

Tutorials for "Concepts of condensed matter
physics"

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Chapter 1

Tutorial #1 - many-body path-integral formalism and the Hubbard-Stratonovich transformation

1.1 References for this tutorial

Altland & Simons Chapter 4 & 6.
Mahan

1.2 Introduction

In this tutorial we have two objectives: (i) to prove the identity (1.9). The motivation will be to show that quantum averages of many-body systems in thermal equilibrium can be computed using Feynman's path integral formalism. (ii) To provide a rigorous formalism in which the phenomenological Ginzburg-Landau (GL) theory of a superconductor can be related to its underlying Fermionic theory. Here the motivation is obvious.

1.3 Coherent-state path-integral

Let me quickly go over a few basic properties of the eigen-state of the annihilation operator a with eigen-value ψ , a.k.a known as the coherent state

$$|\psi\rangle \equiv e^{\zeta\psi a^\dagger}|0\rangle \tag{1.1}$$

where $\zeta = 1$ ($\zeta = -1$) for Bosons (Fermions).

1.3.1 c-numbers

In the simpler case a describes a bosonic degree of freedom and ψ is simply a c-number. We will make use of three basic identities

$$\langle \psi_1 | \psi_2 \rangle = e^{\bar{\psi}_1 \psi_2} \quad (1.2)$$

which implies the second identity (which is known as the resolution of identity)

$$1 = \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} |\psi\rangle\langle\psi| \quad (1.3)$$

Note that ψ is taken to be a vector with a discrete set of components ψ_i corresponding to the underlying Fock space. Thus its continuum limit will be a field, for example $\psi(x)$. Additionally, $\bar{\psi}\psi \equiv \sum_i \bar{\psi}_i \psi_i$ and $d\bar{\psi}d\psi \equiv \prod_i \frac{d\bar{\psi}_i d\psi_i}{\pi}$. The third identity we will need is the Gaussian integral of the complex variables ψ and $\bar{\psi}$

$$\int d\bar{\psi} d\psi e^{-\bar{\psi}A\psi} = \frac{1}{|A|} \quad (1.4)$$

where A is a matrix with a positive definite Hermitian part.

1.3.2 Grassmann numbers

If the operator a describes a fermionic excitation things are a bit more complicated. The eigen-value ψ can not be an ordinary complex number, because the operators a_i anti-commute amongst themselves, such that if the ψ_i 's were simple c-numbers then $\langle \psi | a_i a_j | \psi \rangle = 0$ would directly follow. Thus we need objects that anti-commute, these are known as Grassmann numbers:

$$\psi_i \psi_j = -\psi_j \psi_i \quad (1.5)$$

The operations of integration and derivation with these numbers are defined as follows

$$\int d\psi = 0 ; \int d\psi \psi = 1 \quad (1.6)$$

and

$$\partial_\psi \psi = 1 \quad (1.7)$$

The overlap between two coherent states and the resolution of identity remain in the form of (1.2) and (1.3). The Gaussian integral on the other hand is significantly different

$$\int d\bar{\psi}d\psi e^{-\bar{\psi}A\psi} = |A| \quad (1.8)$$

where A can be any matrix. **Exercise:** use (1.5) and (1.6) to prove (1.8).

1.4 Imaginary-time many-body path-integrals

In what follows we will prove the following identity

$$\mathcal{Z} = \text{Tr}e^{-\beta(\hat{\mathcal{H}}-\mu\hat{N})} = \int \mathcal{D}[\psi, \bar{\psi}] e^{-\int_0^\beta d\tau (\bar{\psi}\partial_\tau\psi + \mathcal{H}[\bar{\psi}, \psi] - \mu N[\bar{\psi}, \psi])} \quad (1.9)$$

where \mathcal{H} and N are the Hamiltonian and particle number respectively and $\psi, \bar{\psi}$ are c-numbers (Grassmann variables) in the case that the particles have Bosonic (Fermionic) mutual statistics. The boundary conditions of this path integral is $\psi(0) = \zeta\psi(\beta)$ and $\bar{\psi}(0) = \zeta\bar{\psi}(\beta)$. As mentioned above, our motivation will be computing expectation values of quantum many-body systems in thermal equilibrium, for example

$$\langle \hat{A} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\psi, \bar{\psi}] A[\psi, \bar{\psi}] e^{-\int_0^\beta d\tau (\bar{\psi}\partial_\tau\psi + \mathcal{H}[\bar{\psi}, \psi] - \mu N[\bar{\psi}, \psi])} \quad (1.10)$$

Question: why did we choose a coherent state path integral? and not say a real space or momentum path integral.

Let start with the definition of the trace

$$\mathcal{Z} = \text{Tr}e^{-\beta(\hat{\mathcal{H}}-\mu\hat{N})} = \sum_n \langle n | e^{-\beta(\hat{\mathcal{H}}-\mu\hat{N})} | n \rangle \quad (1.11)$$

Notice that each term in this sum is the probability amplitude of finding the the system at the same Fock state it started in, i.e. $|n\rangle$, after a time $t = i\hbar\beta$, which, as you know, can be casted to a Feynman path integral. In the first step we will want to "get rid" of the summation over n , to do so we insert the resolution of identity (1.3) into equation (1.11)

$$\mathcal{Z} = \frac{1}{\pi} \int d\bar{\psi}d\psi e^{-\bar{\psi}\psi} \sum_n \langle n | \psi \rangle \langle \psi | e^{-\beta(\hat{\mathcal{H}}-\mu\hat{N})} | n \rangle \quad (1.12)$$

Now need to shift the factor $\langle n | \psi \rangle$ around. In the case of bosonic particles this is just a number and it commutes with anything. In the case of fermions

it is a Grassmann number and therefore collects a minus sign which can be absorbed into the coherent state

$$\mathcal{Z} = \frac{1}{\pi} \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \langle \zeta\psi | e^{-\beta(\hat{\mathcal{H}} - \mu\hat{N})} | \psi \rangle \quad (1.13)$$

