Concepts in Condensed Matter Physics: Tutorial V

Supplementary topics in superconductivity

1 Phonon mediated attractive interaction for electrons

In the previous Tutorial we argued that since electron dynamics are governed by ϵ_F and phonon dynamics by $\omega_D \ll \epsilon_F$, phonons can mediate attractive interaction between electrons. The picture was that an electron passing through some position in space will distort the lattice there by means of Coulomb attraction, and since the lattice relaxation is very slow (compared to the dynamics of the electrons), other electrons will be drawn to the distorted position after the original electron left it. Therefore the instantaneous repulsion between electron is complemented with a retarded attraction through the lattice distortion.

In this section of the tutorial we will support this qualitative picture with a quantitative analysis of an electron-phonon Hamiltonian. We will derive an effective model for the electron in which there is a retarded attractive interaction between them, due to the coupling to the phonons. To be specific, consider the following model of electrons coupled to phonons

$$H = \sum_{\mathbf{k},\sigma} \xi_k c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma} + \sum_{\mathbf{q},\mu} \omega_q b^\dagger_{\mathbf{q},\mu} b_{\mathbf{q},\mu} + \lambda \int d\mathbf{r} \rho(\mathbf{r}) \nabla \cdot \mathbf{u}(\mathbf{r}) + H_{\text{int}} \ , \tag{1}$$

where $\rho(\mathbf{r}) = \sum_\sigma \psi^\dagger_\sigma(\mathbf{r})\psi_\sigma(\mathbf{r})$ is the electron density, ω_q is the phonon dispersion, $\xi_k = \epsilon_k - \epsilon_F$ is the electron dispersion. The electron-phonon couple through the induced charge $\mathbf{P} = -\nabla \cdot \mathbf{u}(\mathbf{r})$, where $\mathbf{u}(\mathbf{r})$ is the lattice displacement field, governed by the coupling constant λ. We also include an unspecified electron-electron interaction, H_{int}, e.g., Coulomb repulsion. We wish to integrate out the phonons to obtain an effective theory for the electrons, in which the phonons are encoded through an effective interaction between electrons.

To do so we first write the coupling term in momentum space and in terms of phonon creation and annihilation operators

$$\rho(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} \rho_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} \ ,$$

$$\Rightarrow \lambda \int d\mathbf{r}\rho(\mathbf{r})\nabla\cdot\mathbf{u}(\mathbf{r}) = \lambda\sum_{\mathbf{q}} q_\mu u_{\mathbf{q},\mu}\rho_{-\mathbf{q}} = \lambda\sum_{\mathbf{q}}\frac{iq_\mu}{\sqrt{2m\omega_q}}(b^\dagger_{-\mathbf{q},\mu}+b_{\mathbf{q},\mu})\rho_{-\mathbf{q}}\ , \tag{2}$$

where $\rho_{\mathbf{q}} = \frac{1}{\sqrt{V}}\sum_{\mathbf{k},\sigma}\bar{\psi}_{\mathbf{k},\sigma}\psi_{\mathbf{k}+\mathbf{q},\sigma}$, is not the electron density in mode $\mathbf{q}$ but rather the $\mathbf{q}$ mode of the electron density in real space, and m is the ion mass. In order to integrate out the phonons, we go to the partition funcation and the action

$$\mathcal{Z} = \int \mathcal{D}\left[\bar{\psi},\psi\right]e^{-\mathcal{S}} \tag{3a}$$

$$\mathcal{S} = \sum_{\mathbf{k},n,\sigma}\bar{\psi}_{\mathbf{k},n,\sigma}(-i\omega_n+\xi_k)\psi_{\mathbf{k},n,\sigma}+\mathcal{S}_{\text{int}}$$
$$+\underbrace{\sum_{\mathbf{q},l,\mu}(-i\omega_l+\omega_q)\phi^*_{\mathbf{q},l,\mu}\phi_{\mathbf{q},l,\mu}+\lambda\sum_{q,l,\mu}\frac{iq_\mu}{\sqrt{2m\omega_q}}\left(\phi_{\mathbf{q},l,\mu}+\phi^*_{-\mathbf{q},-l,\mu}\right)\rho_{-\mathbf{q},-l}}_{\mathcal{S}_{\text{ph}}}\ , \tag{3b}$$

