Concepts in Condensed Matter Physics: Tutorial IV

The BCS Theory of Superconductivity

In the lectures you saw a phenomenological analysis of superconductors. In particular, you saw that given some empirical results, many additional predictions can be made using the Ginzburg-Landau formalism. Historically, this approach has been very successful. However, the theory is still incomplete without a microscopic explanation. In this tutorial we will fill this gap by reviewing the famous BCS theory, established by Bardeen, Cooper, and Schrieffer about 50 years after the initial discovery of superconductivity. Then, we will connect the microscopic picture to the phenomenological one by deriving the Ginzburg-Landau theory.

1 Preliminaries

The BCS theory is based on two important insights:

1. Cooper's realization that attractive interactions between electrons in the vicinity of the Fermi-energy favor the formation of bound states made of two electrons, called cooper pairs.

2. The result that interaction between two electrons, mediated by phonons, can be attractive.

Once one realizes these things, the next step is to assume that the ground state of a many body system with attractive interactions can be described in terms of a condensate of such weakly interacting pairs. The pairs satisfy Bose statistics, giving rise to a physics similar to that of a superfluid, yet different due to the fact that the bosons are now charged. We will see that this picture is capable of explaining superconductivity. We begin by elaborating on the above two crucial points:

1.1 Attractive interaction for fermions

We begin by analyzing the possibility of having attractive interactions between electrons. As it turns out, such electron-electron interaction can originate from electron-phonon coupling, namely mediated by phonons. We will only discuss a very qualitative picture here, but this can be made more rigorous. The idea is that an electron can pass at some time near an ion and distort it from its equilibrium position by attraction. Since the ions are much slower than the electrons $\omega_D^{-1} \gg E_F^{-1}$ the ion will not relax to equilibrium long after the electron passed through. The result is that for a long time (in the electronic scale), at the distorted position, on the path of the original electron, there is a concentration of positive charge attracting other electrons. The net effect is an attractive interaction between the two electrons (which in reality is mediated by the phonons) that overcomes the Coulomb repulsion.

1.2 Formation of bound states: Cooper pairs

To see that pairs of electrons can form bound states, we examine the following toy model. We imagine two electrons, with an attractive interaction between them, on top of a Fermi sea. The two additional electrons do not interact with the Fermi sea electrons, but feel their presence via the Pauli exclusion principle. We would like to find the corresponding two-electron eigenstates.

We assume that the total momentum is zero and that the spin-part of the wavefunction is antisymmetric, such that

$$\psi\left(\mathbf{r}_1, \mathbf{r}_2\right) \propto \sum_{\mathbf{k}} \left[g_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}\right] \left(\frac{|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle}{\sqrt{2}}\right) . \tag{1}$$

The Schrodinger equation for the two electrons is

$$\left[H_0(\mathbf{r}_1) + H_0(\mathbf{r}_2) + V(\mathbf{r}_1 - \mathbf{r}_2)\right] \psi(\mathbf{r_1},\mathbf{r_2}) = E\psi(\mathbf{r_1},\mathbf{r_2}) , \tag{2}$$

and plugging in our anzats, Eq.(1), we find

$$\sum_{\mathbf{k}} g_{\mathbf{k}} \left[H_0(\mathbf{r}_1) + H_0(\mathbf{r}_2) + V(\mathbf{r}_1 - \mathbf{r}_2)\right] e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} = E \sum_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} . \tag{3}$$

For a translation-invariant system H_0 is diagonal in k-space so we can write $H_0(r_i) e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} = \epsilon_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}$ leading to

$$\sum_{\mathbf{k}} g_{\mathbf{k}} \left[2\epsilon_{\mathbf{k}} + V(\mathbf{r}_1 - \mathbf{r}_2)\right] e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} = E \sum_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} , \tag{4}$$

which we can now multiply by $e^{-i\mathbf{q}\cdot(\mathbf{r_1}-\mathbf{r_2})}$ and integrate, over r_1 for example, to get

$$2g_{\mathbf{q}}\epsilon_{\mathbf{q}}\Omega + \sum_{\mathbf{k}} g_{\mathbf{k}} \int d\mathbf{r}_1 V\left(\mathbf{r}_1 - \mathbf{r}_2\right) e^{i(\mathbf{k}-\mathbf{q})(\mathbf{r}_1-\mathbf{r}_2)} = E g_{\mathbf{q}}\Omega \ , \tag{5}$$