Now let us continue to the second step: we divide the imaginary-time evolution operator into M small steps

$$e^{-\beta(\hat{\mathcal{H}} - \mu\hat{N})} = \left[e^{-\delta(\hat{\mathcal{H}} - \mu\hat{N})} \right]^M \quad (1.14)$$

where $\delta = \beta/M$. In the third step we insert M resolutions of identity in the expectation value in equation (1.13)

$$\begin{aligned} \langle \zeta\psi | [e^{-\delta(\hat{\mathcal{H}} - \mu\hat{N})}]^M | \psi \rangle &= \int \prod_{m=1}^M d\bar{\psi}^m d\psi^m e^{-\sum_m \bar{\psi}^m \psi^m} \times \\ \langle \zeta\psi | \psi^1 \rangle \langle \psi^1 | e^{-\delta(\hat{\mathcal{H}} - \mu\hat{N})} | \psi^2 \rangle \langle \psi^2 | e^{-\delta(\hat{\mathcal{H}} - \mu\hat{N})} | \psi^3 \rangle \langle \psi^3 | \dots | \psi^M \rangle \langle \psi^M | e^{-\delta(\hat{\mathcal{H}} - \mu\hat{N})} | \psi \rangle \\ &= \int_{\psi^0 = \zeta\psi^M; \bar{\psi}^0 = \zeta\bar{\psi}^M} \prod_{m=1}^M d\bar{\psi}^m d\psi^m e^{\bar{\psi}^0 \psi^0 - \delta \sum_{m=0}^M (\delta^{-1}(\bar{\psi}^m - \bar{\psi}^{m+1})\psi^m + \mathcal{H}[\bar{\psi}^{m+1}, \psi^m] - \mu N[\bar{\psi}^{m+1}, \psi^m])} \end{aligned} \quad (1.15)$$

where we have denoted $\psi^0 = \zeta\psi^{M+1} = \psi$. Now if we insert this expression in (1.13) we get

$$\mathcal{Z} = \int_{\psi^0 = \zeta\psi^M; \bar{\psi}^0 = \zeta\bar{\psi}^M} \prod_{m=0}^M d\bar{\psi}^m d\psi^m e^{-\delta \sum_{m=0}^M (\delta^{-1}(\bar{\psi}^m - \bar{\psi}^{m+1})\psi^m + \mathcal{H}[\bar{\psi}^{m+1}, \psi^m] - \mu N[\bar{\psi}^{m+1}, \psi^m])} \quad (1.16)$$

Finally, the fourth step, we take $M \rightarrow \infty$ and obtain (1.9), where

$$\mathcal{D}[\bar{\psi}, \psi] \equiv \lim_{M \rightarrow \infty} \prod_{m=0}^M d\bar{\psi}^m \psi^m \quad (1.17)$$

1.5 The Hubbard-Stratonovich transformation

In the class you have seen the static GL theory of a superfluid/superconductor

$$f_{GL} = \int d^d x \left(\alpha |\Delta|^2 + \frac{1}{2m^*} |(\nabla - ie^* \mathbf{A})\Delta|^2 + \beta |\Delta|^4 \right) \quad (1.18)$$

This theory was suggested by Vitaly L. Ginzburg and Lev D. Landau in 1950 on a phenomenological basis and captures almost all the observed phenomena in conventional superconductivity. The field $\Delta(x)$ is bosonic, and the

question is how does the bosonic effective theory emerge from the underlying fermionic one? Here we will not discuss the full procedure including RG from the atomic scale, but rather start at the point where the effective theory is applicable, that is, where the electrons have attractive interactions

$$S = \int d\tau d^d x \left(\bar{\psi} \left(\partial_\tau + ie\phi - \frac{(-i\nabla - e\mathbf{A})^2}{2m} - \mu \right) \psi - g\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow \right) \quad (1.19)$$

and

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}\psi] e^{-S} \quad (1.20)$$

The only part in (1.19) that is really "hard" to diagonalize is the interaction term. To decouple it we use a trick known as the Hubbard-Stratonovich transformation (HST). This is done by using the simple identity

$$\begin{aligned} \int \mathcal{D}[\bar{\Delta}, \Delta] e^{\int d\tau d^d x \left(\frac{1}{g}|\Delta|^2 - \Delta\bar{\psi}_\uparrow\bar{\psi}_\downarrow - \bar{\Delta}\psi_\downarrow\psi_\uparrow \right)} &= \int \mathcal{D}[\bar{\Delta}, \Delta] e^{\int d\tau d^d x \left(\frac{1}{g}|\Delta - g\bar{\psi}_\uparrow\bar{\psi}_\downarrow|^2 - g\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow \right)} \\ &= N e^{-\int d\tau d^d x g\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow} \end{aligned} \quad (1.21)$$

where N is simply the result of the integral over Δ . Thus (1.19) can be rewritten as

$$S_{HS} = \int d\tau d^d x \left(\bar{\psi} \left(\partial_\tau + ie\phi - \frac{(\nabla - ie\mathbf{A})^2}{2m} - \mu \right) \psi + \Delta\bar{\psi}_\uparrow\bar{\psi}_\downarrow + \bar{\Delta}\psi_\downarrow\psi_\uparrow - \frac{1}{g}|\Delta|^2 \right) \quad (1.22)$$

and

$$\mathcal{Z} = \frac{1}{N} \int \mathcal{D}[\bar{\psi}\psi, \bar{\Delta}, \Delta] e^{-S_{HS}} \quad (1.23)$$

Now the Fermionic theory is quadratic

$$S_{HS} = \int d\tau d^d x \left(\bar{\Psi} (\mathcal{G}^{-1} + X) \Psi - \frac{1}{g}|\Delta|^2 \right) \quad (1.24)$$

where

$$\mathcal{G}^{-1} = \begin{pmatrix} [G_0^e]^{-1} & 0 \\ 0 & [G_0^h]^{-1} \end{pmatrix} = \begin{pmatrix} \partial_\tau + ie\phi - \frac{(-i\nabla - e\mathbf{A})^2}{2m} - \mu & 0 \\ 0 & \partial_\tau - ie\phi + \frac{(i\nabla - e\mathbf{A})^2}{2m} + \mu \end{pmatrix} \quad (1.25)$$

and

$$X = \begin{pmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{pmatrix} \quad (1.26)$$

and where the space of these matrices is the Nambu space $\bar{\Psi} = (\bar{\psi}_\uparrow, \bar{\psi}_\downarrow, \psi_\downarrow, -\psi_\uparrow)$. We can perform the path integral over the fermi fields using (1.8)

$$\begin{aligned} \mathcal{Z} &= \frac{1}{N} \int \mathcal{D}[\bar{\psi}\psi, \bar{\Delta}, \Delta] e^{-S_{HS}} = \frac{1}{N} \int \mathcal{D}[\bar{\Delta}, \Delta] \det[\mathcal{G}^{-1} + X] e^{\int d\tau d^d x \frac{1}{g} |\Delta|^2} \\ &= \frac{1}{N} \int \mathcal{D}[\bar{\Delta}, \Delta] e^{-(\int d\tau d^d x \frac{1}{g} |\Delta|^2 - \log \det[\mathcal{G}^{-1} + X])} = \frac{1}{N} \int \mathcal{D}[\bar{\Delta}, \Delta] e^{-(\int d\tau d^d x \frac{1}{g} |\Delta|^2 - \text{Tr} \log[\mathcal{G}^{-1} + X])} \end{aligned} \quad (1.27)$$

