# **Concepts of condensed matter physics -** Exercise #4

Spring 2018

Due date: 07/06/2018

**1.** In this question you will re-derive the BCS theory studied in class and use it to calculate a few properties of superconductors. Our starting point is the Hamiltonian of electrons interacting via an attractive point contact interaction ($g > 0$):

$$\hat{H} = \int d^3x \left[ \sum_{s=\uparrow,\downarrow} c_s^\dagger(x) \left( -\frac{\nabla^2}{2m} - \mu \right) c_s(x) - g c_\uparrow^\dagger(x) c_\downarrow^\dagger(x) c_\downarrow(x) c_\uparrow(x) \right].$$

a. Write the Hamiltonian in momentum space, and then transform it to a quadratic form by assuming the order parameter $\Delta = \frac{g}{\Omega} \sum_k c_{-k\downarrow} c_{k\uparrow}$ is weakly fluctuating (i.e., by performing mean field). Here $\Omega$ is the system's volume.

b. Diagonalize the quadratic Hamiltonian and find the spectrum of excitations.

c. What is the ground state wavefunction? What is the ground state energy? Show that taking $\Delta = 0$ we recover the known non-interacting ground state energy.

d. Using the ground state wavefunction, write a self-consistent equation ("the BSC gap equation") for $\Delta$. Solve this equation for small values of $g$.

e. Extend the gap equation to finite temperatures by promoting the average with respect to the ground state to a thermal average.

f. Find the critical temperature $T_c$ above which superconductivity is destroyed. What is the value of $\Delta$ slightly below the transition?

**2.** In this question you will find the spectrum of the above BCS theory in the presence of spin-orbit and Zeeman coupling. In **one-dimension**, the Hamiltonian is given by:

$$\hat{H} = \int dx \left[ \sum_{s,s'} c_s^\dagger(x) \left( -\frac{\partial_x^2}{2m} + iu\sigma_z^{ss'} \partial_x + B\sigma_x^{ss'} - \mu \right) c_{s'}(x) - g c_\uparrow^\dagger(x) c_\downarrow^\dagger(x) c_\downarrow(x) c_\uparrow(x) \right]$$

a. First, neglecting $g$, diagonalize the quadratic Hamiltonian by going to momentum space. Draw the spectrum (qualitatively) – how does the spin-orbit and Zeeman terms alter the parabolic spectrum of free electrons ($E = \frac{k^2}{2m} - \mu$).

b. Introducing finite $g$ and performing a mean field approximation, write a quadratic Hamiltonian of the form:

$$H = E_0 + \sum_k \vec{\Psi}_k^\dagger h_{BDG}(k) \vec{\Psi}_k,$$

With $\vec{\Psi}_k = \begin{pmatrix} c_\uparrow(k) \\ c_\downarrow(k) \\ c_\downarrow^\dagger(-k) \\ -c_\uparrow^\dagger(-k) \end{pmatrix}$ and $h_{BDG}(k) = \begin{pmatrix} A(k) & D(k) \\ D^\dagger(k) & -\sigma_y A^*(-k)\sigma_y \end{pmatrix}$.

Find the $2 \times 2$ matrices $A(k)$ and $D(k)$.

c. Diagonalize $h_{BDG}$ and find the spectrum of excitations.

d. Show that by changing the ratio $\Delta / B$, we reach a point in which the gap to excitations closes. Draw the spectrum at this point.

3. **Superconductivity on the surface:** In this question you will find that above $H_{c2}$ there is a range of fields for which superconductivity can survive on the surface. Consult "Introduction to superconductivity", by M. Tinkham, page 135.

   a. Start from the Ginzburg-Landau theory of a superconductor and neglect non-quadratic orders near the critical point. Write down the corresponding equations of motion, and using an analogy to the Schrodinger equation, find the critical field $H_{c2}$, above which superconductivity cannot nucleate in the interior of the sample. Write the result in terms of $\phi_0$ and $\xi$. Can you explain the result qualitatively?

   b. Consider the same physical setting with an edge at $x = 0$ (such that for $x > 0$ there is an insulator). Show that the boundary conditions take the form

$\left(\frac{\nabla}{i} - \frac{2\pi A}{\phi_0}\right)\psi\Big|_n = 0$. Show that one can automatically satisfy this boundary condition by considering an auxiliary potential, containing a mirror image of the original potential in the insulating region. Does this affect the solution from part (a) well inside the superconductor (i.e., for $|x| \gg \xi$)?

c. Argue, using the auxiliary potential, that very close to the surface one can find a solution with lower energy, making the critical field higher near the surface.