Changing integration variables to $\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$ and defining

$$V_{\mathbf{k},\mathbf{q}} = \frac{1}{\Omega}\int d\mathbf{r} V\left(\mathbf{r}\right) e^{i(\mathbf{k}-\mathbf{q})\mathbf{r}} \ , \tag{6}$$

we find the following equation:

$$\sum_{\mathbf{k}} g_{\mathbf{k}} V_{\mathbf{k},\mathbf{q}} = g_{\mathbf{q}}\left(E - 2\epsilon_{\mathbf{q}}\right) \tag{7}$$

Obviously the energies depend on the form of the interaction $V_{\mathbf{k},\mathbf{q}}$, but since the phenomena we want to see should be universal for Fermions with attractive interactions, we can pick a simple form such as

$$V_{\mathbf{k},\mathbf{q}} = \begin{cases} -V & E_F < \epsilon_{\mathbf{k}}, \epsilon_{\mathbf{q}} < E_F + \Delta E \\ 0 & \text{Otherwise} \end{cases} \ . \tag{8}$$

This merely means that the attractive interactions only affect electrons occupying states in a small energy shell ΔE above the Fermi-sea. Plugging this form of the interactions into Eq.(7) we find

$$-V\sum_{\mathbf{k}} g_{\mathbf{k}} = \left(E - 2\epsilon_{\mathbf{q}}\right) g_{\mathbf{q}} \ , \tag{9}$$

where the sum over $\mathbf{k}$ is restricted to $\mathbf{k}$ values satisfying the requirement in Eq.(8), namely for which $\epsilon_{\mathbf{k}}$ is withing ΔE from the fermi surface. Dividing the equation by $E - 2\epsilon_{\mathbf{q}}$ and summing of over $\mathbf{q}$ (under the same restrictions), we get

$$-\sum_{\mathbf{q}} \frac{V}{E - 2\epsilon_{\mathbf{q}}} = 1 \ . \tag{10}$$

We now transform the sum over $\vec{k}$ to and integral over energy, introducing the density of states $n(\epsilon)$ to find

$$-\int_{E_F}^{E_F+\Delta E} d\epsilon \frac{Vn(\epsilon)}{E - 2\epsilon} = 1 \ . \tag{11}$$

Since we integrate over a thin shell ΔE we can assume the DOS does not change over it and approximate it by $\nu(E_F)$. Then it can be taken out of the integral which is now easily solved to give

$$\frac{Vn(E_F)}{2}\log\left(\frac{E-2(E_F+\Delta E)}{E-2E_F}\right)=1\ , \tag{12}$$

$$\Rightarrow E=2E_F-2\Delta E e^{-\frac{2}{Vn(E_F)}}. \tag{13}$$

Remember that until now we considered adding two electron on top of the Fermi-sea. Let's now think instead of taking two electron from the Fermi-surface and putting them into this cooper pair bound state. Obviously removing the from the Fermi-surface saves $2E_F$ of energy. Therefore, it is clear that taking pairs of electron from the Fermi-sea and putting them into cooper pairs saves energy in the amount of $2\Delta E e^{-\frac{2}{Vn(E_F)}}>0$, and is thus energetically favorable. This result demonstrates a general principle: if there is an attractive interaction (which can be arbitrarily small) between the electrons, there is an instability towards the formation of cooper pairs. Therefore one can then assume that the ground state of a many-body system with attractive interactions is composed of many weakly interacting pairs.

2 BCS theory of superconductivity

Having the above physics in mind, we postulate that as the system becomes superconducting, there is an instability toward condensation of pairs. To investigate the physics that arises from that, we assume that the ground state of a system with attractive interactions $|\Omega_s\rangle$ is characterized by a macroscopic number of pairs. This means that $\Delta=\frac{g}{\Omega}\sum_{\mathbf{k}}\langle\Omega_s|\,\psi_{-\mathbf{k},\downarrow}\psi_{\mathbf{k}\uparrow}\,|\Omega_s\rangle$, and its complex conjugate $\bar{\Delta}=\frac{g}{\Omega}\sum_{\mathbf{k}}\langle\Omega_s|\,c^{\dagger}_{\mathbf{k}\uparrow}c^{\dagger}_{-\mathbf{k},\downarrow}\,|\Omega_s\rangle$ are non-zero. We regard these quantities as the order parameters of our system.