where ω_n are fermionic Matsubara frequencies, ω_l are bosonic Matsubara frequencies. Let us concentrate on the phononic part of the action and notice that it is quadratic in the phonons. We can therefore complete the square and integrate them out to get an effective action for the fermions. Completing the square requires first writing the phononic action in a "symmetrized" way

$$\mathcal{S}_{\text{ph}} = \frac{1}{2}\sum_{\mathbf{q},l,\mu}\begin{pmatrix}\phi^*_{\mathbf{q},l,\mu} & \phi_{-\mathbf{q},-l,\mu}\end{pmatrix}\begin{pmatrix}-i\omega_l+\omega_q & 0\\ 0 & i\omega_l+\omega_q\end{pmatrix}\begin{pmatrix}\phi_{\mathbf{q},l,\mu}\\ \phi^*_{-\mathbf{q},-l,\mu}\end{pmatrix}$$
$$+\frac{1}{2}\sum_{\mathbf{q},l,\mu}\left[\begin{pmatrix}\phi^*_{\mathbf{q},l,\mu} & \phi_{-\mathbf{q},-l,\mu}\end{pmatrix}\begin{pmatrix}\frac{-i\lambda q_\mu}{\sqrt{2m\omega_q}}\rho_{\mathbf{q},l}\\ \frac{-i\lambda q_\mu}{\sqrt{2m\omega_q}}\rho_{\mathbf{q},l}\end{pmatrix}+\begin{pmatrix}\frac{i\lambda q_\mu}{\sqrt{2m\omega_q}}\rho_{-\mathbf{q},-l} & \frac{i\lambda q_\mu}{\sqrt{2m\omega_q}}\rho_{-\mathbf{q},-l}\end{pmatrix}\begin{pmatrix}\phi_{\mathbf{q},l,\mu}\\ \phi^*_{-\mathbf{q},-l,\mu}\end{pmatrix}\right]\ , \tag{4}$$

where to get this we wrote the action as a sum of two copies of itself (introducing an overall factor of half), and changed $\mathbf{q}\to-\mathbf{q}$ and $\omega_l\to-\omega_l$ in one copy. We now notice that the action is of the form

$$\mathcal{S}_{\text{ph}} = \frac{1}{2}\sum_{\mathbf{q},l,\mu}\left[\vec{\Phi}^\dagger_{\mathbf{q},l,\mu}\mathrm{A}\vec{\Phi}_{\mathbf{q},l,\mu}+\vec{\Phi}^\dagger_{\mathbf{q},l,\mu}\cdot\vec{b}_{\mathbf{q},l,\mu}+\vec{b}^\dagger_{\mathbf{q},l,\mu}\cdot\vec{\Phi}_{\mathbf{q},l,\mu}\right]\ , \tag{5}$$

for which completing the square and integrating out the phonons is easy

$$\mathcal{S}_{\rm ph} = \frac{1}{2}\sum_{\mathbf{q},l,\mu}\left[\vec{\Phi}^{\dagger}_{\mathbf{q},l,\mu}\mathrm{A}\vec{\Phi}_{\mathbf{q},l,\mu} + \vec{\Phi}^{\dagger}_{\mathbf{q},l,\mu}\cdot\vec{b}_{\mathbf{q},l,\mu} + \vec{b}^{\dagger}_{\mathbf{q},l,\mu}\cdot\vec{\Phi}_{\mathbf{q},l,\mu}\right]$$

$$= \frac{1}{2}\sum_{\mathbf{q},l,\mu}\left(\vec{\Phi}^{\dagger}_{\mathbf{q},l,\mu} + \vec{b}^{\dagger}_{\mathbf{q},l,\mu}\mathrm{A}^{-1}\right)\mathrm{A}\left(\vec{\Phi}_{\mathbf{q},l,\mu} + \mathrm{A}^{-1}\vec{b}_{\mathbf{q},l,\mu}\right) - \frac{1}{2}\sum_{\mathbf{q},l,\mu}\vec{b}^{\dagger}_{\mathbf{q},l,\mu}\mathrm{A}^{-1}\vec{b}_{\mathbf{q},l,\mu}$$

$$\sim -\frac{1}{2}\sum_{\mathbf{q},l,\mu}\vec{b}^{\dagger}_{\mathbf{q},l,\mu}\mathrm{A}^{-1}\vec{b}_{\mathbf{q},l,\mu}$$

