Concepts of condensed matter physics

Spring 2021

Exercise #4

Due date: 07/07/2021

1. **Superconductivity on the surface:** In this question you will find that above H_{c2} there is a range of fields for which superconductivity can survive on the surface. Consult "Introduction to superconductivity", by M. Tinkham, page 135.
   - **a.** Start from the Ginzburg-Landau theory of a superconductor and neglect non-quadratic orders near the critical point. Write down the corresponding equations of motion, and using an analogy to the Schrodinger equation, find the critical field H_{c2}, above which superconductivity cannot nucleate in the interior of the sample. Write the result in terms of ϕ_0 and ξ. Can you explain the result qualitatively?
   - **b.** Consider the same physical setting with an edge at $x = 0$ (such that for $x > 0$ there is an insulator). Show that the boundary conditions take the form $\left(\frac{\nabla}{i} - \frac{2\pi A}{\phi_0}\right)\psi\Big|_n = 0$. Show that one can automatically satisfy this boundary condition by considering an auxiliary potential, containing a mirror image of the original potential in the insulating region. Does this affect the solution from part (a) well inside the superconductor (i.e., for $|x| \gg \xi$)?
   - **c.** Argue, using the auxiliary potential, that very close to the surface one can find a solution with lower energy, making the critical field higher near the surface.

2. **Little-Parks effect**

   Consider a superconductor which has the geometry of a ring with radius R and width d.
   - **a.** A flux ϕ penetrates the center of the ring. Write the Ginzburg-Landau theory for the ring, explain what is the condition to be in the quasi 1D limit.
   - **b.** How does T_c depend on ϕ? What is the corresponding coherence length $\xi(\phi)$? Discuss the limit of $R > \xi$ and $< \xi$.
   - **c.** So far we have implicitly ignored phase fluctuations due to vortices slipping in and out of the ring. Qualitatively, when is this a good approximation?

3. **The XY – sine-Gordon duality and the BKT critical behavior**

In this question you are asked to show the equivalence between the XY model and the sine-Gordon model:

$$S_{SG} = \frac{c}{2}\int d^2x \ (\nabla\theta)^2 - g\int d^2x \ \cos\theta \, ,$$

where θ is a non-compact real scalar field, and re-derive the RG equations near the BKT transition.

a. Expand $Z_{SG} = \int D\theta \ e^{-S_{SG}}$ in powers of g explicitly and show that it has the form

$$Z_{SG} = \sum_{n=0}^{\infty} \frac{\left(\frac{g}{2}\right)^{2n}}{(n!)^2} \prod_{j=1}^{2n} \int d^2x_j \, \langle \exp\left(i \sum_{j=1}^{2n} (-1)^j \theta(x_j) \right) \rangle$$

Where the $\langle\rangle$ brackets denote averaging with the free part $S_0 = \frac{c}{2}\int d^2x \ (\nabla\theta)^2$. Hint: recall that the free part is transnationally invariant such that $\langle (\prod_{a=1}^{N} e^{i\theta(x_a)})(\prod_{b=N+1}^{N+M} e^{-i\theta(x_b)}) \rangle$ is non-zero only for $N = M$.

b. Using the properties of the Gaussian average, namely

$$\langle e^A \rangle = e^{\frac{1}{2}\langle A^2 \rangle}$$

for A which is a linear combination of the field θ, and the following identity

$$\langle \left(\theta(x) - \theta(x')\right)^2 \rangle = \frac{C(x-x')}{c} = \frac{1}{2\pi c} \log\left|\frac{x-x'}{\xi}\right| ,$$

show that the partition function may be written as follows

$$Z_{SG} = \sum_{n=0}^{\infty} \frac{\left(\frac{g}{2}\right)^{2n}}{(n!)^2} \prod_{j=1}^{2n} \int d^2x_j \, \exp\left(\frac{1}{2c} \sum_{j<i}^{2n} \sigma_i \sigma_j C(x_i - x_j) \right)$$

where σ_i denotes the sign of the vortex and ξ is a short length cutoff. This is exactly the partition function of the Coulomb gas obtained in class!

c. Repeat the derivation of the RG differential equations near the BKT transition, namely

$$\frac{dy}{dl} = xy \, ; \ \frac{dx}{dl} = y^2 \, .$$

What are x and y in terms of c and g? (Here $l = \log\frac{r}{\xi}$)

d. Use the above equations to determine the screening length ξ_+ on the disordered side close to the transition. Do this by estimating the value of the running parameter l at which x and y reach order 1. Explain physically why ξ_+ is the screening length.

e. Obtain the superfluid stiffness J as a function of $t = T - T_c$ and show that it has a universal jump at T_c.