Concepts of condensed matter physics - Exercise #5

Spring 2014

Due date: 4/06/2014

1. **The XY – sine-Gordon duality and the BKT critical behavior**

In this question you are asked to show the equivalence between the XY model to the sine-Gordon model:

$$S_{SG} = \frac{c}{2}\int d^2x\ (\nabla\theta)^2 - g\int d^2x\ \cos\theta.$$

Where θ is a non-compact real scalar field.

a. Expand $Z_{SG} = \int D\theta\ e^{-S_{SG}}$ in powers of g explicitly and show that it has the form

$$Z_{SG} = \sum_{n=0}^{\infty}\frac{\left(\frac{g}{2}\right)^{2n}}{(n!)^2}\prod_{j=1}^{2n}\int d^2x_j\ \langle\exp\left(i\sum_{j=1}^{2n}(-1)^j\theta(x_j)\right)\rangle$$

Where the $\langle\rangle$ brackets denote averaging with the free part $S_0 = \frac{c}{2}\int d^2x\ (\nabla\theta)^2$. Hint: recall that the free part is transnationally invariant such that $\langle(\prod_{a=1}^{N}e^{i\theta_a})(\prod_{b=1}^{M}e^{-i\theta_b})\rangle$ is non-zero only for $N = M$.

b. Using the properties of the Gaussian average, namely

$$\langle e^A\rangle = e^{\frac{1}{2}\langle A^2\rangle}$$

for A which is a linear combination of the field θ, and the following identity

$$\langle\left(\theta(x) - \theta(x')\right)^2\rangle = \frac{C(x-x')}{c} = \frac{1}{2\pi c}\log\left|\frac{x-x'}{\xi}\right|,$$

show that the partition function may be written as follows

$$Z_{SG} = \sum_{n=0}^{\infty}\frac{\left(\frac{g}{2}\right)^{2n}}{(n!)^2}\prod_{j=1}^{2n}\int d^2x_j\ \exp\left(\frac{1}{2c}\sum_{j<i}^{2n}\sigma_i\sigma_j C(x_i - x_j)\right)$$

where σ_i denotes the sign of the vortex and ξ is a short length cutoff. This is exactly the partition function of the Coulomb gas obtained in class!

c. Repeat the derivation of the RG differential equations near the BKT transition, namely

$$\frac{dy}{dl} = xy\ ;\ \frac{dx}{dl} = y^2$$

What are x and y in terms of c and g? (Here $l = \log\frac{r}{\xi}$)

d. Use the above equations to determine the screening length ξ_+ on the disordered side close to the transition. Do this by estimating the value of the running parameter l at which x and y reach order 1. Explain physically why ξ_+ is the screening length.

e. Use the scaling argument to explain why the singular part of the full action

$$S = \frac{1}{2}\int d^2x\,[-\alpha|\psi|^2 + \beta|\psi|^4 + J\,(\nabla\psi)^2]$$

behaves like $S \sim \frac{1}{\xi^2}$ in the disordered side close to the transition. Deduce from this behavior that the free energy is (or the action is) perfectly analytic to all orders at the transition.

f. Obtain the superfluid stiffness J as a function of $t = T - T_c$ and show that it has a universal jump at T_c.

2. Mean-field approximation of the BKT transition point – In this question you are asked to use the variational principle in order to obtain the critical temperature at which a two-dimensional superconductor undergoes a Berezinksii-Kosterlitz-Thouless (BKT) transition between a state in which the order parameter's correlations decay like a power and a state at which they decay exponentially. We will use the vairational action

$$S_V = \int d^2x\,\left[\frac{c}{2}(\nabla\theta)^2 + \mathrm{m}^2\,\theta^2\right]$$

where m^2 is a variational parameter to estimate the ground state of S_{SG} from question 1. Notice that we can write

$$Z_{SG} = \int D\theta\, e^{-S_{SG}} = \int D\theta\, e^{-S_V} e^{-(S_{SG}-S_V)} = Z_0\langle e^{-(S_{SG}-S_V)}\rangle$$

Where the brackets denote averaging with S_V. The variaitonal principle amounts to making the following approximation.