Therefore the effective bosonic theory is given by

$$\begin{aligned} S_{eff} &= - \int d\tau d^d x \frac{1}{g} |\Delta|^2 - \text{Tr} \log[\mathcal{G}^{-1} + X] \\ &= - \int d\tau d^d x \frac{1}{g} |\Delta|^2 - \text{Tr} \log \mathcal{G}^{-1} - \text{Tr} \log[1 + \mathcal{G}X] \\ &= - \int d\tau d^d x \frac{1}{g} |\Delta|^2 - \text{Tr} \log \mathcal{G}^{-1} - \text{Tr} \left(\mathcal{G}X + \frac{1}{2} \mathcal{G}X\mathcal{G}X + \dots \right) \end{aligned} \quad (1.28)$$

where the coefficients are

$$\text{Tr} \mathcal{G}X = 0 \quad (1.29)$$

$$\frac{1}{2} \text{Tr} \mathcal{G}X\mathcal{G}X = \Pi(\omega, q) = \frac{1}{\beta\Omega} \sum_{\nu k} G_0^e(\nu, k) G_0^h(\omega - \nu, q - k) \quad (1.30)$$

$$\text{Tr} \mathcal{G}X\mathcal{G}X\mathcal{G}X = 0 \quad (1.31)$$

and so on and so forth. The general form of the Fourier transformed Green's functions is

$$G_0^e(\nu, k) = \frac{1}{-i\nu + \epsilon_{\mathbf{k}} - \mu}$$

and

$$G_0^h(\nu, k) = \frac{1}{-i\nu - \epsilon_{\mathbf{k}} + \mu}$$

Here, to simplify i have neglected the electro-magnetic field (ϕ, \mathbf{A}) . The microscopic value of the quadratic GL parameters in (1.18) are then given by

$$\frac{\alpha}{2} = \frac{1}{g} - \Pi(0, 0) \quad (1.32)$$

and

$$\frac{1}{2m^*} = -\frac{1}{2} \partial_q^2 \Pi(q, 0)|_{q=0} \quad (1.33)$$

To compute sums of the form (1.30) we need to integrate over ν and k . Here i have hidden a few subtleties under the rug. First of all upon performing the Fourier transform of the imaginary-time fields

$$\psi(\tau) = \frac{1}{\sqrt{\beta}} \sum_{\nu} \psi(\nu) e^{-i\nu\tau} \quad (1.34)$$

we needed to use the waves of the form $e^{-i\nu\tau}$ that obey the boundary conditions $\psi(0) = \zeta\psi(\beta)$, thus ν is quantized

$$\nu_n = \begin{cases} \frac{2n\pi}{\beta} & \text{Bosons} \\ \frac{(2n+1)\pi}{\beta} & \text{Fermions} \end{cases} \quad (1.35)$$

These imaginary-time frequencies are known as *Matsubara frequencies*. Summing over them is a whole story to itself and i will not specify here how to do it, rather i will state the identity

$$\frac{\zeta}{\beta} \sum_n \frac{1}{-i\nu_n + x} = \begin{cases} n_B(x) & \text{Bosons} \\ n_F(x) & \text{Fermions} \end{cases} \quad (1.36)$$

where $n_B(x)$ ($n_F(x)$) is the Bose (Fermi) distribution function at temperature β^{-1} . Students that wish to understand how to perform these sums are invited to read in Altland & Simons pages 169-172 or the book by Mahan. **Exercise:** use (1.36) to perform the integral over ν in (1.30). Note that since the Greens functions here describe propagation of Fermionic particles ν is a Fermionic Matsubara frequency.

Chapter 2

Tutorial #2 - Renormalization Group approach to interacting electrons and the mean-field gap equation

2.1 References for this tutorial

[1] Altland & Simons pages 276-279

[2] "Renormalization-group approach to interacting fermions" by R. Shankar, Review of modern physics **66**, 129 (1992)

[3] Morel & Anderson, Physics Review, **125**, 1263 (1962)

2.2 Mean-field gap equation

To obtain the mean-field gap equation we return to equation (1.22) which was obtained by performing the HS transformation. The mean-field solution of this action is equivalent to seeking its saddle point, which essentially means neglecting all of the quantum fluctuation effects. To find the saddle point solution for $\Delta(\mathbf{x}, \tau)$ we take a variation of the action with respect to $\bar{\Delta}(\mathbf{x}, \tau)$ and average over Fermi fields.

$$\frac{\delta S_{HS}}{\delta \bar{\Delta}} = -\frac{1}{g} \Delta(\mathbf{x}, \tau) + \langle \psi_{\uparrow}(\mathbf{x}, \tau) \psi_{\downarrow}(\mathbf{x}, \tau) \rangle = 0$$

which gives

$$\Delta(\mathbf{x}, \tau) = g(\tau) \langle \psi_{\uparrow}(\mathbf{x}, \tau) \psi_{\downarrow}(\mathbf{x}, \tau) \rangle = g(\tau) F(\mathbf{x}, \tau) \quad (2.1)$$

where the expectation value is known as the anomalous Green's function. To obtain an expression for F let's look at the full Green's function

$$\mathcal{G}(\mathbf{x}, \tau) = \begin{pmatrix} G^e(\mathbf{x}, \tau) & F(\mathbf{x}, \tau) \\ \bar{F}(\mathbf{x}, \tau) & G^h(\mathbf{x}, \tau) \end{pmatrix} = \begin{pmatrix} \langle \bar{\psi}_\uparrow \psi_\uparrow \rangle & \langle \psi_\uparrow \psi_\downarrow \rangle \\ \langle \bar{\psi}_\downarrow \bar{\psi}_\uparrow \rangle & \langle \psi_\downarrow \bar{\psi}_\downarrow \rangle \end{pmatrix}$$

which is defined by the operatorial relation

$$\begin{pmatrix} \partial_\tau - \frac{\nabla^2}{2m} - \mu & \Delta(\mathbf{x}, \tau) \\ \bar{\Delta}(\mathbf{x}, \tau) & \partial_\tau + \frac{\nabla^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} G^e(\mathbf{x}, \tau) & F(\mathbf{x}, \tau) \\ \bar{F}(\mathbf{x}, \tau) & G^h(\mathbf{x}, \tau) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.2)$$

