Concepts of condensed matter physics - Exercise #5

Spring 2017

Due date: 04/07/2017

1. **The XY – sine-Gordon duality and the BKT critical behavior**

In this question you are asked to show the equivalence between the XY model to the sine-Gordon model:

$$S_{SG} = \frac{c}{2}\int d^2x\ (\nabla\theta)^2 - g\int d^2x\ \cos\theta.$$

Where θ is a non-compact real scalar field.

a. Expand $Z_{SG} = \int D\theta\ e^{-S_{SG}}$ in powers of g explicitly and show that it has the form

$$Z_{SG} = \sum_{n=0}^{\infty}\frac{\left(\frac{g}{2}\right)^{2n}}{(n!)^2}\prod_{j=1}^{2n}\int d^2x_j \left\langle \exp\left(i\sum_{j=1}^{2n}(-1)^j\theta(x_j)\right)\right\rangle$$

Where the $\langle\rangle$ brackets denote averaging with the free part $S_0 = \frac{c}{2}\int d^2x\ (\nabla\theta)^2$.

Hint: recall that the free part is transnationally invariant such that $\langle(\prod_{a=1}^{N} e^{i\theta(x_a)})(\prod_{b=N+1}^{N+M} e^{-i\theta(x_b)})\rangle$ is non-zero only for $N = M$.

b. Using the properties of the Gaussian average, namely

$$\langle e^A\rangle = e^{\frac{1}{2}\langle A^2\rangle}$$

for A which is a linear combination of the field θ, and the following identity

$$\langle\left(\theta(x) - \theta(x')\right)^2\rangle = \frac{C(x-x')}{c} = \frac{1}{2\pi c}\log\left|\frac{x-x'}{\xi}\right|,$$

show that the partition function may be written as follows

$$Z_{SG} = \sum_{n=0}^{\infty}\frac{\left(\frac{g}{2}\right)^{2n}}{(n!)^2}\prod_{j=1}^{2n}\int d^2x_j \exp\left(\frac{1}{2c}\sum_{j<i}^{2n}\sigma_i\sigma_j C(x_i - x_j)\right)$$

where σ_i denotes the sign of the vortex and ξ is a short length cutoff. This is exactly the partition function of the Coulomb gas obtained in class!

c. Repeat the derivation of the RG differential equations near the BKT transition, namely

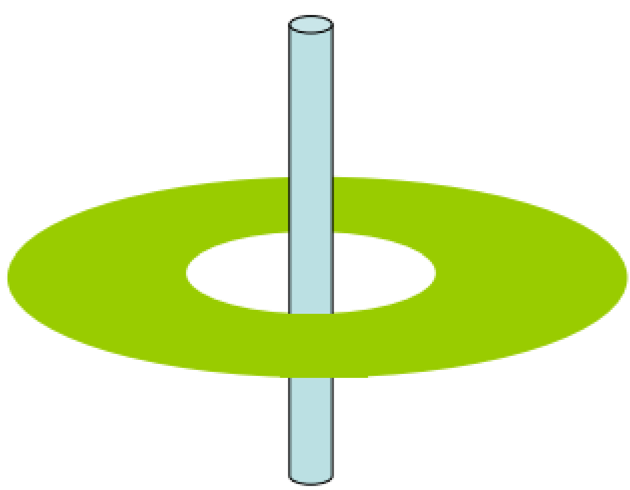
$$\frac{dy}{dl} = xy\,; \;\frac{dx}{dl} = y^2$$

What are x and y in terms of c and g? (Here $l = \log\frac{r}{\xi}$)

d. Use the above equations to determine the screening length ξ_+ on the disordered side close to the transition. Do this by estimating the value of the running parameter l at which x and y reach order 1. Explain physically why ξ_+ is the screening length.

e. Obtain the superfluid stiffness J as a function of $t = T - T_c$ and show that it has a universal jump at T_c.

2. Laughlin's argument and a preview to the fractional quantum Hall effect – Consider a quantum Hall state on an annulus, as shown in the figure below.



Imagine threading magnetic flux through the hole.

a. Show, using classical electrodynamics, that a charge flows from the inner edge to the outer edge as a result of changing the flux.

b. Consider the situation where the flux is increased very slowly from 0 to ϕ_0. Relate the total charge transferred between the edges during the process to the Hall conductance σ_{xy}.

c. Use the above argument and the known properties of the Landau levels to deduce σ_{xy} in cases where an integer number of Landau levels are filled (neglecting interactions). What can you say about the robustness of these results in the presence of interactions?

d. What is the charge that moved from the interior to the exterior if $\sigma_{xy} = \frac{e^2}{3h}$ (this situation corresponds to the $\nu = \frac{1}{3}$ fractional quantum Hall state, observed in experiments). Use the previous sections, and the adiabatic theorem to deduce that the quasiparticles carry fractional charges.

3. **A simple tight-binding model for a 2D Chern insulator -** Discuss σ_{xy} for spin-less particles on a square lattice model that has the following Hamiltonian

$$H = \sum_{\boldsymbol{k}} (\psi_s^+(\boldsymbol{k}) \quad \psi_p^+(\boldsymbol{k}))\ \hat{H}(\boldsymbol{k})\ \begin{pmatrix} \psi_s(\boldsymbol{k}) \\ \psi_p(\boldsymbol{k}) \end{pmatrix}, \text{ where}$$

$$\hat{H}(\boldsymbol{k}) = A\left(\sin k_x \tau_x + \sin k_y \tau_y\right) + \left(m - t\cos k_x - t\cos k_y\right)\tau_z.$$

Here the τ's are Pauli matrices acting in the orbital basis.

a. Find the corresponding real-space representation of the tight-binding Hamiltonian.

b. Discuss σ_{xy} as a function of m

c. Plot the pseudo-spin configuration for different values of $e = m/t$ -- choose them wisely.

d. Assume that the crystal exists only for $x < 0$, and that for $x > 0$ there is vacuum. Write the Schrodinger equation for the single particle solutions near the Fermi energy and (assume that $m > 0$ and that e is close to the critical value).

i. What are the boundary conditions at $x = 0$?

ii. What are the conditions for the existence of a gapless solution on the boundary?

iii. What is the decay length of the wave function?

iv. What happens to the solution at the critical value of the parameter e?

e. Now assume that the crystal exists for all x. Consider the situation where for $x < 0$ the parameter e is slightly larger than the critical value, and for $x > 0$ the parameter e is slightly smaller than it. Find the gapless 1D mode residing on the boundary.

f. Can you generalize the model to one that realizes an arbitrary Chern number?