Concepts of condensed matter physics

Spring 2013

Exercise #2

Due date: 17/04/2013

1. **Decay of the gap function close to a boundary.** In this question we will obtain the qualitative behavior of the superconductors wave function close to a boundary. Consider the following spatially dependent GL theory

$$S_{GL} = \int_0^t dt \int d^d x \left[\psi^+ \left(-i\partial_t - \frac{\nabla^2}{2m^\star}\right)\psi + u\left(|\psi|^2 - \Delta(x)\right)^2\right]$$

where

$$\Delta(x) = \begin{cases} \rho_0, & x < 0 \\ -\rho_0, & x > 0 \end{cases}$$

   a. Derive the classical equation of motion, that is $\frac{\delta S_{GL}}{\delta \psi^+} = 0$.

   b. Assume that ψ depends only on the coordinate x and is time-independent. Now let us assume that at one side far away the system is in the superfluid phase such that $\psi(x = -\infty) = \sqrt{\rho_0}$ and that at the other side, far away, it is in the normal phase such that $\psi(x = \infty) = 0$. First find the static solution of the equation for $x < 0$ following these steps:

      i. First assume that ψ is real, why can you do this?

      ii. Multiply both sides of the equation by $\partial\psi/\partial x$. Now integrate the equation. What should be the constant of integration that obeys the boundary condition at $x \to -\infty$?

      iii. Solve for $\psi(x)$.

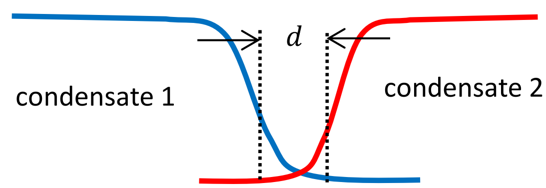
   c. Now seek the solution for $x > 0$. In this region the coefficient in front of $|\psi|^2$ in S_{GL} is positive, therefore we expect ψ to be small and decay towards $x \to \infty$. Thus, here you may neglect the quartic term in S_{GL} for $x > 0$ since it's contribution is small and only slightly modifies the wave function (Bonus: solve exactly and show that this is indeed the case).

d. Now stich the two solutions such that both the wave function and it's first derivative are continues. Remember that you have the constants of integration and a free coefficient (only for $x > 0$) at your disposal.

2. Josephson relation. Consider the real-time action of two decoupled condensates

$$S_{GL} = \sum_{a=1,2} \int_0^t dt[-i\psi_a^+ \partial_t \psi_a + u(|\psi_a|^2 - \rho_0)^2]$$

a. Now let us couple the two condensates by, first assuming that they decay at the points $x_0 = 0$ and $x_0 = d$ space (see the figure and use your results from question 1). Compute the overlap integral, E_J, between the two condensates.



b. The coupling between the condensates has the form

$$-E_J\, \psi_1^+ \psi_2 + h.c.$$

and describes Cooper-pair tunneling. Add this term to the action and write the action in terms of the complex fields ψ_a in a polar representation, *i.e.,*

$\psi_a = \sqrt{\rho_a}\, e^{i\theta_a}$.

c. Perform a canonical transformation to the sum and difference of phases, that is $\theta_\pm = \frac{\theta_1 \pm \theta_2}{\sqrt{2}}$. What are the conjugate fields of $\theta_\pm$? For large E_J the field θ_- is quenched, what is the physical consequence of that?

d. Add to the action the potential term $eV\rho_-$, where $\rho_- = \rho_1 - \rho_2$, and obtain the classical equations of motion for the current $I = e\partial_t \rho_-$ and the phase θ_- ? These equations are known as the Josephson relation. How does the supercurrent I depends on the voltage difference?

3. Persistent currents in a superconducting ring.

Consider the static GL free energy of a superconductor in a ring geometry

$$f_{GL} = a^2 \int_0^{2\pi} d\phi\, R \left[-\psi^+ \frac{(\nabla - ie^\star \boldsymbol{A})^2}{2m^\star} \psi + u(|\psi|^2 - \rho_0)^2 \right].$$

Here ϕ is the angular coordinate along the ring, R is the ring's radius, a^2 is the ring's cross section and we have coupled the superconductor to a magnetic flux in the center of the ring, such that $\boldsymbol{A} = \frac{\Phi}{2\pi R}\,\hat{\phi}$.

a. Use the polar representation of the superconductor in terms of the density ρ and the phase θ and find the physical constraint on the phase θ due to the periodic boundary conditions.

b. Neglecting fluctuations in ρ, find the free energy of the system as a function of the flux Φ. (hint you must minimize the GL theory to determine the angular dependence of θ). Plot the energy qualitatively as a function of Φ .

c. In order to obtain the current in the system let us consider $\boldsymbol{A}$ as a fluctuating field. In such a case the GL theory reads

$$F_{GL} = f_{GL} + \int d^d x \frac{1}{2} (\nabla \times \boldsymbol{A})^2$$

The equation of motion for $\boldsymbol{A}$ is given by the variation $\frac{\delta F_{GL}}{\delta \boldsymbol{A}} = 0$. Use the Maxwell equation $\boldsymbol{J} = \frac{1}{4\pi} \nabla \times \nabla \times \boldsymbol{A}$ to obtain the current density in terms of the parameters of f_{GL} .

d. Use this expression to obtain the current through some cross section of the ring as a function of Φ. What is the unit of the periodicity in Φ ?

e. Bonus: What happens if the ring has one narrow area which forms a Josphson junction (see question 2)?