With the above assumption, we use the usual mean field formulation to transform the interacting Hamiltonian into a quadratic one, neglecting some quantum fluctuations. We start from a system of fermions with attractive contact interactions

$$H=\sum_{\mathbf{k},\sigma}n_{\mathbf{k},\sigma}(\epsilon_{\mathbf{k}}-\mu)-\frac{g}{\Omega}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}\psi^{\dagger}_{\mathbf{k}+\mathbf{q}\uparrow}\psi^{\dagger}_{-\mathbf{k}\downarrow}\psi_{-\mathbf{k}'+\mathbf{q}\downarrow}\psi_{\mathbf{k}'\uparrow}\ . \tag{14}$$

Under our mean-field assumption, $\sum_{\mathbf{k}'}\psi_{-\mathbf{k}'+\mathbf{q}\downarrow}\psi_{\mathbf{k}'\uparrow}$, is governed by small $\mathbf{q}$'s (only very long wavelength fluctuations), and has a mean-field value about which fluctuations are small.

Therefore, we write

$$\sum_{\mathbf{k}'} \psi_{-\mathbf{k}'+\mathbf{q}\downarrow}\psi_{\mathbf{k}'\uparrow} \approx \sum_{\mathbf{k}'} \psi_{-\mathbf{k}'\downarrow}\psi_{\mathbf{k}'\uparrow} = \frac{\Omega\Delta}{g} + \underbrace{\sum_{\mathbf{k}'} \psi_{-\mathbf{k}'\downarrow}\psi_{\mathbf{k}'\uparrow} - \frac{\Omega\Delta}{g}}_{\text{Small}} ,$$

$$\sum_{\mathbf{k}} \psi^{\dagger}_{\mathbf{k}+\mathbf{q}\uparrow}\psi^{\dagger}_{-\mathbf{k}\downarrow} \approx \sum_{\mathbf{k}} \psi^{\dagger}_{\mathbf{k}\uparrow}\psi^{\dagger}_{-\mathbf{k}\downarrow} = \frac{\Omega\bar{\Delta}}{g} + \underbrace{\sum_{\mathbf{k}} \psi^{\dagger}_{\mathbf{k}\uparrow}\psi^{\dagger}_{-\mathbf{k}\downarrow} - \frac{\Omega\bar{\Delta}}{g}}_{\text{Small}} . \tag{15}$$

Plugging these into the Hamiltonian and keeping only the first order terms in the small fluctuations, we get the mean-filed Hamiltonian

$$H_{\text{MF}} = \sum_{\mathbf{k},\sigma} n_{\mathbf{k},\sigma}(\epsilon_{\mathbf{k}} - \mu) + \frac{\Omega}{g}|\Delta|^2 - \Delta \sum_{\mathbf{k}} \psi^{\dagger}_{\mathbf{k}\uparrow}\psi^{\dagger}_{-\mathbf{k}\downarrow} - \bar{\Delta}\sum_{\mathbf{k}} \psi_{-\mathbf{k}\downarrow}\psi_{\mathbf{k}\uparrow} , \tag{16}$$

which is sometimes called the Bogoliubov de-Gennes (BdG) Hamiltonian.

We have transformed the interacting Hamiltonian into a quadratic mean-field Hamiltonian that captures the correct ordering in our system. Note, however, that this form is dramatically different than the type of mean-field Hamiltonians we usually write as it doesn't conserve the number of particles. The number of particles is indeed not conserved, but the parity of that number (i.e., the number of particles mod 2) remains a good quantum number. We would now like to diagonalize the BDG Hamiltonian. To do so, we define the Nambu-spinor $\Psi_{\mathbf{k}} = \begin{pmatrix} \psi_{\mathbf{k}\uparrow} & \psi^{\dagger}_{-\mathbf{k}\downarrow} \end{pmatrix}^T$, in terms of which the Hamiltonian is given by

$$H = \frac{\Omega}{g}|\Delta|^2 + \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu) + \sum_{\mathbf{k}} \Psi^{\dagger}_{\mathbf{k}} h_{BdG} \Psi_{\mathbf{k}} ,$$

$$h_{BdG} = \begin{pmatrix} \epsilon_{\mathbf{k}} - \mu & -\Delta \\ -\bar{\Delta} & -(\epsilon_{\mathbf{k}} - \mu) \end{pmatrix} . \tag{17}$$