Omitting the \mathbf{x} and τ indices, we can write two relations

$$\left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) F + \Delta G^h = 0 \quad (2.3)$$

$$\bar{\Delta} F + \left(\partial_\tau + \frac{\nabla^2}{2m} + \mu \right) G^h = 1 \quad (2.4)$$

Solving for F we obtain

$$\begin{aligned} \left[\bar{\Delta} - \left(\partial_\tau + \frac{\nabla^2}{2m} + \mu \right) \Delta^{-1} \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \right] F &= 1 \rightarrow \quad (2.5) \\ F &= \left[|\Delta|^2 - \Delta \left(\partial_\tau + \frac{\nabla^2}{2m} + \mu \right) \Delta^{-1} \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \right]^{-1} \Delta \end{aligned}$$

Using (2.1) we obtain the desired result, a closed equation for the gap

$$\Delta = g \left[|\Delta|^2 - \Delta \left(\partial_\tau + \frac{\nabla^2}{2m} + \mu \right) \Delta^{-1} \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \right]^{-1} \Delta \quad (2.6)$$

Generally, this equation is hard to solve. However, it can greatly simplify if Δ is rather flat for times longer than $t_D \sim 1/\omega_D$ and varies slowly in space. In this case we can neglect the spatial and temporal dependence of Δ in the denominator. The idea is that $\Delta \ll \omega_D$ such that for frequencies of the order of ω_D the gap Δ is just a small correction, and thus it can be substituted by Δ_0 which is its value at $\omega = 0$. Equation (2.6) assumes the form

$$\Delta(\mathbf{x}, \tau) = g(\tau) \frac{1}{|\Delta_0|^2 - \left(\partial_\tau + \frac{\nabla^2}{2m} + \mu \right) \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right)} \Delta(\mathbf{x}, \tau) \quad (2.7)$$

To solve, we transform to Matsubara frequencies and momentum space

$$\Delta(\omega) = -\frac{1}{\beta\Omega} \sum_{\omega} \sum_{\mathbf{k}} \frac{g(\omega - \nu)\Delta(\nu)}{\nu^2 + \xi_{\mathbf{k}}^2 + |\Delta_0|^2} \quad (2.8)$$

where ν and ω are bosonic and fermionic Matsubara frequencies, $\xi = k^2/2m - \mu$, Ω is the volume of the system and we have used the convolution theorem. Notice that we have taken g (and also Δ) to be a function of time but independent on space. This mimics a local interaction with retardation. We will demonstrate the retardation effect in the exercise. For now let us keep get a feeling of this equation by considering Δ and g which are time independent, equation (2.7) assumes the form

$$\frac{1}{g} = \sum_{\nu \mathbf{k}} \frac{1}{\nu^2 + \xi^2 + |\Delta|^2}$$

First we sum over the fermionic Matsubara frequencies, we can use the identity (1.36) and the fact that

$$\frac{1}{\omega^2 + x^2} = \left(\frac{1}{i\omega + x} \right) \left(\frac{1}{-i\omega + x} \right) = -\frac{1}{2x} \left(\frac{1}{i\omega + x} - \frac{1}{i\omega - x} \right) \quad (2.9)$$

to obtain

$$\frac{1}{g} = \frac{1}{\Omega} \sum_{\mathbf{k}} \frac{1 - 2n_F(\sqrt{\xi^2 + |\Delta|^2})}{2\sqrt{\xi^2 + |\Delta|^2}} = \frac{1}{\Omega} \sum_{\mathbf{k}} \frac{\tanh\left(\frac{\beta E_k}{2}\right)}{2E_k} \quad (2.10)$$

where $E_k = \sqrt{\xi^2 + |\Delta|^2}$.

Let us use the gap equation to obtain the transition temperature, namely where $\Delta = 0$. In this case we have

$$\frac{1}{g} = \int_0^{\omega_D} d\xi \nu(\xi) \frac{\tanh\left(\frac{\beta_c \xi}{2}\right)}{\xi} \approx \nu \int_0^{\beta_c \omega_D/2} dx \frac{\tanh(x)}{x} \approx \nu \log \beta_c \omega_D \quad (2.11)$$

where $\nu(\xi)$ is the DOS and ν is the DOS at Fermi level. Inverting this equation we obtain

$$k_B T_c = \omega_D e^{-\frac{1}{\nu g}} \quad (2.12)$$

2.3 RG

2.3.1 Introduction to RG

In what follows we will derive the Gell-Mann differential equation that governs the renormalization group flow of the cooper channel interaction term in a fermionic theory with a Fermi surface

$$\frac{dg}{d \log D} = \frac{\nu}{2} g^2 \quad (2.13)$$

where D is the bandwidth, ν is the DOS at Fermi level and V is the flowing interaction constant describing interactions in the cooper channel

$$H_I = g \int d^d k d^d k' \psi_{\uparrow}^{\dagger}(\mathbf{k}) \psi_{\downarrow}^{\dagger}(-\mathbf{k}) \psi_{\downarrow}(-\mathbf{k}') \psi_{\uparrow}(\mathbf{k}') \quad (2.14)$$

Before we dive into the specifics of how to technically obtain this equation let us first understand the general picture.

Just from it's name you can understand that RG contains two ingredients: (i) renormalization of coupling constants and (ii) a scale transformation which transforms the renormalized action to a form which is identical to the original one. Let us see how this works:

renormalization of coupling constants. Generally, interactions can scatter particles from any two states in momentum space to any other two, as long as momentum is conserved. However, due to the Pauli principle at low temperatures $k_B T \ll \epsilon_F$ only processes where the two incoming and outgoing particles lie in a small energy shell around the Fermi energy $\epsilon_F \pm k_B T$ will be important. In such a case one can define a high energy cutoff $D \gg k_B T$ ¹. Now, consider this situation in the context of second order perturbation theory, we can integrate out processes in which a pair scatters to two high-energy states $\epsilon_k \sim D$ and then scatter back down to the Fermi surface. The energy correction in second order perturbation theory has the general form of some coupling constant squared over the energy of the virtual high-energy state. In our case we have two scattering events for the numerator, i.e. g^2 , and an energy denominator of order D , that is

$$g(D - \delta D) \rightarrow g(D) - \frac{\nu (g(D))^2}{D} \quad (2.15)$$

where i have inserted the DOS to set the units straight. In RG we take $\delta D \ll D$ and obtain a differential equation of the form (2.13), indeed (2.15) reduces to something of the form of (2.13) in this limit.