$$Z_{SG} \approx Z_0 e^{-\langle(S_{SG}-S_V)\rangle} \equiv Z_V.$$

a. Show that $F_{SG} \leq F_V$, where F_{SG} is the free energy defined by the relation $F_{SG} = -T\log\left[Z_0\langle e^{-(S_{SG}-S_V)}\rangle\right]$ and therefore $F_V = F_0 + T\langle(S_{SG} - S_V)\rangle$.

b. Compute $\langle(S_{SG} - S_V)\rangle$ and minimize the free energy F_V

with respect to m. Here you will need to introduce some ultraviolet cutoff Λ which represents a microscopic scale at which the theory breaks down.

c. Show that $m = \Lambda^{\alpha}$, find α. By taking $\Lambda \to \infty$ find the critical value of c separating the massive and massless phases.

3. Dissipative quantum tunneling – In question 1 of problem set 3 you have obtained the Josephson relation for the supercurrent through a weak link between two superconductors. There, we have completely neglected the effect of spatial fluctuations in the superconducting leads. In this question we take them into account and see that they dissipate the current. Consider two one-dimensional superfluids connected by a weak link at $x = 0$, which are described by the following Lagrangian

$$L = \sum_{a=1,2} \left[\psi_a^{+} \left(\partial_\tau - \frac{\partial_x^2}{2m} \right) \psi_a + u(|\psi_a|^2 - \rho_0)^2 \right] - E_J(\psi_1^{+}\psi_2 + \mathrm{H.c.})\delta(x)$$

a. Use the polar representation $\psi_a = \sqrt{\rho_a} e^{i\phi_a}$ and the transformation to the relative and sum representation of phase and density $\phi = (\phi_1 - \phi_2)/2$, $\Phi = \phi_1 + \phi_2$, $\rho = \rho_1 - \rho_2$ and $\mathrm{P} = (\rho_1 + \rho_2)/2$, to show that in the limit of $\sqrt{\langle \rho^2 \rangle} \ll \langle \mathrm{P} \rangle = \rho_0$ the Lagrangian reduces to

$$L = i\rho\partial_\tau\phi + \frac{1}{2m}\left(\frac{(\partial_x\rho)^2}{2\rho_0} - 2\rho_0(\partial_x\phi)^2 \right) + \frac{u}{2}\rho^2 + g\cos 2\phi(0,\tau)$$

What is g?

b. Integrate out the fluctuations of ρ and obtain the effective Lagrangian

$$L = \frac{1}{2\mathrm{v}K}\left[(\partial_\tau\phi)^2 + (\mathrm{v}\,\partial_x\phi)^2\right] + g\cos 2\phi(0,\tau)$$

What are v and K?

c. Discuss the renomalization group flow of the coupling constant g. Note that the perturbation g is localized at $x = 0$, therefore, the first step is to integrate all modes at $x \neq 0$ to obtain an effective 1+0 dimensional action describing the junction. Use the relation between the Green's function $G(\tau) = \langle \phi\,(0,\tau)\phi(0,0) \rangle$ and the action that generates it

$$L_0 = \phi_0(\tau) G^{-1}(\tau - \tau')\phi_0(0)$$

to obtain the L_0. What is $G(\omega)$? What is the mechanical analog of this system?

Now perform the RG transformation according to the following step:

i. Define a cutoff D and separate the modes $\phi_0(\omega)$ into two types: "slow" where $|\omega| < D/b$ and "fast" where $D/b < |\omega| < D$, where $b = 1 + \delta D/D$ and δD is athin shell to be integrated out.

ii. Integrate out the "fast" modes up to lowest order needed in g such that you obtain an effective action for the "slow" modes.

iii. Redefine the frequency $\omega' = \alpha\omega$ such that the limits of integration are stretched over to span over $-D < |\omega'| < D$. What is α? Complete the RG transformation by redefining the slow fields $\phi_0(\omega')$ such that the action $S_0 = \int_{-D}^{D} d\omega'\, L_0(\omega')$ stays invariant.

iv. Obtain a differential equation by taking $\delta D \to 0$. What is the critical value of K? What is the behavior of g for $K > K_c$ and $K < K_c$.

d. Draw the flow diagram in the plane of g and K.