To explicitly see that this is correct we plug in the definition of $\Psi_{\mathbf{k}}$:

$$H = \frac{\Omega}{g}|\Delta|^2 + \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu) + \sum_{\mathbf{k}} \Psi^{\dagger}_{\mathbf{k}} h_{BdG} \Psi_{\mathbf{k}}$$

$$= \frac{\Omega}{g}|\Delta|^2 + \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu) + \sum_{\mathbf{k}} \left[(\epsilon_{\mathbf{k}} - \mu)\left(\psi^{\dagger}_{\mathbf{k}\uparrow}\psi_{\mathbf{k}\uparrow} - \psi_{-\mathbf{k}\downarrow}\psi^{\dagger}_{-\mathbf{k}\downarrow}\right) - \left(\Delta\psi^{\dagger}_{\mathbf{k}\uparrow}\psi^{\dagger}_{-\mathbf{k}\downarrow} + \bar{\Delta}\psi_{-\mathbf{k}\downarrow}\psi_{\mathbf{k}\uparrow}\right)\right]$$

$$= \sum_{\mathbf{k},\sigma} n_{\mathbf{k},\sigma}(\epsilon_{\mathbf{k}} - \mu) + \frac{\Omega}{g}|\Delta|^2 - \Delta \sum_{\mathbf{k}} \psi^\dagger_{\mathbf{k}\uparrow}\psi^\dagger_{-\mathbf{k}\downarrow} - \bar{\Delta} \sum_{\mathbf{k}} \psi_{-\mathbf{k}\downarrow}\psi_{\mathbf{k}\uparrow} \ . \tag{18}$$

Now, the matrix h_{BDG}, being hermitian, can always be diagonalized by a unitary transformation such that (assuming Δ is real)

$$U h_{BDG} U^{-1} = \begin{pmatrix} \lambda_{\mathbf{k}} & 0 \\ 0 & -\lambda_{\mathbf{k}} \end{pmatrix} \ ,$$

$$U \begin{pmatrix} \psi_{\mathbf{k}\uparrow} \\ \psi^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} \equiv \begin{pmatrix} c_{\mathbf{k},1} \\ c_{\mathbf{k},2} \end{pmatrix} \equiv \chi_{\mathbf{k}} \ . \tag{19}$$

The unitary transformation can be parametrized by

$$U = \begin{pmatrix} \cos\theta_{\mathbf{k}} & \sin\theta_{\mathbf{k}} \\ \sin\theta_{\mathbf{k}} & -\cos\theta_{\mathbf{k}} \end{pmatrix} \ , \tag{20}$$

where $\tan(2\theta_{\mathbf{k}}) = -\frac{\Delta}{\epsilon_{\mathbf{k}}-\mu}$, and the eigenvalues are $\lambda_{\mathbf{k}} = \sqrt{\Delta^2 + (\epsilon_{\mathbf{k}} - \mu)^2}$. In terms of these, the Hamiltonian takes the diagonal form

$$H = \frac{\Omega}{g}|\Delta|^2 + \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu) + \sum_{\mathbf{k}} \left(\lambda_{\mathbf{k}} c^\dagger_{\mathbf{k},1} c_{\mathbf{k},1} - \lambda_{\mathbf{k}} c^\dagger_{\mathbf{k},2} c_{\mathbf{k},2} \right) \ . \tag{21}$$

Taking $\epsilon_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m}$, we get the dispersion shown in Fig. 1a.

