Notation and preliminaries: Unless explicitly stated, an algorithm is allowed to be randomized (and for all inputs x succeeds with probability $\geq 2/3$). The $\tilde{O}(\cdot)$ notation hides polylogarithmic (in n) factors.

1 Large k to Small k

(a) Suppose that there is an $\varepsilon > 0$ such that 10-SUM can be solved in $O(n^{5-\varepsilon})$ time. Prove that there is a $\delta > 0$ such that 100-SUM can be solved in $O(n^{50-\delta})$ time.

(b) Let $k \geq 3$ be a constant that is a multiple of 3. Design an algorithm that decides if a graph on n nodes contains a k-clique in $\tilde{O}(n^{\omega k/3})$ time.

2 APSP and Triangles in Medium Sparsity Graphs

The input in each of the following problems is a weighted graph $G = (V, E, w), w : E \to [1, U], U = n^{O(1)}$ on $n = |V|$ nodes and $m = |E|$ edges. Assuming the APSP Conjecture that we saw in class (for dense graphs), prove or disprove the following claims about the medium sparsity regime:[1]

(a) APSP can be computed in $\tilde{O}(n^{2+1/3})$ time when $m = O(n^{4/3})$.

(b) Min-Triangle can be computed in $\tilde{O}(n^2)$ time when $m = O(n^{4/3})$.

(c) APSP can be computed in $\tilde{O}(n^2)$ time when $m = O(n^{4/3})$.

(d) Min-Triangle can be computed in $\tilde{O}(n^{4/3})$ time when $m = O(n^{4/3})$.

3 3SUM-Listing

Prove that if 3SUM is in $O(n^{2-\varepsilon})$ time, then the following *3SUM-Listing* problem is in $\tilde{O}(n^{2-\varepsilon/2})$ time: Given three sets of n integers $A, B, C \subseteq [-n^3, n^3]$ return all numbers $a \in A$ such that there exist $b \in B, c \in C$ with $a + b + c = 0$.

Hint: If we partition each set arbitrarily into n/s groups of size s, then a 3-sum can lie in any of the $(n/s)^3$ triples of groups. There is a more clever (randomized) partitioning into $O(n/s)$ groups of size $O(s)$ in which there is a subset of only $O\left((n/s)^2\right)$ triples of groups such that any 3-sum is contained in one of these triples with constant probability.[2]

[1] Medium sparsity is a general term for regimes where the graphs are not dense ($m = n^{2-o(1)}$) nor sparse ($m = n^{1+o(1)}$). Specifically, in the following questions we are only interested in the special case where $m = \Theta(n^{4/3})$. However, it is important to note that you may only assume the standard APSP Conjecture and not some "medium sparsity" version of it.

[2] Solutions that are correct assuming the existence of such a partitioning (without proof) will receive most points.