$$= -\frac{1}{2}\sum_{\mathbf{q},l,\mu}\begin{pmatrix} \frac{i\lambda q_\mu}{\sqrt{2m\omega_q}}\rho_{-\mathbf{q},-l} & \frac{i\lambda q_\mu}{\sqrt{2m\omega_q}}\rho_{-\mathbf{q},-l}\end{pmatrix}\begin{pmatrix}\frac{1}{-i\omega_l+\omega_q} & 0 \\ 0 & \frac{1}{i\omega_l+\omega_q}\end{pmatrix}\begin{pmatrix}\frac{-i\lambda q_\mu}{\sqrt{2m\omega_q}}\rho_{\mathbf{q},l} \\ \frac{-i\lambda q_\mu}{\sqrt{2m\omega_q}}\rho_{\mathbf{q},l}\end{pmatrix}$$

$$= -\frac{1}{2}\sum_{\mathbf{q},l}\frac{\lambda^2 q^2}{2m\omega_q}\rho_{\mathbf{q},l}\left[\frac{1}{-i\omega_l+\omega_q} + \frac{1}{i\omega_l+\omega_q}\right]\rho_{-\mathbf{q},-l}$$

$$= -\frac{1}{2}\sum_{\mathbf{q},l}\frac{\lambda^2 q^2}{2m\left(\omega_l^2+\omega_q^2\right)}\rho_{\mathbf{q},l}\rho_{-\mathbf{q},-l}\ . \tag{6}$$

Therefore, the effective interaction for the electrons is

$$\mathcal{S}_{\rm eff} = \sum_{\mathbf{k},n,\sigma}\bar{\psi}_{\mathbf{k},n,\sigma}(-i\omega_n+\xi_k)\psi_{\mathbf{k},n,\sigma} + \sum_{\mathbf{q},l}\left[g_0(\mathbf{q}) - \frac{1}{2}\frac{\lambda^2 q^2}{2m\left(\omega_l^2+\omega_q^2\right)}\right]\rho_{\mathbf{q},l}\rho_{-\mathbf{q},-l}\ , \tag{7}$$

where $g_0(\mathbf{q})$ is the original electron-electron interaction (say Coulomb repulsion). The effective electron-electron has also a contribution from the phonons, and is given by

$$g_{\rm eff}(\mathbf{q},i\omega_l) = g_0(\mathbf{q}) - \frac{1}{2}\frac{\lambda^2 q^2}{2m\left(\omega_l^2+\omega_q^2\right)}\ . \tag{8}$$

Wick rotating the effective interaction to real time, i.e., taking $\omega_l \to -i\omega - \delta$, we find that for every momentum mode $\mathbf{q}$ the phonon-mediated correction to the interaction is attractive at low enough frequency: $\omega < \omega_q$. Since acoustic phonons have $\omega_q\ q$, it is either high energy acoustic phonons, or, more commonly, optical phonons that generate the attractive interaction. We can thus estimate $\omega_q \sim \omega_D$ and the effective interaction

$$g(\mathbf{q},\omega) = g_0(\mathbf{q}) - \frac{1}{2}\frac{\lambda^2 q^2}{2m\left(\omega_D^2-\omega^2\right)} \tag{9}$$

is still repulsive for $\omega > \omega_D$, but may become attractive for $\omega < \omega_D$ (if this negative contribution overcomes the original repulsive interaction.This dependence on ω is exactly the retardation effect.

2 RG equation for the Cooper channel

In this section we will begin from a generic interacting Hamiltonian and derive the RG equation governing the interactions in the Cooper channel. We will disregard all other channels, and thus won't be able to say anything about what happens there. Therefore, our picture will be incomplete, and we will not know what actually happens to the system. However, we will be able to say if (and when) the Cooper channel is important.

We begin our discussion from the following Hamiltonian of electrons with density-density interaction $V(\mathbf{r}_1 - \mathbf{r}_2)$, e.g., Coulomb repulsion:

$$H = H_0 + H_{\rm int} \ , \tag{10a}$$

$$H_0 = \int d\mathbf{r} \sum_\sigma \left[\psi^\dagger_\sigma(\mathbf{r}) \left(-\frac{\nabla^2}{2m} - \mu \right) \psi_\sigma(\mathbf{r}) \right] \ , \tag{10b}$$

$$H_{\rm int} = \frac{1}{2} \int d\mathbf{r}_1 d\mathbf{r}_2 \sum_{\sigma_1,\sigma_2} \left[\psi^\dagger_{\sigma_1}(\mathbf{r}_1) \psi^\dagger_{\sigma_2}(\mathbf{r}_2) V(\mathbf{r}_1 - \mathbf{r}_2) \psi_{\sigma_2}(\mathbf{r}_2) \psi_{\sigma_1}(\mathbf{r}_1) \right] \ . \tag{10c}$$