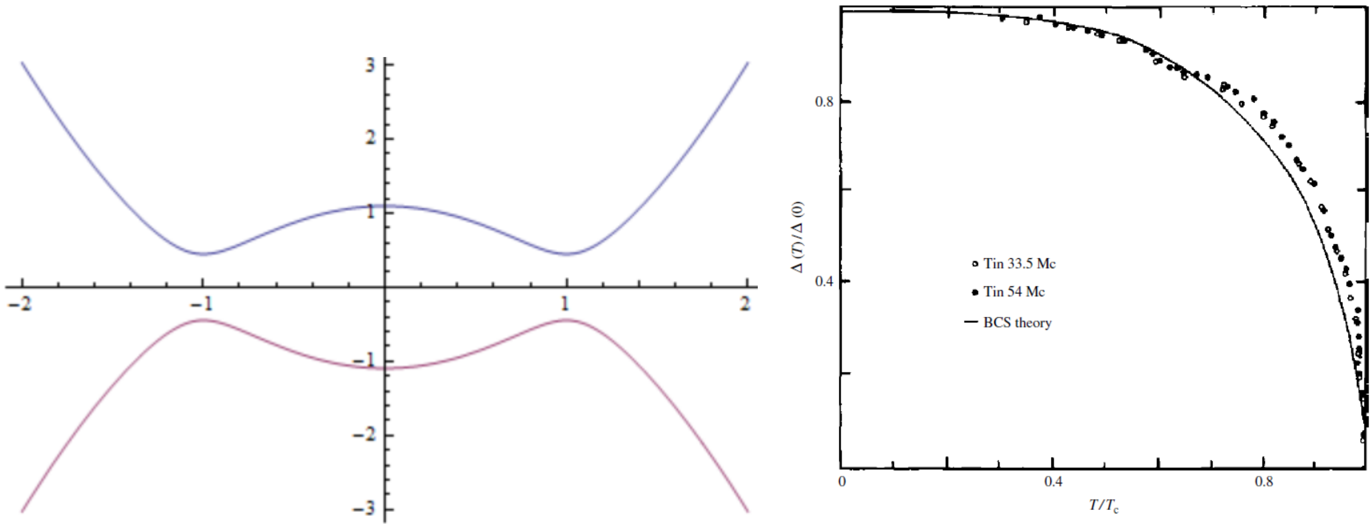
It is now simple to identify the ground state: it is the state in which all the negative energy states are occupied (all c_2 states) and the positive energy states are empty (all c_1 states), that is

$$|g.s.\rangle = \prod_{\mathbf{k}} c^\dagger_{\mathbf{k},2} c_{\mathbf{k},1} |0\rangle \propto \prod_{\mathbf{k}} \left(\cos\theta_{\mathbf{k}} - \sin\theta_{\mathbf{k}} \psi^\dagger_{\mathbf{k}\uparrow}\psi^\dagger_{-\mathbf{k}\downarrow} \right) |0\rangle \ , \tag{22}$$

where $|0\rangle$ is the vacuum of the Fock space spanned by $\psi^\dagger_{k,\sigma}$, i.e., for all k and σ, $\psi_{k,\sigma} \,|\, 0 \,\rangle = 0$, and the corresponding ground state energy is

$$E_{g.s.} = \frac{\Omega}{g}|\Delta|^2 + \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu - \lambda_{\mathbf{k}}) \ . \tag{23}$$

The excited states ware obtained from either destroying a $c_{\mathbf{k}2} = \sin(\theta_k)\,\psi_{k,\uparrow} - \cos(\theta_k)\,\psi^\dagger_{-k,\downarrow}$ quasi-particle, or create a $c_{\mathbf{k}1} = \cos(\theta_k)\,\psi_{k,\uparrow} + \sin(\theta_k)\,\psi^\dagger_{-k,\downarrow}$ quasi-particle - both with an en-



(a) (b)

Figure 1: (a) Spectrum of BdG Hamiltonian for $\epsilon_{\vec{k}} = \frac{\vec{k}^2}{2m}$, and (b) Comparison between BCS gap equation and experimental data.

ergy cost of $\lambda_{\mathbf{k}}$. Crucially, there is a gap Δ to excitations. This gap is essential for superconductivity. Recall that Δ was defined as the expectation value $\Delta = \frac{g}{\Omega}\sum_{\mathbf{k}} \langle\Omega_s|\, \psi_{-\mathbf{k},\downarrow}\psi_{\mathbf{k}\uparrow}\, |\Omega_s\rangle$. We are now in a position to write a self-consistent equation for it. All we need to do is to write the $\psi_{k,\sigma}$'s in terms of the $c_{k,i}$'s for which it is easy to compute. We find

$$\Delta = \frac{g}{\Omega}\sum_{\mathbf{k}} \langle\Omega_s|\, \psi_{-\mathbf{k},\downarrow}\psi_{\mathbf{k}\uparrow}\, |\Omega_s\rangle = -\frac{1}{2}\frac{g}{\Omega}\sum_{\mathbf{k}} \sin\left(2\theta_{\mathbf{k}}\right) = \frac{g}{\Omega}\sum_{\mathbf{k}} \frac{\Delta}{2\lambda_{\mathbf{k}}} \ . \tag{24}$$

