Concepts of condensed matter physics

Spring 2013

Exercise #4

Due date: 19/05/2013

1. **Microscopic theory of superconductivity - BCS** (reference: "Introduction to superconductivity" by Michael Tinkham pages 59 - 64). The objective of this question is to gain intuition into the superconducting state using a mean-field approach. You are asked to decouple the interacting model using mean-field, then to diagonalize it, obtain the gap out of a self-consistency equation and then finally compute the energy reduction due to the formation of superconductivity. Let us consider the Hamiltonian of interacting Fermions

$$H = \sum_{\substack{|k-k_F|<\Lambda \\ \sigma}} \xi_k \, \psi^{+}_{k\sigma}\psi_{k\sigma} - g \sum_{\substack{|k-k_F|<\Lambda \\ |p-k_F|<\Lambda}} \psi^{+}_{p\uparrow}\psi^{+}_{-p\downarrow}\psi_{-k\downarrow}\psi_{k\uparrow}$$

Where the dispersion is given by $\xi_k = \frac{k^2}{2m} - \mu \approx v_F(k - k_F)$.

The sums over momentum states are cut off by $\Lambda = \omega_{\mathrm{D}}/v_F$ where ω_D is a typical energy scale of the phonon spectrum (i.e. the Debye frequency) and $v_F = k_F/m$ is the group velocity at Fermi level. Here g is the strength of the attractive interactions between electrons, which is generated by lattice vibrations. The logic behind the cutoff Λ is that the phonon-mediated interactions are attractive only for excitations that are close to the Fermi energy – to gain intuition why? read section 3.2 (pgaes 46-48) of "Introduction to superconductivity" by Michael Tinkham.

   a. Let us try and solve the Hamiltonian using mean-field. Assume that the operator $\psi_{-k\downarrow}\psi_{k\uparrow}$ has a finite mean, such that

$$\frac{1}{\Omega}\sum_{k}\langle\psi_{-k\downarrow}\psi_{k\uparrow}\rangle = \Delta/g$$

where Ω is the volume. The physics behind this mean-field is that pairs of fermions with opposite momenta and spin develop correlations, similar to a bound-state between two identical particles. Let us assume that the fluctuations around this mean-field are small such that we can write

$$\frac{1}{\Omega}\sum_k \psi_{-k\downarrow}\psi_{k\uparrow} = \frac{\Delta}{g} + \delta$$

where $\delta = \frac{1}{\Omega}\sum_k \psi_{k\uparrow}\psi_{-k\downarrow} - \frac{\Delta}{g}$ is considered to be small, and thus to lowest order we can keep only first order terms in δ. Apply this mean-field and neglect the quadratic terms in δ, in order to obtain a quadratic mean field Hamiltonian H_{MF}. Write H_{MF} explicitly in terms of Δ (Note: do not omit constant terms!)

b. To diagonalize the Hamiltonian we will use the Bogoliubov transformation

$$\psi_{k\uparrow} = u_k^*\gamma_k + v_k\eta_k^+$$

$$\psi_{-k\downarrow}^+ = -v_k^*\gamma_k + u_k\eta_k^+$$

where $|u_k| + |v_k| = 1$ are the coefficients of a canonical transformation which will be determined in the next section. Show that γ and η obey Fermionic commutation relations.

c. Find u_k and v_k such that H_{MF} will be diagonal, i.e. will assume the form

$$H_{MF} = \sum_k E_k(\gamma_k^+\gamma_k + \eta_k^+\eta_k) + E_{gs}$$

This can be done by solving the equations

$$[H_{MF}, \gamma_k] = -E_k\gamma_k$$

$$[H_{MF}, \eta_k] = -E_k\eta_k$$

Find the value of E_{gs} by plugging in the expressions for u_k and v_k.

Draw the quasi-particle dispersion E_k as a function of k (use the full free electron dispersion $\xi_k = \frac{k^2}{2m} - \mu$). The Hamiltonian you have obtained should have two bands (i.e. dispersion relations), one for the γ particles and one for the η particles, which are gapped from zero energy, what is the magnitude of the gap? Is this state a band insulator?

