Concepts in Condensed Matter Physics: Exercise 1

Due date: May 1st 2025

1 The "Half-Haldane" Model

Consider the following tight binding model:

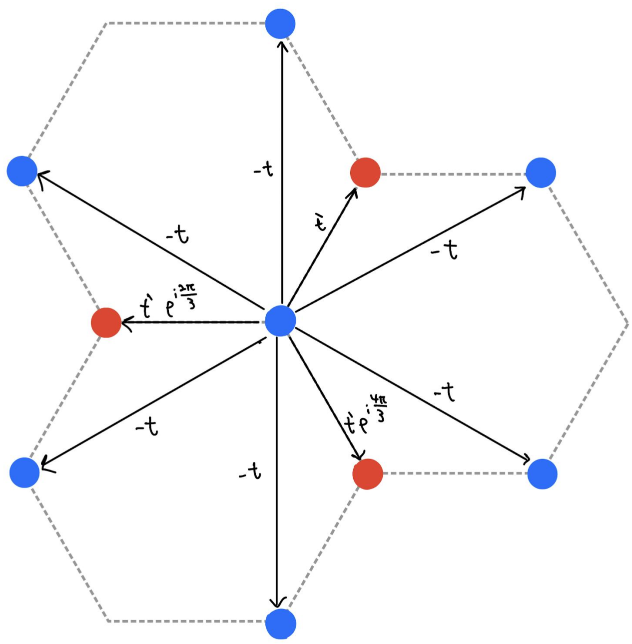

Figure 1: Sublattice hopping elements

We denote the lattice constant a as the distance between neighboring A (blue) atoms.

1. Write (in real space) the tight-binding Hamiltonian using sublattice spinors (in a similar manner to what you saw in class). Notice that t acts only between A atoms.

2. Consider the case of $t' = 0$. Express the single particle Hamiltonian in momentum space and find analytical expression for the dispersion of both bands $\epsilon_A(\mathbf{k})$, $\epsilon_B(\mathbf{k})$. Plot their dispersion along the 1d line $K - \Gamma - K'$.

3. Consider finite t' and write down the full tight-binding in momentum space.

4. Expand the tight-binding for small momenta up to second order in k.

5. Does a small t' gaps out the $t' = 0$ band crossing in the spectrum? Based on the form of the $A - B$ hybridization you found in (4), express qualitatively how does the lower band spinor vary as function of k. Specifically, how should it look in the vicinity of the $t = 0$ band crossing? How does it look as function of angle?

6. What should be the Chern number of this model? Does it depend on the sign of t'? If you are able to argue the correct answer based on the small t' limit it is sufficient. If not, you should calculate the Chern number explicitly. You are allowed to do it numerically as long as you explain exactly what you calculated.

2 Spin-wave dispersion

(Consult "Interacting Electrons and Quantum Magnetism" by A. Auerbach pages 123 - 126). In this question you are asked to derive the spin-wave dispersion of the two-dimensional Heisenberg model on a square lattice with antiferromagnetic coupling (i.e. $J > 0$). The Hamiltonian of such a model is given by

$$H = J \sum_{\langle i,j \rangle} \left[S_i^z S_j^z + \frac{1}{2} \left(S_i^+ S_j^- + S_i^- S_j^+ \right) \right]. \tag{1}$$

The $\langle i, j \rangle$ brackets denote summation over nearest neighbors.

1. Separate the lattice into two sub-lattices, A and B, such that all the neighbors of an A site are B's and vice versa. Now take $\left\langle S_j^z \right\rangle = \eta(j)$, with $\eta(j) = 1$ if $j \in A$ and $\eta(j) = -1$ if $j \in B$. What is the energy of this configuration?

2. Show that the mean-field solution you have obtained is not an eigenstate of the Hamiltonian, and thus is not the true ground state.

3. Now let us refine the solution by accounting for quantum fluctuations. First apply a rotation of π about the x axis to all spins on sub-lattice B, $\vec{S}_j \to \vec{\tilde{S}}_j$ (the idea is that we expand the Hamiltonian around the mean-field solution, where we have assumed that the spins are aligned along the z direction and anti-parallel to all their nearest neighbors). Now let us assume that all spins are fluctuating weakly around $\left\langle \tilde{S}_i^z \right\rangle \approx \frac{1}{2}$, such that we may introduce the Holstein-Primakoff bosons

$$S^z = \frac{1}{2} - n_b,$$

$$S^+ = \sqrt{1 - n_b} b,$$

$$S^- = b^\dagger \sqrt{1 - n_b},$$

where $n_b = b^\dagger b$. Apply this transformation to the Hamiltonian.

4. Diagonalize the bosonic Hamiltonian using a Bogoliubov transformation. (note that in the limit of weak fluctuations $\langle n_b \rangle \ll 1$). Plot the spin-wave dispersion schematically.

5. In class you have derived the spin-wave dispersion of the ferromagnetic model (i.e. $J < 0$). Discuss the difference in the long wave-length dependence of the fluctuations.