We define the (bosonic) operator creating two electrons with arbitrary spins at arbitrary positions

$$\Phi^\dagger_{\sigma_1,\sigma_2}(\mathbf{r}_1, \mathbf{r}_2) = \psi^\dagger_{\sigma_1}(\mathbf{r}_1)\, \psi^\dagger_{\sigma_2}(\mathbf{r}_2) \ , \tag{11}$$

and its hermitian conjugate, annihilating two such fermions. Transforming to the center of mass coordinate $\mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}$ and relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ we can write

$$\Phi^\dagger_{\sigma_1,\sigma_2}(\mathbf{r}, \mathbf{R}) = \psi^\dagger_{\sigma_1} \left(\mathbf{R} + \frac{\mathbf{r}}{2} \right) \psi^\dagger_{\sigma_2} \left(\mathbf{R} - \frac{\mathbf{r}}{2} \right) \ , \tag{12}$$

in terms of which the interaction Hamiltonian is

$$H_{\rm int} = \frac{1}{2} \int d\mathbf{r} d\mathbf{R} V(\mathbf{r}) \Phi^\dagger_{\sigma_1,\sigma_2}(\mathbf{r}, \mathbf{R}) \Phi_{\sigma_1,\sigma_2}(\mathbf{r}, \mathbf{R}) \ . \tag{13}$$

Notice that we have written the interaction in a suggestive way, namely, focusing on the cooper channel.[1] For short-range interactions, since we are interested in long-wavelength physics, we can approximate $V(\mathbf{r}) = g\delta(\mathbf{r})$ to obtain

$$H_{\text{int}} = g \int d\mathbf{R}\Phi^{\dagger}_{\uparrow,\downarrow}(0,\mathbf{R})\Phi_{\uparrow,\downarrow}(0,\mathbf{R}) \ , \tag{14}$$

where we have specified to electrons of opposite spin since for $r = 0$ same spin contributions vanish. It is important to note that even for the long-range Coulomb interaction this approximation is valid. The reason is that in metals, where the fermions form a Fermi surface state (as in H_0), the long-range Coulomb interaction is screened, and becomes effectively short-range.

We now go to the partition function, and perform a Hubbard-Stratonovich transformation, introducing the bosonic field $\Delta(R,\tau)$ which decouples the interactions in the Cooper channel to get

$$\mathcal{Z} = \int \mathcal{D}\left[\bar{\psi},\psi,\bar{\Delta},\Delta\right] \exp\left[-\mathcal{S}_{\text{H.S.}}\right] \ , \tag{15a}$$

$$\begin{aligned}
\mathcal{S}_{\text{H.S.}} =& \int d\mathbf{R}d\tau \sum_{\sigma} \bar{\psi}_{\sigma}(\mathbf{R},\tau)\left[\partial_{\tau} - \frac{\nabla^2}{2m} - \mu\right]\psi_{\sigma}(\mathbf{R},\tau) + \frac{1}{g}\int d\mathbf{R}d\tau \bar{\Delta}(\mathbf{R},\tau)\Delta(\mathbf{R},\tau) \\
&+ \int d\mathbf{R}d\tau \left[i\bar{\Delta}(\mathbf{R},\tau)\psi_{\downarrow}(\mathbf{R},\tau)\psi_{\uparrow}(\mathbf{R},\tau) + i\Delta(\mathbf{R},\tau)\bar{\psi}_{\uparrow}(\mathbf{R},\tau)\bar{\psi}_{\downarrow}(\mathbf{R},\tau)\right] \\
=& \frac{1}{g}\sum_{\mathbf{Q},\omega_m} \bar{\Delta}(\mathbf{Q},\omega_m)\Delta(\mathbf{Q},\omega_m) + \sum_{\sigma,\mathbf{k},\omega_n} \left(-i\omega_n + \xi_k\right) \bar{\psi}_{\sigma}(\mathbf{k},\omega_n)\psi_{\sigma}(\mathbf{k},\omega_n) \\
&+ \frac{i}{\sqrt{\beta V}} \sum_{\mathbf{k},\omega_n}\sum_{\mathbf{Q},\omega_m} \bar{\Delta}(\mathbf{Q},\omega_m)\psi_{\downarrow}(\mathbf{Q}-\mathbf{k},\omega_m-\omega_n)\psi_{\uparrow}(\mathbf{k},\omega_n) \\
&+ \frac{i}{\sqrt{\beta V}} \sum_{\mathbf{k},\omega_n}\sum_{\mathbf{Q},\omega_m} \Delta(\mathbf{Q},\omega_m)\bar{\psi}_{\uparrow}(\mathbf{k},\omega_n)\bar{\psi}_{\downarrow}(\mathbf{Q}-\mathbf{k},\omega_m-\omega_n) \ ,
\end{aligned} \tag{15b}$$