By transforming the sum over $\mathbf{k}$ into an integral over energy, using the density of states $n(\xi)$, and recalling that the attractive interaction occurs only at a thin shell of order ω_D around the Fermi-energy, we write

$$1 = \frac{g}{2}\int_{-\omega_D}^{\omega_D} d\xi \frac{n(\xi)}{\sqrt{\Delta^2+\xi^2}} \approx gn(0)\int_0^{\omega_D} \frac{d\xi}{\sqrt{\Delta^2+\xi^2}} = gn(0)\sinh^{-1}\left(\frac{\omega_D}{\Delta}\right) \ . \tag{25}$$

We can solve this for Δ, assuming that it is small compared to ω_D, finding

$$\Delta \approx 2\omega_D e^{-\frac{1}{gn(0)}} \ , \tag{26}$$

which is indeed much smalled than ω_D.

Finally, it is instructive to find the critical temperature from this formalism. To do this we need to write the self-consistency equation at finite temperatures. We can use the machinery we already have and write Δ as a sum of Matsubara frequencies using the coherent state path integral formulation. However, since we understand the excitations of the problem 21, we can do something simpler and immediately write

$$\left\langle c^\dagger_{\mathbf{k}1} c_{\mathbf{k}1} \right\rangle = n_F(\lambda_{\mathbf{k}}) \ , \tag{27}$$

$$\left\langle c^\dagger_{\mathbf{k}2} c_{\mathbf{k}2} \right\rangle = 1 - n_F(\lambda_{\mathbf{k}}) \ . \tag{28}$$

Plugging this into the definition of Δ, we get the finite temperature self consistent equation

$$\Delta = \frac{g}{\Omega} \sum_{\mathbf{k}} \langle \psi_{-\mathbf{k},\downarrow} \psi_{\mathbf{k}\uparrow} \rangle = \frac{g\Delta}{2\Omega} \sum_{\mathbf{k}} \frac{1 - 2n_F(\lambda_{\mathbf{k}})}{\lambda_{\mathbf{k}}} = \frac{g\Delta}{2\Omega} \sum_{\mathbf{k}} \frac{\tanh\left(\frac{\beta\lambda_{\mathbf{k}}}{2}\right)}{\lambda_{\mathbf{k}}} \ , \tag{29}$$

$$\Rightarrow 1 = \frac{g}{2\Omega} \sum_{\mathbf{k}} \frac{\tanh\left(\frac{\beta\lambda_{\mathbf{k}}}{2}\right)}{\lambda_{\mathbf{k}}} . \tag{30}$$

As before, we will transform the sum over $\mathbf{k}$ into an integral over energy, again using the density of states $n(\xi)$ to obtain the well-known BCS gap equation

$$1 = gn(0) \int_0^{\omega_D} d\xi \frac{\tanh\left(\frac{\beta\sqrt{\Delta^2+\xi^2}}{2}\right)}{\sqrt{\Delta^2+\xi^2}} . \tag{31}$$

A comparison between this mean-field approximate solution and experimental measurements is shown in Fig.1b.

Above the critical temperature $\Delta = 0$, and close to the critical temperature, on the superconducting side, $\Delta \approx 0$. Therefore, We can now find the critical temperature by setting $\Delta = 0$ into the gap equation. We find

$$1 = gn(0) \int_0^{\omega_D} d\xi \frac{\tanh\left(\frac{\beta\xi}{2}\right)}{\xi} \ ,$$

$$\Rightarrow T_c = C\omega_D e^{-\frac{1}{gn(0)}} \ , \tag{32}$$

where C is some numerical factor of order 1.

To summarize this part, we now have a microscopic theory that explains the condensation of pairs and the emerging gap to excitations. However, this picture doesn't actually allow us to find the electromagnetic response of the system. To capture this part, we need to include an additional degree of freedom in our picture: the Goldstone mode associated with changing the phase of Δ. Such a treatment necessarily goes beyond the above mean field treatment, which treats Δ as a constant. This is the focus of the next section.

3 Deriving the Ginzburg-Landau theory for superconductivity

To make contact with the phenomenological analysis, and include the phase mode in the analysis, we turn to derive the Ginzburg-Landau functional from the microscopics using the Hubbard-Stratonovich transformation. This is very similar in spirit to what we already saw in the second tutorial when we discussed magnetism.