d. Use the mean-field Hamiltonian to compute the value of Δ in a self-consistent manner, i.e. compute (following equation 3.33 in Tinkham's book)

$$\Delta = \frac{g}{\Omega}\sum_k \langle \psi_{-k\downarrow}\psi_{k\uparrow}\rangle$$

by plugging in the expression for the Bogoliubov quasi-particles and using the diagonal form of H_{MF}. (comments: (1) Remember that the sum here is limited to $k \in (k_F - \Lambda, k_F + \Lambda)$ (or equivalently the energy scale ω_D). (2) To perform this calculation you will need to assume that the density of states of the fermions $N(\epsilon - \mu)$ is constant in the window of integration, i.e. $(\epsilon_F - \omega_D, \epsilon_F + \omega_D)$, such that you can substitute it with $N(0)$ (it will be easier to perform all integrals over energy momentum). (3) Notice that in the tutorial we have obtained the same equation, and solved it).

e. The constant E_{gs} term is the ground state energy of the superconductor. Compute the energy reduction in the superconducting state, i.e. show that at zero temperature (following pages 57-58 in Tinkham's book)

$$\delta E_{gs} = E_{gs} - \langle FS|H_{MF}|FS\rangle = -\Omega N(0)\frac{|\Delta|^2}{2}$$

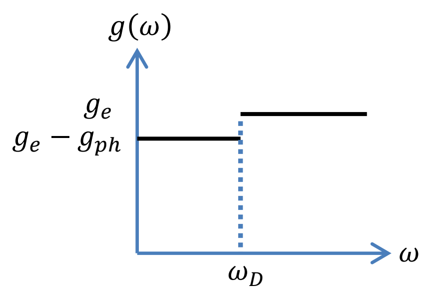
where $|FS\rangle = \prod_{|k|<k_F} \psi^+_{k\uparrow}\psi^+_{k\downarrow}|0\rangle$ is a full Fermi sea. Comments: use all comments from section d + assume that the gap is small, i.e. $\Delta \ll \omega_D$.

2. **Anderson-Morel gap equation - a way to understand the importance of retardation.** The objective of this question is to resolve one of the biggest questions in superconductivity. Namely, *how do two electrons which repel one another very strongly on the microscopic scale end up forming a bound-state at low energies?* To do so we will use the gap equation obtained in the tutorial (read the tutorial notes on-line thoroughly)

$$\Delta(\omega) = -\int_0^{\mu} dz \int_0^{\infty} d\xi \frac{g(\omega - z)\Delta(z)}{z^2 + \xi^2 + |\Delta_0|^2}$$

where

$$g(\omega) = \begin{cases} g_e - g_{ph}, & \omega < \omega_D \\ g_e, & \omega > \omega_D \end{cases}$$



is a frequency dependent (dimensionless) interaction (it is dimensionless because we have absorbed a DOS factor due to transforming from an integration over *k* to integration over ξ). g_e is the repulsive Coulomb interaction. g_{ph} is the attractive phonon-mediated interaction that appears only for $\omega < \omega_D$ due to retardation effects (here we have chosen a notation where all g's are positive). ξ is the fermion dispersion with constant DOS $N(0)$.

We will seek a solution for the gap which has the same form, i.e.,

$$\Delta(\omega) = \begin{cases} \Delta_0, & \omega < \omega_D \\ -\Delta_1, & \omega > \omega_D \end{cases}$$

where $\Delta_1 > 0$ and $\Delta_2 > 0$ are to be determined from the gap equation.

   a. Obtain the following equations for Δ_0 and Δ_1

$$\Delta_0 = (g_{ph} - g_e)\Delta_0 \log\frac{\omega_D}{\Delta_0} + g_e\Delta_1 \log\frac{\mu}{\omega_D}$$

$$-\Delta_1 = -g_e\Delta_0 \log\frac{\omega_D}{\Delta_0} + g_e\Delta_1 \log\frac{\mu}{\omega_D}$$

The first (second) equation is obtained by taking $\omega < \omega_D$ ($\omega > \omega_D$).

b. Solve the equations. Show that

$$\Delta_0 = \omega_D e^{-\frac{1}{g^*}}$$

what is g^*? Explain your result physically, how does g_e effect the value of Δ_0? Compare to section d in question 1. The realistic regime is where $g_e \gg g_{ph}$, is there a solution in this regime? Discuss the limits of superconductivity

c. Now let us obtain the same result from the point view of RG. Use the RG equation

$$\frac{d\tilde{g}}{d\log D} = N(0)\,\tilde{g}^2$$

to determine the gap. First, write this equation in a dimensionless form using $g = N(0)\tilde{g}$. Next, Integrate the equation from μ down to ω_D with a repulsive interaction $g(\mu) = g_e$. Now add a negative contribution g_{ph} and then continue the integration from ω_D to the scale at which g.