where $\xi_k = \frac{k^2}{2m} - \epsilon_F$. We are interested in the low-energy physics, and so we expand in $\mathbf{Q}$ and ω_m keeping only the zeroth-order term. Effectively this means setting $\mathbf{Q} = 0$ and $\omega_m = 0$ to get

$$\mathcal{S}_{\text{H.S.}} = \frac{1}{g}\bar{\Delta}\Delta + \sum_{\sigma,\mathbf{k},\omega_n} \left(-i\omega_n + \xi_k\right) \bar{\psi}_{\sigma}(\mathbf{k},\omega_n)\psi_{\sigma}(\mathbf{k},\omega_n)$$

[1] Everything we did so far is of course exact, and so even if the cooper channel is not the important channel, doing so is still correct. However, going forward with the derivation requires some assumptions which we will perform on this channel. By making these assumptions on this specific form of the Hamiltonian we are essentially focus on the cooper channel.

$$+\frac{i}{\sqrt{\beta V}}\sum_{\mathbf{k},\omega_n}\bar{\Delta}\psi_\downarrow(-\mathbf{k},-\omega_n)\psi_\uparrow(\mathbf{k},\omega_n)+\frac{i}{\sqrt{\beta V}}\sum_{\mathbf{k},\omega_n}\Delta\bar{\psi}_\uparrow(\mathbf{k},\omega_n)\bar{\psi}_\downarrow(-\mathbf{k},-\omega_n)\ , \tag{16}$$

This step is where we specifically consider the physics of the cooper channel. Up to now everything we did was exact and could capture any physical effect. In this step we focus on the cooper channel, analyzing what happens specifically there [2].

To perform the RG analysis we introduce a high-energy cutoff Ω for the electrons, i.e., $|\xi_k|\leq\Omega$. The cutoff dependent action takes to form

$$\mathcal{S}(\Omega)=\frac{1}{g(\Omega)}\bar{\Delta}\Delta+\sum_{\sigma,\omega_n}\sum_{|\xi_k|<\Omega}\left(-i\omega_n+\xi_k\right)\bar{\psi}_{\sigma,\mathbf{k},\omega_n}\psi_{\sigma,\mathbf{k},\omega_n}$$

$$+\frac{i}{\sqrt{\beta V}}\sum_{\omega_n}\sum_{|\xi_k|<\Omega}\left[\bar{\Delta}\psi_{\downarrow,-\mathbf{k},-\omega_n}\psi_{\uparrow,\mathbf{k},\omega_n}+\Delta\bar{\psi}_{\uparrow,\mathbf{k},\omega_n}\bar{\psi}_{\downarrow,-\mathbf{k},-\omega_n}\right]$$

$$=\frac{1}{g(\Omega)}\bar{\Delta}\Delta+\sum_{\sigma,\omega_n}\sum_{|\xi_k|<\Omega}\begin{pmatrix}\bar{\psi}_{\uparrow,\mathbf{k},\omega_n} & \psi_{\downarrow,-\mathbf{k},-\omega_n}\end{pmatrix}\begin{pmatrix}-i\omega_n+\xi_k & \frac{i}{\sqrt{\beta V}}\Delta\\ \frac{i}{\sqrt{\beta V}}\bar{\Delta} & -i\omega_n-\xi_k\end{pmatrix}\begin{pmatrix}\psi_{\uparrow,\mathbf{k},\omega_n}\\ \bar{\psi}_{\downarrow,-\mathbf{k},-\omega_n}\end{pmatrix}\ . \tag{17}$$

