D$ with a repulsive interaction $g(\mu) = g_e$. Now add a negative contribution $g_{ph}$ and then continue the integration from $\omega_D$ down to the some cutoff $k_B T_c$ where $g(k_B T_c)$ is of order 1 and the RG equation loses its validity.