Concepts of condensed matter physics - Exercise #6

Spring 2018

Due date: 05/07/2018

1. **Quantization of inter-layer Hall conductivity –** Consider two parallel two-dimensional conductors (e.g. two layers of graphene separated by a thin insulating buffer). In the presence of a perpendicular magnetic field the generalized conductivity tensor has the following form

$$J_i^a = \sigma_{ij}^{ab} E_j^b$$

where $a = 1, 2$ denotes the layer number and J_i^a and E_i^a are the i'th component of the in-plane current density and electric field. Show that (also) the off-diagonal Hall conductivity σ_{xy}^{12} is quantized following the arguments presented in class. Do so following these steps:

   a. Write the generalized linear response formula for this off diagonal element, $\sigma_{xy}^{12} = \frac{J_x^1}{\epsilon_y^2}$.

   b. Using the appropriate toroidal geometry, write this element in terms of derivatives of the ground state with respect to magnetic fluxes threaded through the torus' holes.

   c. Generalize the argument given in class to prove that σ_{xy}^{12} is quantized. What happens when the two layers are decoupled?

2. **A simple tight-binding model for a 2D topological insulator -** Discuss σ_{xy} for spin-less particles on a square lattice model that has the following Hamiltonian

$$H = \sum_{\boldsymbol{k}} (\psi_s^+(\boldsymbol{k}) \quad \psi_p^+(\boldsymbol{k}))\ \widehat{H}(\boldsymbol{k})\ \begin{pmatrix} \psi_s(\boldsymbol{k}) \\ \psi_p(\boldsymbol{k}) \end{pmatrix}, \text{ where}$$

$$\widehat{H}(\boldsymbol{k}) = A\left(\sin k_x \tau_x + \sin k_y \tau_y\right) + \left(m - t \cos k_x - t \cos k_y\right)\tau_z.$$

Here the τ's are Pauli matrices acting in the orbital basis.

   a. Find the corresponding real-space representation of the tight-binding Hamiltonian.

   b. Discuss σ_{xy} as a function of m

c. Plot the pseudo spin configuration as a function of $e = m/t$ for different values of e_a -- choose them wisely.

d. Assume that the crystal exists only for $x < 0$, and that for $x > 0$ there is vacuum. Write the Schrodinger equation for the single particle solutions near the Fermi energy and (assume that $m > 0$ and that e is close to the critical value).

    i. What are the boundary conditions at $x = 0$?

    ii. What are the conditions for the existence of a gapless solution on the boundary?

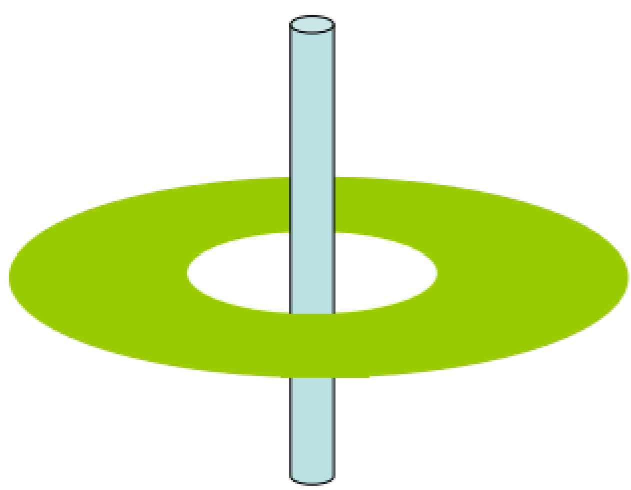
    iii. What is the decay length of the wave function?

    iv. What happens to the solution at the critical value of the parameter e?

e. Now assume that the crystal exists for all x. Consider the situation where for $x < 0$ the parameter e is slightly larger than the critical value, and for $x > 0$ the parameter e is slightly smaller than it. Find the gapless 1D mode residing on the boundary.

f. Can you generalize the model to one that realizes an arbitrary Chern number?

3. **Laughlin's argument and a preview to the fractional quantum Hall effect –** Consider a quantum Hall state on an annulus, as shown in the figure below.



Imagine threading magnetic flux through the hole.

a. Show, using classical electrodynamics, that a charge flows from the inner edge to the outer edge as a result of changing the flux.

b. Consider the situation where the flux is increased very slowly from 0 to ϕ_0. Relate the total charge transferred between the edges during the process to the Hall conductance σ_{xy}.

c. Use the above argument and the known properties of the Landau levels to deduce σ_{xy} in cases where an integer number of Landau levels are filled (neglecting interactions). What can you say about the robustness of these results in the presence of interactions?

d. What is the charge that moved from the interior to the exterior if $\sigma_{xy} = \frac{e^2}{3h}$ (this situation corresponds to the $\nu = \frac{1}{3}$ fractional quantum Hall state, observed in experiments). Use the previous sections, and the adiabatic theorem to deduce that the quasiparticles carry fractional charges.