The partition function is given by

$$\mathcal{Z} = \int \mathcal{D}\left[\bar{\psi}, \psi\right] e^{-\int_0^\beta d\tau dx \left\{\sum_\sigma \bar{\psi}_\sigma \left[\partial_\tau + ie\phi + \frac{1}{2m}(-i\nabla - e\mathbf{A})^2 - \mu\right]\psi_\sigma - g\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow\right\}} , \tag{33}$$

where we have introduced coupling to the electromagnetic field in the form of the minimal coupling: $\partial_\tau \to \partial_\tau + ie\phi$, and $-i\nabla \to -i\nabla - e\mathbf{A}$. The first step towards obtaining the Ginzburg-Landau theory is to decouple the interacting term by introducing a Hubbard-Stratonovich field Δ

$$e^{\int_0^\beta d\tau \int dx g\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow} \sim \int \mathcal{D}\left[\bar{\Delta}, \Delta\right] e^{-\int_0^\beta d\tau \int dx \left[\frac{|\Delta|^2}{g} - \left(\bar{\Delta}\psi_\downarrow\psi_\uparrow + \Delta\bar{\psi}_\uparrow\bar{\psi}_\downarrow\right)\right]} . \tag{34}$$

The resulting action is identical to the mean-field action we had in the previous section *if* we treat Δ as a constant field, thus interpreting it Δ as the superconducting order parameter. However, now we will not do that, but instead treat it as a dynamical field, with amplitude and phase fluctuations.

The second step is to integrate out the fermions. To do so we define the Nambu-spinor $\Psi = \begin{pmatrix} \psi_\uparrow & \bar{\psi}_\downarrow \end{pmatrix}^T$ in terms of which the partition function is

$$\mathcal{Z} = \int \mathcal{D}\left[\bar{\psi}, \psi\right] \mathcal{D}\left[\bar{\Delta}, \Delta\right] e^{-\mathcal{S}} ,$$

$$\mathcal{S} = \int_0^\beta d\tau \int dx \left[\frac{|\Delta|^2}{g} - \bar{\Psi}\mathcal{G}^{-1}\Psi \right] , \tag{35}$$

$$\mathcal{G}^{-1} = \begin{pmatrix} \left[G^{(p)}\right]^{-1} & \Delta \\ \bar{\Delta} & \left[G^{(h)}\right]^{-1} \end{pmatrix} , \tag{36}$$

where

$$\left[G^{(p)}\right]^{-1} = -\partial_\tau - ie\phi - \frac{1}{2m}\left(-i\nabla - e\mathbf{A}\right)^2 + \mu , \tag{37}$$

$$\left[G^{(h)}\right]^{-1} = -\partial_\tau + ie\phi + \frac{1}{2m}\left(i\nabla - e\mathbf{A}\right)^2 - \mu . \tag{38}$$

Integrating out the fermions, to get an effective action for Δ is simple, and the result is

$$\mathcal{Z} = \int \mathcal{D}\left[\bar{\Delta}, \Delta\right] e^{-\mathcal{S}} ,$$

$$\mathcal{S} = \int_0^\beta d\tau \int dx \left[\frac{|\Delta|^2}{g} \right] + \log\left[\det\left(\mathcal{G}^{-1}\right)\right] . \tag{39}$$

Again, the mean-field results can be obtained from this effective theory by deriving the equations of motion, and neglecting quantum fluctuations in Δ. Doing so will reproduce the gap equation we got in the mean-field analysis in section 2. However, here we want to go beyond mean-field by considering the effects of fluctuations. To do so We will assume that Δ is small, which is true close to the transition, and expand $\log\left[\det\left(\mathcal{G}^{-1}\right)\right] = \mathrm{tr}\left[\log\left(\mathcal{G}^{-1}\right)\right]$ to lowest orders in Δ. First we write

$$\mathcal{G}^{-1} = \mathcal{G}_0^{-1} + \hat{\Delta} = \mathcal{G}_0^{-1}\left(1 + \mathcal{G}_0\hat{\Delta}\right) , \tag{40}$$

where $\mathcal{G}_0^{-1} \equiv \mathcal{G}^{-1}(\Delta = 0)$, and $\hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{pmatrix}$, such that

$$\begin{aligned} \mathrm{tr}\left[\log\left(\mathcal{G}^{-1}\right)\right] &= \mathrm{tr}\left[\log\left(\mathcal{G}_0^{-1}\right)\right] + \mathrm{tr}\left[\log\left(1 + \mathcal{G}_0\hat{\Delta}\right)\right] \\ &= \mathrm{tr}\left[\log\left(\mathcal{G}_0^{-1}\right)\right] - \sum_{n=0}^{\infty} \frac{1}{2n} \mathrm{tr}\left[\left(\mathcal{G}_0\hat{\Delta}\right)^{2n}\right] . \end{aligned} \tag{41}$$