Scale transformation. The process of integrating out states in the small shell $D - \delta D < \epsilon < D$ modifies the theory. It is important to complete the process by bringing it back to it's original form, such that the physics predicted by the renormalized action will be identical to that of the original one up to a change in the couplings. To do this we re-scale the energy and momenta in the system, for example

$$\omega \rightarrow \omega' = \frac{D}{D - \delta D} \omega \approx \left(1 + \frac{\delta D}{D}\right) \omega = b\omega \quad (2.16)$$

¹Note that here we take temperature as a typical energy scale that cuts off the flow, and actually we are going to do our analysis at zero temperature where something else will replace this cutoff.

where ω is energy, for example the frequency dependence of the Fermi operators. As we will see this re-scaling may also lead to a flow of the coupling constants in the theory.

Comment: If one wishes to perform the RG process in Hamiltonian formalism he must keep in mind that the energy are also be renormalized. In other words the Schrodinger equation itself is modified and therefore the wave-function must be renormalized to compensate this.

2.3.2 Specifics of the RG transformation

Let us now start with the detailed process. **The first step** is to divide the action into to two parts

$$S = S_0 + S_I \quad (2.17)$$

where S_0 is the part of the action we can diagonalize and will be called the *fixed point* part and S_I represents all the additional terms that will "flow" and will be called the *couplings* part. In the case of interacting Fermions with a Fermi surface the fixed point part is given by

$$S_0 = \int d\omega \int d^d k \bar{\psi} \left(i\omega + \frac{k^2}{2m} - \epsilon_F \right) \psi \quad (2.18)$$

where $\psi = (\psi_\uparrow, \psi_\downarrow)^T$. It turns out that the strong renormalization effects occur close to the Fermi energy where $D \ll \epsilon_F$. Therefore we can linearize the spectrum around this point

$$S_0 = \int d\Omega k_F^{d-1} \int d\omega \int_{-\Lambda}^{\Lambda} d\delta k \bar{\psi} (i\omega + v_F \delta k) \psi \quad (2.19)$$

where $d\Omega$ represents integration over the d -dimensional solid angle, $v_F = k_F/m$, $\delta k = k - k_F$, $k_F = \sqrt{2m\epsilon_F}$ and Λ is a momentum cutoff of the linear spectrum. The bandwidth for the RG will be taken to be $D = v_F \Lambda$.

Our coupling term will simply be the contact interactions between the fermions

$$S_I = U_4 \int d1d2d3d4 \psi_\uparrow^\dagger(4)\psi_\downarrow^\dagger(3)\psi_\downarrow(2)\psi_\uparrow(2)\delta(1+2-3-4) \quad (2.20)$$

where $dj = d\omega_j d^d k_j$, $\psi(j) = \psi(\omega_j, \mathbf{k}_j)$ and $\delta(1+2-3-4) = \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)$. However, writing the interaction in this form it does not contain any information about the cutoff Λ . Actually limiting the phase space we allow this interaction to scatter particles limits it greatly, especially

in the limit $\Lambda \ll k_F$. To see why we may consider the delta function for the sum of incoming and outgoing momenta

$$\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \approx \frac{1}{k_F} \delta(\mathbf{n}_1 + \mathbf{n}_2 - \mathbf{n}_3 - \mathbf{n}_4)$$

where \mathbf{n}_j is unit vector pointing in the direction of the momentum vector \mathbf{k}_j . This delta function is limited to three cases

$$(I) \text{ Direct} \quad \theta_{14} = \theta_{23} = 0 \quad (2.21)$$

$$(II) \text{ Exchange} \quad \theta_{13} = \theta_{24} = 0 \quad (2.22)$$

$$(III) \text{ Cooper} \quad \theta_{12} = 0 \ \& \ \theta_{34} = 0 \quad (2.23)$$

where θ_{ij} is the angle between \mathbf{n}_i and \mathbf{n}_j . The idea is that the interaction term can be decoupled into these three cases, each of which will start from the coupling strength U_4 . Then we can focus only on the renormalization of these interactions, which will evolve differently under the RG process. In Shankars review he denotes the direct interactions by the function $F(\theta_{12})$ which, indeed can only be a function of θ_{12} . Similarly the exchange interaction is weighted by $-F(\theta_{12})$ and the Cooper channel by $V(\theta_{13})$.

Overall, the partition function is then given by

$$\mathcal{Z} = \int \mathcal{D}[\psi, \bar{\psi}] e^{-S_0 - S_I} \quad (2.24)$$

The second step is to divide the action into *fast* and *slow* parts, lets start with the fixed point part

$$\begin{aligned} S_0^f &= \int d\omega \int d\Omega \int_{-\Lambda}^{-\Lambda/b} dk k^{d-1} \bar{\psi}^f (i\omega + v_F k) \psi^f \\ &+ \int d\omega \int d\Omega \int_{\Lambda/b}^{\Lambda} dk k^{d-1} \bar{\psi}^f (i\omega + v_F k) \psi^f \end{aligned} \quad (2.25)$$

and

$$S_0^s = \int k_F^{d-1} d\Omega \int d\omega \int_{-\Lambda/b}^{\Lambda/b} d\delta k \bar{\psi}^s (i\omega + v_F \delta k) \psi^s \quad (2.26)$$

where $b = 1 + \delta\Lambda/\Lambda$ is larger than 1. This part separates nicely since $S_0 = S_0^f + S_0^s$. However, the interaction term mixes between fast and slow modes such that we have

$$S_I = S_I^{ss} + S_I^{sf} + S_I^{ff} \quad (2.27)$$

where the important term has four Fermion terms of the form

$$S_I^{sf} \sim \bar{\psi}_\uparrow^f \bar{\psi}_\downarrow^f \psi_\downarrow^s \psi_\uparrow^s + \bar{\psi}_\uparrow^f \bar{\psi}_\downarrow^s \psi_\downarrow^s \psi_\uparrow^f + \bar{\psi}_\uparrow^s \bar{\psi}_\downarrow^f \psi_\downarrow^s \psi_\uparrow^f + \dots \quad (2.28)$$

