Concepts of condensed matter physics - Exercise #5

Spring 2018

Due date: 21/06/2018

1. **Integer Quantum Hall effect in Graphene**

   Consider the low-energy effective theory of graphene in a perpendicular magnetic field.

   a. Find the Landau-levels in graphene close to half-filling. To do so, use the full low energy Hamiltonian, and write it in real space. Then introduce the electromagnetic potential using the minimal substitution. Plot the density of states for positive/negative energies.

   b. What is the degeneracy of each Landau level? Is it different compared to a quadratic dispersion relation?

   c. So far, we have ignored the spin degrees of freedom. Discuss qualitatively what happens when these are included. How does the Zeeman effect change the picture?

2. **The XY – sine-Gordon duality and the BKT critical behavior**

   In this question you are asked to show the equivalence between the XY model to the sine-Gordon model:

$$S_{SG} = \frac{c}{2}\int d^2x\ (\nabla\theta)^2 - g\int d^2x\ \cos\theta\,,$$

   where θ is a non-compact real scalar field, and re-derive the RG equations near the BKT transition.

   a. Expand $Z_{SG} = \int D\theta\ e^{-S_{SG}}$ in powers of g explicitly and show that it has the form

$$Z_{SG} = \sum_{n=0}^{\infty} \frac{\left(\frac{g}{2}\right)^{2n}}{(n!)^2} \prod_{j=1}^{2n} \int d^2x_j \left\langle \exp\left(i\sum_{j=1}^{2n} (-1)^j \theta(x_j) \right) \right\rangle$$

Where the ⟨⟩ brackets denote averaging with the free part $S_0 = \frac{c}{2}\int d^2x\ (\nabla\theta)^2$.

Hint: recall that the free part is transnationally invariant such that $\langle(\prod_{a=1}^{N} e^{i\theta(x_a)})(\prod_{b=N+1}^{N+M} e^{-i\theta(x_b)})\rangle$ is non-zero only for $N = M$.

b. Using the properties of the Gaussian average, namely

$$\langle e^A\rangle = e^{\frac{1}{2}\langle A^2\rangle}$$

for A which is a linear combination of the field θ, and the following identity

$$\langle\left(\theta(x) - \theta(x')\right)^2\rangle = \frac{C(x - x')}{c} = \frac{1}{2\pi c}\log\left|\frac{x - x'}{\xi}\right|,$$

show that the partition function may be written as follows

$$Z_{SG} = \sum_{n=0}^{\infty}\frac{\left(\frac{g}{2}\right)^{2n}}{(n!)^2}\prod_{j=1}^{2n}\int d^2x_j \exp\left(\frac{1}{2c}\sum_{j<i}^{2n}\sigma_i\sigma_j C(x_i - x_j)\right)$$

where σ_i denotes the sign of the vortex and ξ is a short length cutoff. This is exactly the partition function of the Coulomb gas obtained in class!

c. Repeat the derivation of the RG differential equations near the BKT transition, namely

$$\frac{dy}{dl} = xy\ ;\ \frac{dx}{dl} = y^2\ .$$

What are x and y in terms of c and g? (Here $l = \log\frac{r}{\xi}$)

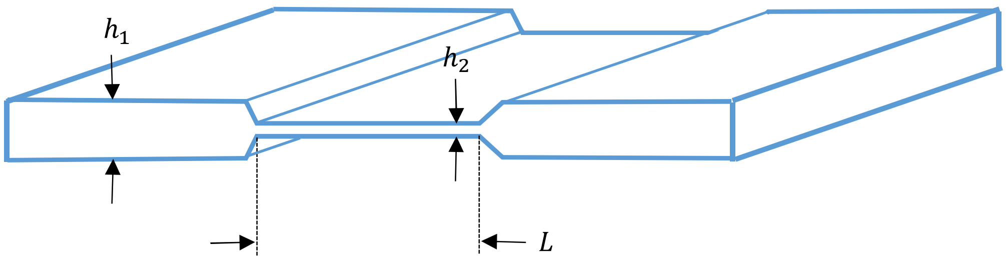
d. Use the above equations to determine the screening length ξ_+ on the disordered side close to the transition. Do this by estimating the value of the running parameter l at which x and y reach order 1. Explain physically why ξ_+ is the screening length.

e. Obtain the superfluid stiffness J as a function of $t = T - T_c$ and show that it has a universal jump at T_c.

3. Trenched 2D superconductor

Consider a two dimensional superconducting slab which has a spatially dependent height (See figure).

$$h(x) = \begin{cases} h_2, & |x| < L/2 \\ h_1, & otherwise \end{cases}$$



Assume $h_2 < h_1 < \xi$, where ξ is the coherence length, such that the superconductor is effectively two-dimensional everywhere (and $h_1 > h_2$).

a. Given the corresponding Ginzburg-Landau theory

$$F_{\rm GL} = \int dxdyh(x)\left[\frac{1}{2m}|\nabla\Psi|^2 - \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4\right],$$

i. Write down the coherence length ξ and the core energy of a vortex.

ii. How does the core energy depend on h and ξ?

iii. In which region of the sample is the fugacity of a vortex, y, largest?

iv. What is the superfluid stiffness J?

b. Draw the RG flow diagram of the inverse superfluid stiffness $1/J$ and vortex fugacity, y, for a two-dimensional superfluid at finite temperature. How is this flow modified if the vortices are bound to move along a line at $x = 0$?

c. Qualitatively draw the starting point (microscopic scale) for the flow of each region. Roughly, up to what length scale, L^*, does this flow diagram describe the evolution of the couplings in the thin region ($|x| < L/2$) of the slab?

d. How do you expect the flow to change for length scales larger than L^*?

e. Depending on the initial conditions what are the possible scenarios and what is their physical meaning?