

A 2-dimensional topological superconductor:

Consider the following 2D tight binding model of spinless electrons with hopping on a square lattice and superconducting terms of the form:

$$H = H_t + H_\mu + H_\Delta$$

$$H_t = - \left(\sum_{i,j} t \psi_{i,j}^\dagger \psi_{i+1,j} + t \psi_{i,j}^\dagger \psi_{i,j+1} \right) + h.c.$$

$$H_\mu = - \sum_{i,j} \mu \psi_{i,j}^\dagger \psi_{i,j}$$

$$H_\Delta = \frac{\Delta}{2} \sum_{i,j} (\psi_{i,j}^\dagger \psi_{i+1,j}^\dagger - \psi_{i,j}^\dagger \psi_{i-1,j}^\dagger + i \psi_{i,j}^\dagger \psi_{i,j+1}^\dagger - i \psi_{i,j}^\dagger \psi_{i,j-1}^\dagger) + h.c.$$

- a) Explain why in this case we cannot write on-site superconducting terms.
- b) Transforming to k -space $\psi_{i,j}^\dagger = \frac{1}{\sqrt{N}} \sum_k \Psi^\dagger(k) e^{i\vec{k} \cdot (i,j)}$, write the Hamiltonian in the form

$$H = \frac{1}{2} \sum_k \Psi^\dagger(k) h(k) \Psi(k),$$

with $\Psi = (\psi(k), \psi^\dagger(-k))^T$ and $h(k) = \begin{pmatrix} \varepsilon(k) & d(k) \\ d^*(k) & -\varepsilon(-k) \end{pmatrix}$

Diagonalize it and find the spectrum. Is there a gap to excitations? If so, find the values of $\frac{t}{\mu}$ in which the gap closes.

- c) Study the Chern number of the model as a function of $\frac{t}{\mu}$.
- d) Find a low energy theory near the critical point with a positive $\frac{t}{\mu}$. To do so, expand the Hamiltonian to first order in momentum (around which point?), and get the form

$$h(k) = \alpha \sigma_x k_y + \beta \sigma_y k_x + m \sigma_z.$$

What are α, β, m ? (for consistency, make sure $m = 0$ in the transition).

- e) Using the above continuum model in real space ($k \rightarrow -i\partial_x$), examine the situation where the parameter m is spatially dependent:

$$m = \begin{cases} m_0, x > 0 \\ -m_0, x < 0 \end{cases}.$$

Show that the system has a zero energy mode localized near $x = 0$. What is its localization length? What happens to it at the critical point.

$$H = H_z + H_n + H_o$$

$$H_z = -t \sum_{ij} [\psi_{ij}^\dagger \psi_{i+1,j} + \psi_{ij}^\dagger \psi_{i,j+1} + h.c.]$$

$$H_n = -\mu \sum_{ij} \psi_{ij}^\dagger \psi_{ij}$$

$$H_o = \Delta \sum_{ij} [\psi_{ij}^\dagger \psi_{i+1,j}^\dagger + i \psi_{ij}^\dagger \psi_{i,j+1}^\dagger] + h.c.$$

a) fermions are spinless
and $\psi_{ij}^\dagger \psi_{ij}^\dagger = 0!$

b)

$$H_z = -\frac{t}{N} \sum_{ij} \sum_{k,q} [\psi_k^\dagger \psi_q e^{-iq_x r} +$$

$$\psi_{k+q}^\dagger \psi_q e^{-iq_y r}] e^{i(k-q) \cdot r} =$$

$$= -t \sum_k \psi_k^\dagger \psi_k [2 \cos(k_x) + 2 \cos(k_y)]$$

$$H_M = -\mu \sum_k \psi_k^\dagger \psi_k$$

$$H_D = \frac{D}{2} \sum_{ij} [\psi_{ij}^\dagger \psi_{i+1,j}^\dagger - \psi_{ij}^\dagger \psi_{i-1,j}^\dagger + i \psi_{ij}^\dagger \psi_{i,j+1}^\dagger - i \psi_{ij}^\dagger \psi_{i,j-1}^\dagger] + h.c. =$$

$$= \frac{D}{2N} \sum_{ij} \sum_{kq} \psi_k^\dagger \psi_q^\dagger [e^{iq_x} - e^{-iq_x} + i e^{iq_y} - i e^{-iq_y}] e^{i(\xi + k) \cdot r} + h.c.$$

$$= -D \sum_k \psi_k^\dagger \psi_{-k}^\dagger [i \sinh(k_x) - \sinh(k_y)] + h.c.$$

define $\Psi = (\psi_k \quad \psi_{-k}^\dagger)^T$

and:

