Notation and preliminaries: We use the notation $[n] = \{1, \ldots, n\}$. Unless explicitly stated, an algorithm is allowed to be randomized (and for all inputs x succeeds with probability $\geq 2/3$).

1 Amplifying the Success Probability of Randomized Algorithms

In this question we will see how to turn an algorithm that makes two-sided error with probability $1/3$ *into one that (a) has much smaller error probability and (b) only has one-sided error.*

Let $k \in \{3, 4, 5, \ldots\}$ be a fixed constant. A k-clique in a graph is a set of k (distinct) nodes such that all pairs are neighbors. Suppose that there is a randomized algorithm $\mathcal{A}$ that decides if a graph on n nodes contains a k-clique in $O(T(n))$ time and is correct with probability at least $2/3$.

(a) Design an algorithm $\mathcal{A}'$ that takes $O(T(n) \cdot \log n)$ time and errs only with probability at most $1/n^{10}$.
(*Hint:* Repeat and take the majority. Use a Chernoff bound for the analysis.)

(b) Design an algorithm $\mathcal{A}''$ that takes $O(T(n) \cdot \log^2 n)$ time and errs only with probability $1/n^9$, and in addition, must be correct when it outputs *yes*.
(*Hint:* Use an idea we saw in class to obtain a witness. Use a Union Bound for the analysis.)

Remark You are encouraged to read (and use anything from) this short *primer to randomness* from the fine-grained complexity course at Max-Plank Institute: (click here).

2 Convolution 3SUM to 3SUM

Prove that if 3-SUM on n numbers in $[-U, +U]$ can be solved in $T(n, U)$ time, then the following problem can be solved in $T(O(n), O(nU))$ time.

Convolution 3SUM: Given three arrays A, B, C each containing n numbers in $[-U, +, U]$ decide if there are two numbers $i, j \in [n]$ such that $A[i] + B[j] + C[i + j] = 0$.

3 Triangle Detection to 3SUM

A triangle in a graph is a 3-clique. Prove that if 3SUM can be solved in $T(n, U)$ time then one can detect if a graph on n nodes and m edges has a triangle in $O(\log n) \cdot T(O(m), O(n^3))$ time.

4 3SUM to Sumset Size

The *sumset* $X+Y$ of two sets of integers X, Y is the set of all pairwise sums $X+Y = \{x+y \mid x \in X, y \in Y\}$. The size of the sumset $|X + Y|$ is at most n^2 but could be smaller.

Prove that Sumset Size is 3SUM-Hard: if we can compute the size of the sumset of two sets in $O(n^{2-\varepsilon})$ time, for some $\varepsilon > 0$, then the 3SUM Conjecture is false.