We now split the action into a low energy part, $|\xi_k|\leq\Omega-d\Omega$, and high energy part, $\Omega-d\Omega<|\xi_k|\leq\Omega$, and integrate out the high energy electrons to obtain the action at the scale $\Omega-d\Omega$

$$\mathcal{Z}=\int_{|\xi_k|\leq\Omega}\mathcal{D}\left[\bar{\psi},\psi\right]\exp\left[-\mathcal{S}(\Lambda)\right]=\int_{|\xi_k|\leq\Omega-d\Omega}\mathcal{D}\left[\bar{\psi},\psi\right]\int_{\Omega-d\Omega<|\xi_k|\leq\Omega}\mathcal{D}\left[\bar{\psi},\psi\right]\exp\left[-\mathcal{S}(\Omega)\right]$$

$$=\int_{|\xi_k|\leq\Omega-d\Omega}\mathcal{D}\left[\bar{\psi},\psi\right]\exp\left[-\mathcal{S}(\Omega-d\Omega)\right] \tag{18a}$$

$$\mathcal{S}(\Omega-d\Omega)=\frac{1}{g(\Omega)}\bar{\Delta}\Delta-\sum_{\sigma,\omega_n}\sum_{\Omega-d\Omega<|\xi_k|\leq\Omega}\log\left[-\omega_n^2-\xi_k^2+\frac{1}{\sqrt{\beta V}}\bar{\Delta}\Delta\right]$$

$$+\sum_{\omega_n}\sum_{|\xi_k|<\Omega-d\Omega}\begin{pmatrix}\bar{\psi}_{\uparrow,\mathbf{k},\omega_n} & \psi_{\downarrow,-\mathbf{k},-\omega_n}\end{pmatrix}\begin{pmatrix}-i\omega_n+\xi_k & \frac{i}{\sqrt{\beta V}}\Delta\\ \frac{i}{\sqrt{\beta V}}\bar{\Delta} & -i\omega_n-\xi_k\end{pmatrix}\begin{pmatrix}\psi_{\uparrow,\mathbf{k},\omega_n}\\ \bar{\psi}_{\downarrow,-\mathbf{k},-\omega_n}\end{pmatrix}\ . \tag{18b}$$

Notice that the second part is already of the same form as the original action. To proceed,

[2] More generally one can decouple the Hamiltonian in many channels simultaneously, and in each of them consider the low-energy long-wavelength limit of the relevant Hubbard-Stratonovich field, to find the RG euqations of that specific channel.

and find a form similar to the original action also for the first part, we expand the logarithm and identify the coupling constant, g, at the new scale

$$
\begin{aligned}
&\frac{1}{g(\Omega)}\bar{\Delta}\Delta - \sum_{\omega_n}\sum_{\Omega-d\Omega<|\xi_k|\le\Omega}\log\left[-\omega_n^2-\xi_k^2+\frac{1}{\beta V}\bar{\Delta}\Delta\right] \\
&= \frac{1}{g(\Omega)}\bar{\Delta}\Delta - \sum_{\omega_n}\sum_{\Omega-d\Omega<|\xi_k|\le\Omega}\log\left[1-\frac{1}{\beta V}\frac{\bar{\Delta}\Delta}{\omega_n^2+\xi_k^2}\right] + \text{const} \\
&\approx \left[\frac{1}{g(\Omega)}+\frac{1}{\beta V}\sum_{\omega_n}\sum_{\Omega-d\Omega<|\xi_k|\le\Omega}\frac{1}{\omega_n^2+\xi_k^2}\right]\bar{\Delta}\Delta \equiv \frac{1}{g(\Omega-d\Omega)}\bar{\Delta}\Delta \ . \qquad (19)
\end{aligned}
$$

So we have found the following equation describing the scale dependence of the coupling constant

$$
\frac{1}{g(\Omega-d\Omega)} = \frac{1}{g(\Omega)} + \frac{1}{\beta V}\sum_{\omega_n}\sum_{\Omega-d\Omega<|\xi_k|\le\Omega}\frac{1}{\omega_n^2+\xi_k^2} \ . \qquad (20)
$$