$$H_D + H_M = \sum_k \psi_k^\dagger \psi_k \xi(k)$$

$$\xi(k) = \frac{D}{2N} [e^{ik_x} - e^{-ik_x} + i e^{ik_y} - i e^{-ik_y}]$$

$$= \frac{1}{2} \sum_{\mathbf{k}} [T_{\mathbf{k}} T_{\mathbf{k}}^{\dagger} \epsilon(\mathbf{k}) + T_{-\mathbf{k}} T_{\mathbf{k}}^{\dagger} \epsilon(\mathbf{k})]$$

$$= \frac{1}{2} \sum_{\mathbf{k}} [\psi_{\mathbf{k}}^{\dagger} \psi_{\mathbf{k}} \epsilon(\mathbf{k}) - \psi_{-\mathbf{k}} \psi_{-\mathbf{k}}^{\dagger} \epsilon(-\mathbf{k})] + \frac{1}{2} \sum_{\mathbf{k}} \epsilon(\mathbf{k}) =$$

$$= \frac{1}{2} \sum_{\mathbf{k}} \Psi^{\dagger}(\mathbf{k}) \begin{pmatrix} \epsilon(\mathbf{k}) & 0 \\ 0 & -\epsilon(\mathbf{k}) \end{pmatrix} \Psi(\mathbf{k}) + \frac{1}{2} \sum_{\mathbf{k}} \epsilon(\mathbf{k})$$

$$H_0 = \frac{1}{2} \sum_{\mathbf{k}} \Psi^{\dagger}(\mathbf{k}) \begin{pmatrix} 0 & d(\mathbf{k}) \\ d^{\dagger}(\mathbf{k}) & 0 \end{pmatrix} \Psi(\mathbf{k})$$

$$d(\mathbf{k}) = 2\Delta [\sin(k_y) - i \sin(k_x)]$$

$$\Rightarrow H = \frac{1}{2} \sum_{\mathbf{k}} \Psi^{\dagger}(\mathbf{k}) h_{\text{BdG}}(\mathbf{k}) \Psi(\mathbf{k}) + \frac{1}{2} \sum_{\mathbf{k}} \epsilon(\mathbf{k})$$

$$1 \quad \dots \quad T_{\mathbf{k}} \rightarrow$$

$$n_B d\epsilon(k) = d(k) \cdot \sigma$$

$$d_z(k) = -2t \cos(k_x) - 2t \cos(k_y)$$

↗

$$d_x(k) = 2\Delta \sin(k_y)$$

$$d_y(k) = 2\Delta \sin(k_x)$$

$$\Rightarrow E = \pm |\vec{d}(k)| = \pm \sqrt{\epsilon^2(k) + 4\Delta^2 [\sin^2(k_x) + \sin^2(k_y)]}$$

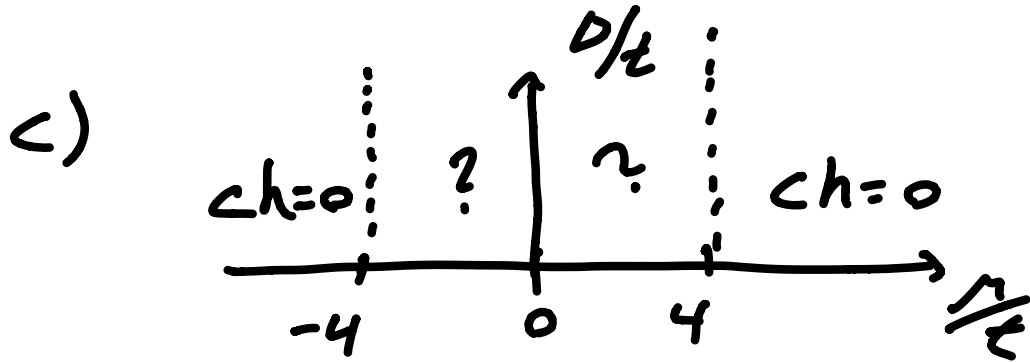
$$\pm \sqrt{[2t \cos(k_x) + 2t \cos(k_y) + \mu]^2 + 4\Delta^2 \sin^2(k_x) + 4\Delta^2 \sin^2(k_y)}$$

$$E \stackrel{?}{=} 0 \Rightarrow k_x = 0, \pi$$

$$k_y = 0, \pi$$

$$\Rightarrow \begin{cases} \vec{k} = (0, 0) \\ \vec{k} = (0, \pi) \end{cases} \Rightarrow \frac{t}{\mu} = -\frac{1}{4}$$

