**Concepts of condensed matter physics -** Exercise #5

Spring 2021

Due date: 21/07/2021

**1.** In this question you will find the spectrum of the above BCS theory in the presence of spin-orbit and Zeeman coupling. In **one-dimension**, the Hamiltonian is given by:

$$\hat{H} = \int dx \left[ \sum_{s,s'} c_s^\dagger(x) \left( -\frac{\partial_x^2}{2m} + iu\sigma_z^{ss'}\partial_x + B\sigma_x^{ss'} - \mu \right) c_{s'}(x) - g c_\uparrow^\dagger(x) c_\downarrow^\dagger(x) c_\downarrow(x) c_\uparrow(x) \right]$$

a. First, neglecting $g$, diagonalize the quadratic Hamiltonian by going to momentum space. Draw the spectrum (qualitatively) – how do the spin-orbit and Zeeman terms alter the parabolic spectrum of free electrons ( $E = \frac{k^2}{2m} - \mu$ )?

b. Introducing finite $g$ and performing a mean field approximation, write a quadratic Hamiltonian of the form:

$$H = E_0 + \frac{1}{2}\sum_k \vec{\Psi}_k^\dagger h_{BDG}(k) \vec{\Psi}_k,$$

With $\vec{\Psi}_k = \begin{pmatrix} c_\uparrow(k) \\ c_\downarrow(k) \\ c_\downarrow^\dagger(-k) \\ -c_\uparrow^\dagger(-k) \end{pmatrix}$ and $h_{BDG}(k) = \begin{pmatrix} A(k) & D(k) \\ D^\dagger(k) & -\sigma_y A^*(-k)\sigma_y \end{pmatrix}$.

Find the $2 \times 2$ matrices $A(k)$ and $D(k)$.

c. Diagonalize $h_{BDG}$ and find the spectrum of excitations. Find the self-consistent gap equation and solve it.

d. Show that by changing the ratio $\Delta / B$, we reach a point in which the gap to excitations closes. Draw the spectrum at this point.

2. **RG analysis for the onset of Superconductivity in the presence of disorder**

Consider a thin three-dimensional superconducting material, with dimensions $W \times L \times L$ where$W \ll L$. The interaction term $H_{Inter} = u_b \sum_{k,p} c_{k\downarrow}^{\dagger} c_{-k\uparrow}^{\dagger} c_{-p\uparrow} c_{p\downarrow}$ leads to a pairing instability. Assume that the bare interaction parameter $u_b$ follows

$$u_b = \begin{cases} u_0 & for \quad \omega_D < \omega < E_F \\ u_0 - u_{ph} & for \quad \omega < \omega_D \end{cases},$$

where $\omega_D$ is the phonon Debye frequency, $E_F$ is the Fermi energy and $u_0 > u_{ph} > 0$. The RG flow of the running coupling constant $u$ is described by $\frac{d\Gamma}{dl} = \beta(\Gamma), with \Gamma = \nu u$ where $l = log\left(\frac{\Omega}{\omega}\right)$, $\nu$ the density of states and $\Omega$ is some ultra-violate cutoff.

   a. What is the function$\beta(\Gamma)$? Using its solution find the transition temperature, $T_c$, of the following clean systems ($\nu$ is the density of states of the material):
      i. $log\left(\frac{E_F}{\omega_D}\right) \approx 5, \Gamma_0 = \nu u_0 = 0.7$, and$\Gamma_{ph} = \nu u_{ph} = 0.3$.
      ii. $log\left(\frac{E_F}{\omega_D}\right) \approx 5, \Gamma_0 = \nu u_0 = 0.8$, and$\Gamma_{ph} = \nu u_{ph} = 0.04$.

   Now we would like to include the effect of disorder, which become important for frequencies below the Thouless energy given by$E_{Th} = \hbar \frac{D}{W^2} < \omega_D$, where $D$ is the diffusion constant.

   b. Based on the diffusion equation,$(\partial_t - D\partial_x^2)n = 0$, explain what is the physical origin of the Thouless energy.

   It can be shown that, due to the diffusive motion of the electrons, for $\omega < E_{Th}$ the RG equation is modified according to $\beta(\Gamma) \to g + \beta(\Gamma)$where $g = \pi \frac{e^2}{h} R$ and $R$ is the resistance of the sample.

   c. Discussed qualitatively the effect of disorder, via $g$, on the flow equation. What is the critical value of $g$ that will make the superconducting phase disappear?

      Bonus: what happens if $E_{Th} > \omega_D$?

   d. Assume that the interaction parameter at the Thouless energy can be expressed as

$$\Gamma(E_{Th}) \equiv \Gamma_{Th} = \frac{-1}{log\left(\frac{E_{Th}}{T_c^0}\right)},$$

where $T_c^0$ is the critical temperature in the clean case, and find an analytic expression for $T_c$ as a function of $u_{Th}$, and $g$. Assume that at $T_c$ the interaction parameter $u$, flows to $-\infty$.

You may find the following integral helpful:

$\int_{-\infty}^{-|\Gamma|} \frac{d\Gamma}{g-\Gamma^2} = \frac{1}{2\sqrt{g}} log\left(\frac{1-\frac{\sqrt{g}}{|\Gamma|}}{1+\frac{\sqrt{g}}{|\Gamma|}}\right)$, for $g < \Gamma^2$.

**e.** Draw qualitatively $T_c$ as a function of $g$. You may assume $\Gamma_{Th} = -0.1$ .