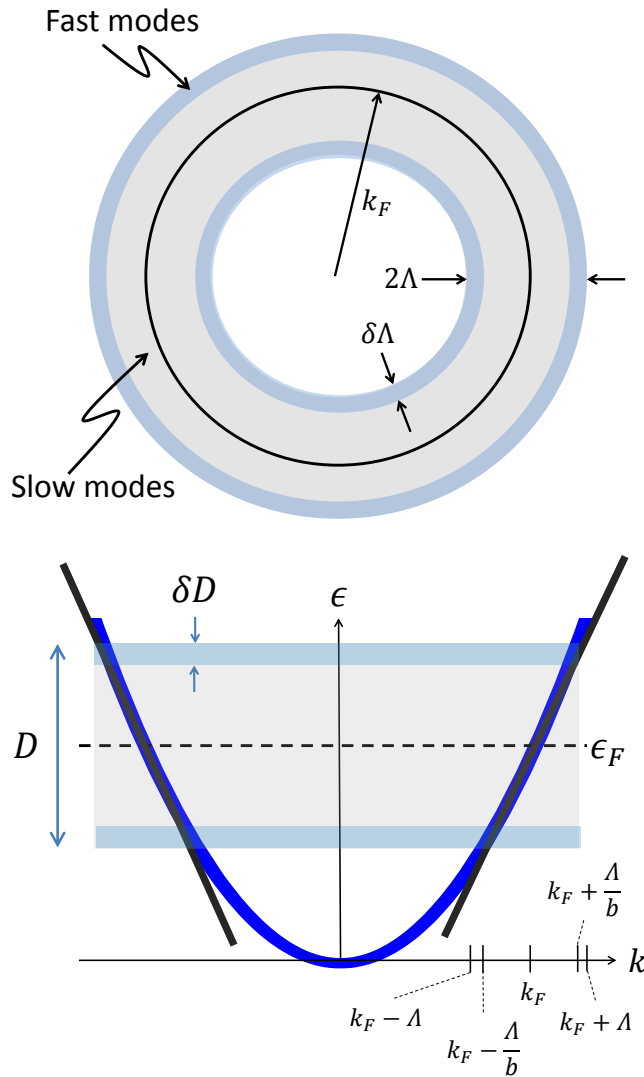


Figure 2.1: **(Top)** The Fermi surface and the states in momentum space we wish to consider. The grey shaded region are the band width $\Lambda = D/v_F$ and the blue strips are the states we call fast modes which we integrate out. **(Bottom)** The dispersion from side view with the linearization close to the chemical potential (the black dashed line is the chemical potential.)

i.e. involves two fast and two slow particles. Over all the partition function has the form

$$\mathcal{Z} = \int \mathcal{D}[\psi, \bar{\psi}] \exp \left[-S_0^s - S_0^f - S_I^{ss} - S_I^{sf} - S_I^{ff} \right] \quad (2.29)$$

The third step is to integrate out the fast modes. Formally, the partition function assumes the form

$$\mathcal{Z} = \int \mathcal{D}[\psi, \bar{\psi}] e^{-S_0^s - S_I^{ss}} \langle e^{-S_I^{sf} - S_I^{ff}} \rangle_f \quad (2.30)$$

where the $\langle \rangle_f$ brackets denote averaging over fast-modes. Of course this averaging can not be done exactly, otherwise we would simply solve the problem, but it can be done perturbatively. The idea is to use the cumulant expansion which relates the expectation value of an exponential to the exponential of expectation values

$$\langle e^{g\hat{O}} \rangle_f = e^{g\langle \hat{O} \rangle_f + \frac{g^2}{2} (\langle \hat{O}^2 \rangle_f - \langle \hat{O} \rangle_f^2) + \dots} \quad (2.31)$$

In our case $g\hat{O} = -S_I^{fs} - S_I^{ff}$. Here we will focus only on the first non-trivial correction, which is of order U_4^2 . Note that from all the possible diagrams we will only be interested in those that have two incoming and two outgoing slow modes since they are the ones that renormalize the interaction term (see top panel of Fig.[2.2]). **Comment: Terms that have an odd number of slow modes have also an odd number of fast ones and therefore vanish under averaging due to symmetry. Terms that have only two slow modes will renormalize the chemical potential, this will not interest us here since it is self consistently set to satisfy particle number.** The term $-\langle \hat{O} \rangle_f^2$ essentially cancels all diagrams in $\langle \hat{O}^2 \rangle_f$ which are the square of some simpler diagram, i.e. disconnected diagrams. Thus only connected diagrams will interest us. Given all these simplifications the only two diagrams that contribute to second order (one-loop) are presented in the middle panel of Fig.[2.2]. To see why lets look at the formal expression

$$\frac{U_4^2}{2} \left(\langle \hat{O}^2 \rangle_f - \langle \hat{O} \rangle_f^2 \right) = \frac{U_4^2}{2} \int_{|\delta k| < \Lambda/b} d1d2d3d4 \int_{\Lambda > |\delta k| > \Lambda/b} d5d6d7d8 \times \quad (2.32)$$

$$\bar{\psi}_\uparrow^s(4) \bar{\psi}_\downarrow^s(3) \langle \bar{\psi}_\uparrow^f(8) \bar{\psi}_\downarrow^f(7) \psi_\downarrow^f(6) \psi_\uparrow^f(5) \rangle_f \psi_\downarrow^s(2) \psi_\uparrow^s(1) \delta(\text{vertex1}) \delta(\text{vertex2})$$

This is the most general way of writing a one-loop diagram with the structure of the top panel diagram in Fig.[2.2]. The two ways in which this term can be contracted are schematically presented in the bottom of Fig.[2.2].

Let us postpone the actual calculation of these diagrams to later on. For now we can say that if we have indeed computed these diagrams then they give us the following renormalization

$$U_4 \rightarrow U'_4 = U_4 + \frac{U_4^2}{2} (ZS' + \text{BCS}) \quad (2.33)$$

Note that this equation is schematic, actually only special types of interactions will be renormalized.

The fourth step is to stretch the momenta and frequency such that the fixed point part goes back to itself. This can be done by performing a change of variables

$$\omega' = b\omega \ ; \ \delta k' = b\delta k \quad (2.34)$$

which will give us

$$S_0^s = b^{-3} \int k_F^{d-1} d\Omega \underbrace{\int d\omega' \int_{-\Lambda}^{\Lambda} d\delta k'}_{b^{-2}} \underbrace{\bar{\psi}(i\omega' + v_F k') \psi}_{b^{-1}} \quad (2.35)$$