The final step is to resolve the sums, by first approximating the sum over ω_n with an integral, and then transform from sum over k to integral over ξ (using the the density of states)

$$
\begin{aligned}
&\sum_{\omega_n}\sum_{\Omega-d\Omega<|\xi_k|\le\Omega}\frac{1}{\beta V}\frac{1}{\omega_n^2+\xi_k^2} \approx \frac{1}{V}\sum_{\Omega-d\Omega<|\xi_k|\le\Omega}\int\frac{d\omega}{2\pi}\frac{1}{\omega^2+\xi_k^2} = \frac{1}{2V}\sum_{\Omega-d\Omega<|\xi_k|\le\Omega}\frac{1}{|\xi_k|} \\
&= \nu_0\int_{\Omega-d\Omega}^{\Omega}\frac{d\xi}{\xi} = \nu_0\log\left(\frac{\Omega}{\Omega-d\Omega}\right) = -\nu_0\log\left(\frac{\Omega-d\Omega}{\Omega}\right) = -\nu_0\log\left(1-\frac{d\Omega}{\Omega}\right) \approx \nu_0\frac{d\Omega}{\Omega} \ , \qquad (21)
\end{aligned}
$$

where ν_0 is the density of states (normalized by spin and volume) on the Fermi surface. The resulting RG equation is therefore

$$
\begin{aligned}
&\frac{1}{g(\Lambda)} - \frac{1}{g(\Lambda-d\Lambda)} = -\nu_0\frac{d\Lambda}{\Lambda} \\
&\Rightarrow \Lambda\frac{d}{d\Lambda}\left[\frac{1}{g(\Lambda)}\right] = -\nu_0 \\
&\Rightarrow \frac{d}{d\log(\Lambda)}\left[\frac{1}{g(\Lambda)}\right] = -\nu_0 \ . \qquad (22)
\end{aligned}
$$

The solution of the RG equation is

$$g(\Lambda) = \frac{g_0}{1 + \nu_0 g_0 \log\left(\frac{\Lambda_0}{\Lambda}\right)} \;, \tag{23}$$

where $g_0 = g(\Lambda_0)$. What does this mean? If the sign of g_0 is positive, namely, the interactions are repulsive then g decreases with decreasing scale Λ, making the interactions irrelevant at low energy. However, if the sign of g_0 is negative, namely, the interactions are attractive then g remains negative but grows in its absolute value, making the interactions relevant and ultimately dominant. We know that attractive interactions are a result of phonon mediated interaction between electrons, and thus occur only below $\sim \omega_D$. How can interactions in such a small energy range overcome the repulsive interaction in the much larger range? To answer this, let us look again at the RG equation (22), and integrate it from Ω_0 all the way down to ω_D

$$g(\omega_D) = \frac{g_0}{1 + \nu_0 g_0 \log\left(\frac{\Omega_0}{\omega_D}\right)} \;. \tag{24}$$

Ω_0 should be of the order of the Fermi energy, which is typically roughly two orders of magnitude larger than ω_D leading to $\log\left(\frac{\Lambda_0}{\omega_D}\right) \sim \log\left(\frac{\epsilon_F}{\omega_D}\right) \sim 5$. Define a dimensionless interaction strength $\Gamma = \nu_0 * g$ we find that $g(\omega_D)$ is quite small

$$\Gamma(\omega_D) \sim \frac{\Gamma_0}{1 + 5\Gamma_0} \leq \frac{1}{5} \;, \tag{25}$$

and can thus be overcome by some week phonon mediated attractive interaction. Therefore, at this point we add the attractive interaction due to phonons, $\Gamma'(\omega_D) = \Gamma(\omega_D) - \Gamma_{\rm ph}$, and keep integrating the RG equation to even lower energies

$$\Gamma(\omega) = \frac{\Gamma'}{1 + \Gamma' \log\left(\frac{\omega_D}{\omega}\right)} \;, \tag{26}$$

where $\Gamma' \equiv \Gamma'(\omega_D) = \Gamma(\omega_D) - \Gamma_{\rm ph}$. If indeed $\Gamma' < 0$, then $g(\omega)$ diverges (in its absolute value) as ω is reduced toward the value at which $\Gamma' \log\left(\frac{\omega_D}{\omega}\right) = -1$. This divergence is a sign of some sort of instability or phase transition. We can estimate the critical temperature using the scale at which the interactions are of order -1 (not small anymore),

$$\Gamma(T_C) = -1 = \frac{-|\Gamma'|}{1 - |\Gamma'| \log\left(\frac{\omega_D}{T_C}\right)} \;,$$

$$\Rightarrow T_C \sim \omega_D \exp\left[-\frac{1}{|\Gamma'|}\right] = \omega_D \exp\left[-\frac{1}{\nu_0 |g'|}\right] . \tag{27}$$

This is indeed the same T_C we got from BSC theory (perhaps up to a numerical prefactor)!