We will not calculate the traces here, but those who are interested in such details are referred to Altland & Simons, chapter 6. Neglecting temporal fluctuations (making it a semi-classical

Ginzburg-Landau theory) the result is

$$\mathcal{S}_{GL} = \beta \int dx \left[\frac{r}{2} |\Delta|^2 + \frac{c}{2} |(\partial_x - 2ie\mathbf{A}) \Delta|^2 + u |\Delta|^4 \right] , \tag{42}$$

with $r = n\frac{T-T_c}{T_C}$. This is exactly the phenomenological theory you saw in class.

Let's see how the unique experimental properties of superconductors arise from this action. Below T_c we have $r < 0$ so the potential $\frac{r}{2} |\Delta|^2 + u |\Delta|^4$ has a minimum at $|\Delta|^2 = \sqrt{\frac{-r}{4u}} = \Delta_0^2$. However, the phase of Δ, i.e., the Goldstone mode, is not determined by the potential. Therefore, we write $\Delta = e^{2i\theta}\Delta_0$ in the Ginzburg-Landau actionto get

$$\mathcal{S}_{GL} = 2c\Delta_0^2\beta \int dx \left(\partial_x\theta - e\mathbf{A}\right)^2 . \tag{43}$$

We want to find the electromagnetic response of the system, so we need to treat it as dynamical field. Therefore, we should also add its kinetic term $\mathcal{S}_{Maxwell} = \frac{\beta}{2} \int dx (\nabla \times \mathbf{A})^2$ (assuming $\phi = 0$, and the field is static), such that the total action is

$$\frac{\mathcal{S}[A,\theta]}{\beta} = \int dx \left[2c\Delta_0^2 \left(\partial\theta - e\mathbf{A}\right)^2 + \frac{1}{2} \left(\nabla \times \mathbf{A}\right)^2 \right] . \tag{44}$$

In order to get an effective action for the gauge field A we integrate over the Goldstone mode. You already saw that explicitly in class, so I will not repeat this here, but the result is that after integrating out θ, the electromagnetic field acquires a mass (Higgs mechanism)

$$\frac{\mathcal{S}[A]}{\beta} = \frac{1}{2} \int dx \left[\frac{\rho_0}{m}\mathbf{A}^2 + \partial_i\mathbf{A}\partial_i\mathbf{A} \right] , \tag{45}$$

where we have adopted the notations used in class. Deriving the equations of motion, we get $\frac{\rho_0}{m}\mathbf{A} = \nabla^2\mathbf{A}$, and taking the curl we reproduce the London equation

$$\frac{\rho_0}{m}\mathbf{B} = \nabla^2\mathbf{B} , \tag{46}$$

which was discussed in class. In particular, it was already shown that it results in the decay of the magnetic field as we go into the bulk of the superconductor.

The second effect we want to see is the zero DC resistivity. To do that, we find the electric current

$$\mathbf{j}(\mathbf{r}) = \frac{\delta}{\delta\mathbf{A}(\mathbf{r})} \int dx \frac{\rho_0}{2m}\mathbf{A}^2 = \frac{\rho_0}{m}\mathbf{A} \tag{47}$$

Taking the time-derivative, working in a gauge where $\phi = 0$ and $\mathbf{E} = -i\partial_\tau \mathbf{A}$ we find that the electric field satisfies

$$\mathbf{E} = -i\frac{m}{\rho_0}\partial_\tau \mathbf{j} \ . \tag{48}$$

Therefore, for a constant DC current there will be no electric field. A system with a finite DC current and zero electric field has, by definition, zero resistivity. This can be seen from Ohm's law

$$\begin{aligned} j &= \sigma E \\ \Rightarrow E &= \frac{1}{\sigma} j \ , \end{aligned} \tag{49}$$

For a finite j, in order to get $E = 0$ we need $\sigma \to \infty$.