$$\begin{cases} \vec{k} = (\pi, 0) \Rightarrow \mu = 0 \\ \vec{k} = (\pi, \pi) \Rightarrow \frac{t}{\mu} = \frac{1}{4} \end{cases}$$



i) can choose D, μ, t in each region and calculate:

$$Ch = \frac{1}{2\pi} \int d^2 k F_{x,y}(k)$$

$$F_{x,y}(k) = \frac{1}{2} \epsilon_{\mu\nu\rho} \hat{\partial}_\mu \partial_x \hat{\partial}_\nu \partial_y \hat{\partial}_\rho$$

$$\hat{\partial}_\mu = \frac{\vec{\partial}_\mu}{|\partial|}$$

* ii) expand around $\mu = 4t$

com.
at end $\vec{k} = (\pi, \pi) + \vec{p}$

$$\vec{d} = \begin{pmatrix} -2\Delta P_y \\ -2\Delta P_x \\ 4t - \mu \\ -\frac{1}{t} \end{pmatrix} \Rightarrow |\vec{d}| = 2\sqrt{\Delta^2 P_x^2 + \Delta^2 P_y^2 + (t - \frac{\mu}{4})^2}$$

$$\partial_\mu |\vec{d}| = \frac{4\Delta^2 P_\mu}{|\vec{d}|}$$

$$\partial_\mu \hat{d}_x = \partial_\mu \frac{d_x}{|\vec{d}|} = \frac{\partial_\mu d_x}{|\vec{d}|} - \frac{d_x}{|\vec{d}|^2} \partial_\mu |\vec{d}|$$

$$= \frac{\partial_\mu d_x}{|\vec{d}|} - \frac{4d_x \Delta^2 P_\mu}{|\vec{d}|^3}$$

$$\partial_x \hat{d}_x = \frac{8\Delta^3 P_x P_y}{|\vec{d}|^3}$$

$$\partial_x \hat{d}_y = -\frac{2\Delta |\vec{d}|^2}{|\vec{d}|^3} + \frac{8\Delta^3 P_x^2}{|\vec{d}|^3}$$

$$\partial_x \hat{d}_z = -\frac{4\Delta P_x \Delta^2}{|\vec{d}|^3}$$

$$\partial_y \hat{d}_x = -\frac{2\Delta |d|^2}{|d|^3} + \frac{8\Delta^3 p_y^2}{|d|^3}$$

$$\partial_y \hat{d}_y = \frac{8\Delta^3 p_x p_y}{|d|^3}$$

$$\partial_y \hat{d}_z = -\frac{4\epsilon \Delta^2 p_y}{|d|^3}$$

$$F_{x,y}(p) = \frac{4\epsilon \Delta^2}{[4\Delta^2 p^2 + \epsilon^2]^{3/2}}$$

$$Ch = \frac{1}{2\pi} \int d^2 p F_{x,y}(p) =$$

$$= \frac{1}{2\pi} \int p dp d\theta F_{x,y}(p) = \int_0^\infty p dp F_{x,y}(p)$$

$$= \frac{1}{2} \frac{\epsilon}{|\Delta|} \int_0^\infty dp \frac{p}{[p^2 + (\frac{\epsilon}{2\Delta})^2]^{3/2}} =$$

$$= \frac{1}{2} \frac{\epsilon}{|\Delta|} \left| \frac{\Delta}{\epsilon} \right| = \frac{\text{sign}(\epsilon)}{2}$$

$$\Rightarrow \Delta ch_{\mu=4t} = ch(\varepsilon > 0) - ch(\varepsilon < 0)$$

$$= 1$$

for $\mu = 0$, $\vec{K}_{\pm} = (0, \pi) + \vec{p}$
 $(\pi, 0) + \vec{p}$

$$\vec{d}_{\pm} = \begin{pmatrix} \pm 2 \Delta p_y \\ \mp 2 \Delta p_x \\ -\mu \end{pmatrix}$$

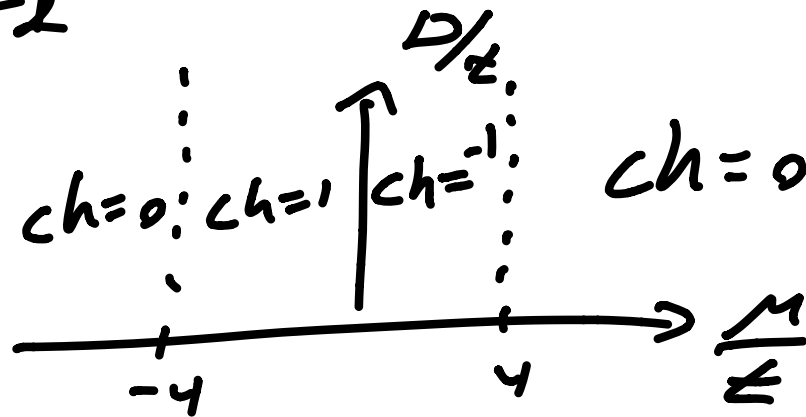
$$F_{x,y}^{\pm}(p) = - \frac{4\mu \Delta^2}{[4p^2 \Delta^2 + \mu^2]^{3/2}}$$