Thus, the fixed point part will be invariant under this transformation if

$$\psi' = b^{-\frac{3}{2}} \psi \quad (2.36)$$

Using this scaling we find that the interaction term

$$S_I^{ss} = U_4 \int_{|k| < \Lambda/b} \underbrace{d1d2d3d4}_{b^{-8}} \underbrace{\bar{\psi}_\uparrow(4)\bar{\psi}_\downarrow(3)\psi_\downarrow(2)\psi_\uparrow(1)}_{b^6} \underbrace{\delta(1+2-3-4)}_{b^{-2}} \quad (2.37)$$

goes back to its original form (2.20). In other words the four fermion interaction term is also invariant under the RG transformation????!! Well, we haven't computed the higher order corrections yet, so actually it is invariant up to the order of U_4 , which is also called the tree-level. When a coupling term is invariant under the RG transformation up to tree-level we call it marginal. **Exercise:** show that the term $\mu \int d\omega d^d k \bar{\psi} \psi$ grows by a factor of b under the transformation (and is thus relevant) and that $\lambda \int d\omega d^d k k^4 \bar{\psi} \psi$ shrinks by a factor of b^{-1} (and is thus irrelevant). **Comment:** It will be instructive to show that fermi interactions with more than 4 fields are irrelevant. To see this let us write a 6 field interaction

$$S_6 = U_6 \int \underbrace{d1d2d3d4d5d6}_{b^{-12}} \underbrace{\bar{\psi}(6)\bar{\psi}(5)\bar{\psi}(4)\psi(3)\psi(2)\psi(1)}_{b^9} \underbrace{\delta(1+2+3-4-5-6)}_{b^2} \quad (2.38)$$

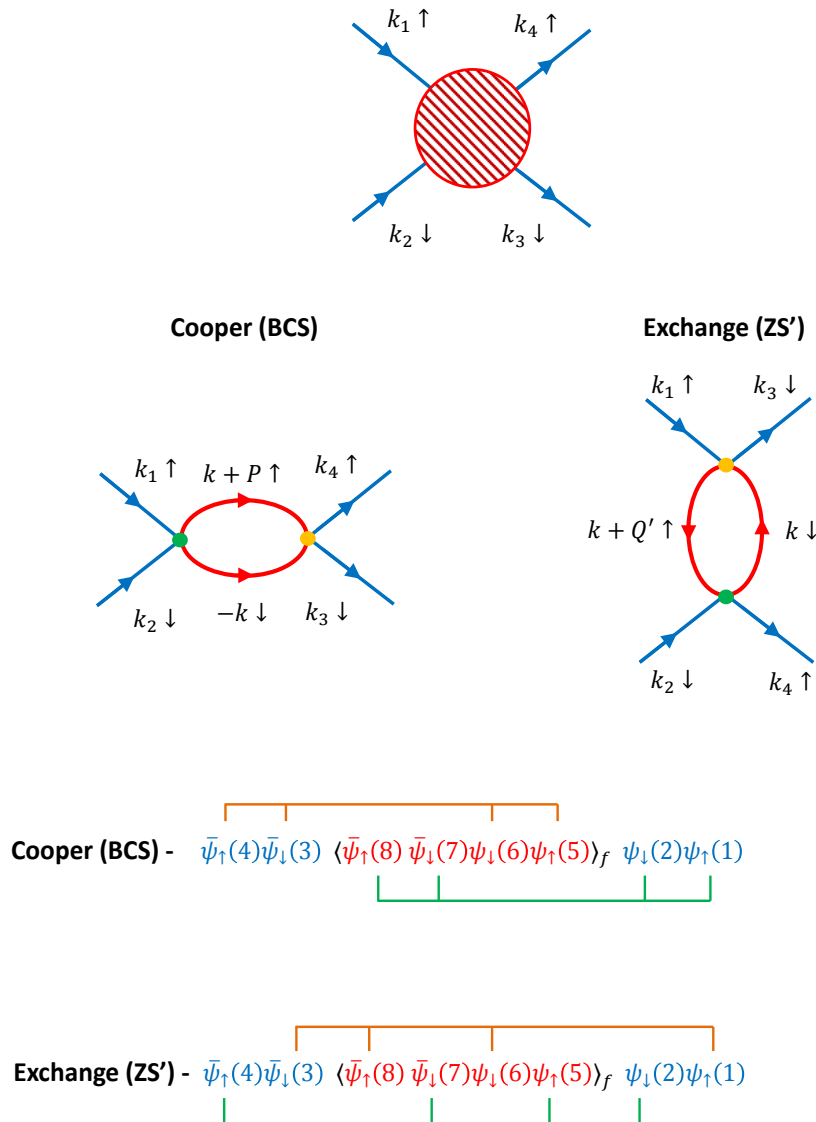


Figure 2.2: **(Top)** The generic structure of a diagram that renormalize S_I^{ss} , which contains two incoming and two outgoing slow modes with any number of fast modes in the middle. The red circle represents any diagram in-which all propagators belong to fast modes and is not simply a square of a simpler diagram. **(Middle)** The two important diagrams at one-loop level. Here blue (red) lines represent propagators of slow (fast) modes. The intermediate momenta are $P = k_1 + k_2$, $Q' = k_1 - k_3$ and k is the loop momentum which is summed over. **(Bottom)** The two different ways in-which the one-loop correction can be contracted with two connected vertices and their corresponding diagrams.

overall this gives b^{-1} such that at order one the differential equation is $dU_6/d \log b = -U_6$. Thus, overall both the coupling and the fixed point part are go back to their original form due to the stretching. We can sum our results with the following equation

$$\begin{aligned} U_4(D - \delta D) &= U_4(D) + \frac{U_4^2(D)}{2} (\text{ZS}' + \text{BCS}) \\ &\rightarrow \frac{dU_4}{dD} = -\frac{U_4^2}{2} (\text{ZS}' + \text{BCS}) \end{aligned} \quad (2.39)$$

Again remember that this equation is schematic.

Finally, let us return to step two and compute the ZS' and BCS diagrams. As an example we will do this in two-dimensions which can be generalized to any dimension. Let us start with the ZS' diagram

$$\text{ZS}' = -\frac{k_F}{8\pi^2} \int d\omega \int_0^{2\pi} d\phi \int_{\delta\Lambda} d\delta k \frac{1}{i\omega - v_F \delta k} \frac{1}{i(\omega + \nu) - v_F(|\mathbf{k} + \mathbf{Q}'| - k_F)} \quad (2.40)$$

where $\mathbf{Q}' = \mathbf{k}_1 - \mathbf{k}_3$, $\nu = \omega_1 - \omega_3$ and $\delta\Lambda$ denotes summation over regions where both $\Lambda/b < |\delta k| < \Lambda$ and $\Lambda/b < ||\mathbf{k} + \mathbf{Q}'| - k_F| < \Lambda$ (i.e. the overlap between the blue shaded areas in Fig.[2.3]). We will consider only static corrections to the interactions such that $\omega_1 = \omega_2 = \omega_3 = \omega_4 = 0$ such that $\nu = 0$. Thus we have

$$\text{ZS}' = -\frac{k_F}{8\pi^2 v_F} \int d\omega \int_0^{2\pi} d\phi \int_{\delta\Lambda} d\delta k \frac{n_F(\delta k) - n_F(|\mathbf{k} + \mathbf{Q}'| - k_F)}{|\mathbf{k} + \mathbf{Q}'| - k_F - \delta k} \quad (2.41)$$

where we have used the identity (1.36)

$$\int d\omega \frac{1}{i\omega - x_1} \frac{1}{i\omega - x_2} = \int d\omega \frac{1}{x_2 - x_1} \left(\frac{1}{i\omega - x_1} - \frac{1}{i\omega - x_2} \right) = \frac{n_F(x_1) - n_F(x_2)}{x_2 - x_1}$$

