MEAN-FIELD GAP EQUATION

In this supplement we will obtain the self-consistent mean-field gap equation for the case in-which the gap function depends on frequency. To obtain the mean-field gap equation we consider a system of fermions Fermions with an attractive interaction g (see tutorial #4) and decouple it using the HS transformation into the BCS channel

$$\begin{aligned} S_{HS} = \int \frac{d\tau d^d x}{(2\pi)^{d+1}} \Bigg[\sum_{s=\uparrow,\downarrow} \bar{\psi}_s(\boldsymbol{x},\tau) \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi_s(\boldsymbol{x},\tau) \\ + \Delta(\boldsymbol{x},\tau)\bar{\psi}_\uparrow(\boldsymbol{x},\tau)\bar{\psi}_\downarrow(\boldsymbol{x},\tau) + \bar{\Delta}(\boldsymbol{x},\tau)\psi_\downarrow(\boldsymbol{x},\tau)\psi_\uparrow(\boldsymbol{x},\tau) - \frac{|\Delta(\boldsymbol{x},\tau)|^2}{g} \Bigg] \\ = \int \frac{d\tau d^d x}{(2\pi)^{d+1}} \left[\bar{\Psi}(\boldsymbol{x},\tau)\mathcal{G}^{-1}\Psi(\boldsymbol{x},\tau) - \frac{|\Delta(\boldsymbol{x},\tau)|^2}{g} + const \right] \end{aligned} \tag{1}$$

where $\bar{\Psi} = (\bar{\psi}_\uparrow, \psi_\downarrow)$ and

$$\mathcal{G}^{-1} = \begin{pmatrix} \partial_\tau - \frac{\nabla^2}{2m} - \mu & \Delta(\boldsymbol{x},\tau) \\ \bar{\Delta}(\boldsymbol{x},\tau) & \partial_\tau + \frac{\nabla^2}{2m} + \mu \end{pmatrix}$$

The mean-field solution of this action is equivalent to seeking it's saddle point, which essentially means neglecting all of the quantum fluctuation effects. To find the saddle point solution for $\Delta(\boldsymbol{x},\tau)$ we take a variation of the action with respect to $\bar{\Delta}(\boldsymbol{x},\tau)$ and average over Fermi feilds.

$$\frac{\delta S_{HS}}{\delta \bar{\Delta}} = -\frac{1}{g}\Delta(\boldsymbol{x},\tau) + \langle \psi_\uparrow(\boldsymbol{x},\tau)\psi_\downarrow(\boldsymbol{x},\tau)\rangle = 0$$

which gives

$$\Delta(\boldsymbol{x},\tau) = g(\tau)\langle \psi_\uparrow(\boldsymbol{x},\tau)\psi_\downarrow(\boldsymbol{x},\tau)\rangle = g(\tau)F(\boldsymbol{x},\tau) \tag{2}$$

where the expectation value, which we have denoted by F, is known as the anomalous Green's function. To obtain an expression for F let's look at the definition of the Green's function

$$\mathcal{G}(\boldsymbol{x},\tau) = \begin{pmatrix} G^e(\boldsymbol{x},\tau) & F(\boldsymbol{x},\tau) \\ \bar{F}(\boldsymbol{x},\tau) & G^h(\boldsymbol{x},\tau) \end{pmatrix} = \begin{pmatrix} \langle \bar{\psi}_\uparrow \psi_\uparrow \rangle & \langle \psi_\uparrow \psi_\downarrow \rangle \\ \langle \bar{\psi}_\downarrow \bar{\psi}_\uparrow \rangle & \langle \psi_\downarrow \bar{\psi}_\downarrow \rangle \end{pmatrix}$$

which is defined by the operatorial relation $\mathcal{G}^{-1}\mathcal{G} = \mathbb{1}$ or more explicitly

$$\begin{pmatrix} \partial_\tau - \frac{\nabla^2}{2m} - \mu & \Delta(\boldsymbol{x},\tau) \\ \bar{\Delta}(\boldsymbol{x},\tau) & \partial_\tau + \frac{\nabla^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} G^e(\boldsymbol{x},\tau) & F(\boldsymbol{x},\tau) \\ \bar{F}(\boldsymbol{x},\tau) & G^h(\boldsymbol{x},\tau) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

To shorten notations we will from now on omit the $\boldsymbol{x}$ and τ indices (Keep in mind that the multiplications in this matrix equation are matrix multiplications in the space of τ and $\boldsymbol{x}$.). The equation above gives us the identities

$$\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) F + \Delta G^h = 0 \tag{4}$$

$$\bar{\Delta}F + \left(\partial_\tau + \frac{\nabla^2}{2m} + \mu \right) G^h = 1 \tag{5}$$

Solving for F we obtain

$$\begin{aligned} \left[\bar{\Delta} - \left(\partial_\tau + \frac{\nabla^2}{2m} + \mu \right) \Delta^{-1} \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \right] F = 1 \rightarrow \\ F = \left[|\Delta|^2 - \Delta \left(\partial_\tau + \frac{\nabla^2}{2m} + \mu \right) \Delta^{-1} \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \right]^{-1} \Delta \end{aligned} \tag{6}$$

Using (2) we obtain the desired result, a closed equation for the gap

$$\Delta = g \left[|\Delta|^2 - \Delta \left(\partial_\tau + \frac{\nabla^2}{2m} + \mu \right) \Delta^{-1} \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \right]^{-1} \Delta \tag{7}$$

Generally, this equation is hard to solve. However, it can greatly simplify if Δ is rather flat for times longer than $t_D \sim 1/\omega_D$ and varies slowly in space. In this case we can neglect the spatial and temporal dependance of Δ in the denominator. The idea is that $\Delta \ll \omega_D$ such that for frequencies of the order of ω_D the gap Δ is just a small correction, and thus it can be substituted by Δ_0 which is it's value at $\omega = 0$. Equation (7) assumes the form

$$\Delta(\boldsymbol{x}, \tau) = g(\tau) \frac{1}{|\Delta_0|^2 - \left(\partial_\tau + \frac{\nabla^2}{2m} + \mu\right)\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)} \Delta(\boldsymbol{x}, \tau) \tag{8}$$

To solve, we transform to Matsubara frequencies and momentum space

$$\Delta(\omega) = -\frac{1}{\beta\Omega} \sum_{\omega} \sum_{\boldsymbol{k}} \frac{g(\omega - \nu)\Delta(\nu)}{\nu^2 + \xi_{\boldsymbol{k}}^2 + |\Delta_0|^2} \tag{9}$$

where ν and ω are bosonic and fermionic Matsubara frequencies, $\xi = k^2/2m - \mu$, Ω is the volume of the system and we have used the convolution theorem. Notice that we have taken g (and also Δ) to be a function of time but independent on space. This mimics a local interaction with retardation. We will demonstrate the retardation effect in the exercise.