$$ch = -\frac{1}{2} \frac{\mu}{|\Delta|} \int_0^{\infty} dp \frac{p}{p^2 + (\frac{\mu}{2\Delta})^2}$$

$$= -\frac{1}{2} \frac{\mu}{|\Delta|} \left(\frac{\Delta}{\mu} \right) = -\frac{1}{2} \operatorname{sgn}(\mu)$$

$$17ch = [ch] - 1 - ch - 17$$

$$\psi(x,y) = [\psi(x, \mu=0) - \psi(x, \mu=0)] = -2$$



$$d) \frac{\mu}{\epsilon} = 4 \Rightarrow \vec{k} = (\pi, \pi) + \vec{p}$$

$$h = \vec{j} \cdot \vec{\sigma} = -2 \Delta p_y \sigma_x - 2 \Delta p_x \sigma_y - \epsilon \sigma_z$$

$$\Rightarrow \alpha = -2\Delta = \beta$$

$$m = -\epsilon = 4t - \mu = -\epsilon_\mu$$

$$e) h(x,y) = i 2 \Delta \sigma_x \partial_y + 2 \Delta \sigma_y i \partial_x - \epsilon(x) \sigma_z$$

$$h(x,y) \psi(x,y) = 0$$

$$\psi(x, y) = e^{ik_y y} \chi(x)$$

$$(-2\Delta k_y \sigma_x + 2\Delta i \sigma_y \partial_x - \epsilon(x) \sigma_z) \chi = 0$$

$$2\Delta \partial_x \chi = A \chi$$

$$A = -\epsilon(x) i \sigma_y \sigma_z - 2\Delta k_y i \sigma_y \sigma_x$$

$$= \epsilon(x) \sigma_x - 2\Delta k_y \sigma_z$$

$$= \begin{pmatrix} -2\Delta k_y & \epsilon(x) \\ \epsilon(x) & 2\Delta k_y \end{pmatrix}$$

$$\partial_x U \chi = \frac{1}{2\Delta} \underbrace{U A U^\dagger}_1 U \chi$$

simpler: set $k_y = 0$

$$(2\Delta i \sigma_y \partial_x - \epsilon(x) \sigma_z) \chi = 0$$

$$\Rightarrow \partial_x \chi = \frac{\xi(x)}{2\Delta} \sigma_x \chi$$

$$\Rightarrow \chi_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \partial_x \chi_{\pm} = \pm \frac{\xi(x)}{2\Delta} \chi_{\pm}$$

$$\chi_{\pm} = \chi_{\pm}(x=0) e^{\pm \frac{1}{2\Delta} \int_0^x dx \xi(x)}$$

$$\xi = m = \begin{cases} m_0 & x > 0 \\ -m_0 & x < 0 \end{cases}$$

$$\chi_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\frac{1}{2\Delta} \int_0^x dx \xi(x)}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\frac{m_0}{2\Delta} |x|}$$

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\frac{m_0}{2\Delta} |x|} e^{ik_y y}$$

$$\hbar \psi = E \psi = 2\Delta k_y \psi$$

$$E = 2\Delta K_y$$

$$\gamma = \frac{2\Delta}{m_0} \rightarrow \infty$$

⊛ you need only follow
Dirac cones:

$$\mu = 4\varepsilon \Rightarrow -2\Delta(k_y\sigma^x + k_x\sigma^y) + \varepsilon\sigma^z$$

$$\text{rotate: } \sigma_x \rightarrow \sigma_y \rightarrow \sigma_y$$

$$\sigma_y \rightarrow -\sigma_x \rightarrow \sigma_x$$

$$\sigma_z \rightarrow \sigma_z \rightarrow -\sigma_z$$

$$\Rightarrow \vec{k} \cdot \vec{\sigma} + \frac{\varepsilon}{2\Delta} \sigma_z = \vec{k} \cdot \vec{\sigma} + m\sigma_z$$

$$\Delta h = n(m > 0) - n(m < 0) = 1$$

$$\Rightarrow n(m < 0) = n(\varepsilon < 0) =$$

$$\underline{= h(m, 0) - 1 = -1}$$

$$n = 0 \Rightarrow \dots$$

$$n = -4t \rightarrow \dots$$