Now the integration is only over the region in-which both $\Lambda/b < |\delta k| < \Lambda$ and $\Lambda/b < ||\mathbf{k} + \mathbf{Q}'| - k_F| < \Lambda$. It is easy to see from Fig.[2.3] that this diagram will have contributions of order $\delta\Lambda$ only for $\mathbf{Q}' = 0$. This means that this diagram only contributes to processes where $\mathbf{k}_1 = \mathbf{k}_3$, i.e. it renormalizes the exchange interaction $-F(\theta_{12})$ (in Shankar's notations). Unfortunately, as \mathbf{Q}' is taken to zero the window of integration given by the Fermi functions focusses around the Fermi surface, a region where only slow modes exist.

We are left with the BCS diagram which has the form

$$\text{BCS} = -\frac{k_F}{8\pi^2} \int d\omega \int_0^{2\pi} d\phi \int_{\delta\Lambda} d\delta k \frac{1}{-i\omega - v_F \delta k} \frac{1}{i\omega - v_F(|\mathbf{k} + \mathbf{P}| - k_F)} \quad (2.42)$$

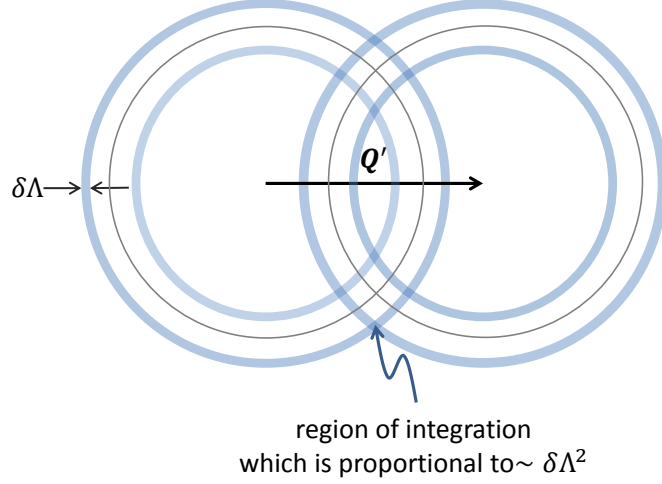


Figure 2.3: The overlap between the strips of fast particles that integrated out in the exchange channel with finite momentum Q' . The diagram shows that aside from $Q' = 0$ all Q' 's contribute $\delta\Lambda^2$ which can be neglected.

Again, in the limit $\delta\Lambda \rightarrow 0$ only the $P = 0$ (i.e. $\mathbf{k}_1 = -\mathbf{k}_2$) will contribute. Thus only the interaction between pairs with opposite momenta will be renormalized, this is nothing but the Cooper channel denoted by $V(\theta_{13})$ (in Shankar's notations). Since we are interested in contact interaction and $V(\theta_{13})$ is isotropic we will denote it by a constant g . Let us go back to computing the BCS diagram, we use the identity

$$\begin{aligned} \int d\omega \frac{1}{i\omega - x_1} \frac{1}{-i\omega - x_2} &= - \int d\omega \frac{1}{x_2 + x_1} \left(\frac{1}{i\omega - x_1} - \frac{1}{i\omega + x_2} \right) \quad (2.43) \\ &= - \frac{n_F(x_1) - n_F(-x_2)}{x_1 + x_2} = \frac{1 - n_F(x_1) - n_F(x_2)}{x_1 + x_2} \end{aligned}$$

Thus, as we take $P \rightarrow 0$ we obtain

$$\text{BCS} = -\frac{k_F}{8\pi^2 v_F} \int d\omega \int_0^{2\pi} d\phi \int_{\Lambda/b < |\delta k| < \Lambda} d\delta k \frac{1 - 2n_F(\delta k)}{2\delta k} = -\frac{\nu}{2} \log b \quad (2.44)$$

where we have used the identity $\nu = k_F/2\pi v_F$. Finally, finally we have arrived at the desired result. If we denote V the running coupling constant standing in front of an interaction in the Cooper channel (2.14) then we have

$$g(\Lambda - \delta\Lambda) = g(\Lambda) - \frac{\nu}{2}g^2(\Lambda) \log\left(1 + \frac{\delta\Lambda}{\Lambda}\right) \quad (2.45)$$

as we take $\delta\Lambda$ to zero we obtain

$$\frac{dg}{d\log\Lambda} = \frac{dg}{d\log D} = \frac{\nu}{2}g^2 \quad (2.46)$$

or in a unitless form

$$\frac{d\Gamma}{d\log\Lambda} = \frac{\Gamma^2}{2} \quad (2.47)$$

where $\Gamma = \nu g$.

2.3.3 The superconducting transition temperature

It would be a waste to do all the hard work with no play. Let us see what this equation can do. Let us assume that around $D = \omega_D$ the interaction (2.14) in the Cooper channel becomes attractive but small. Now we reduce the cutoff from ω_D towards zero. Of course the Gell-Mann equation (2.47) predicts that V will become more and more negative until it will diverge to $-\infty$. However, we must keep in mind that our RG analysis holds only if Γ is small, namely if $\Gamma < 1$. Therefore let us stop the cutoff reduction when $\Gamma = 1$ and denote that bandwidth as $D_{final} = k_B T_c$ such that by integrating both sides of (2.47) we have

$$\int_{\omega_D}^{k_B T_c} \frac{d\Gamma}{\Gamma^2} = \frac{1}{\Gamma(\omega_D)} - 1 = \frac{1}{2} \log \frac{\omega_D}{k_B T_c}$$

which gives

$$k_B T_c \approx \omega_D e^{-\frac{2}{\nu|g(\omega_D)|}} \quad (2.48)$$

Equivalently we can write an expression for the interaction as a function temperature

$$g(T) = \frac{g(\omega_D)}{1 + \nu g(\omega_D) \log \frac{\omega_D}{T}} \quad (2.49)$$