- Recap
- Quantum?
- Algorithmic success stories
Take a problem $X$ in P, say in $O(n^2)$ time.

And prove that:

“$X$ probably cannot be solved in $O(n^{2-\varepsilon})$ time.”
Fine-Grained Complexity or: Hardness in P

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And prove that:

“$X$ probably cannot be solved in $O(n^{2-\varepsilon})$ time.”

Why care about $n$ vs. $n^2$ vs. $n^3$...?
Local Alignment

Input: two (DNA) sequences of length n and a scoring matrix.

Output: The optimal alignment of two substrings.

Typically: \( n >> 10^6 \)

[Smith-Waterman ’81] \( O(n^2) \) with dynamic programming - too slow!
Why care about $n$ vs. $n^2$ vs. $n^3$...?
Here’s one example where it matters...

BLAST: A heuristic, linear time algorithm for Local Alignment.

95k citations!

Are there fast algorithms with optimality guarantees?

Fine-Grained Complexity has the answers.
“For large computing problems, 43 percent of algorithm families had year-on-year improvements that were equal to or larger than the much-touted gains from Moore’s Law.”
How do we get $n^2$ and $n^3$ lower bounds?

NP-hardness is not fine-grained enough...
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Lower bounds for restricted algorithms?

e.g. $\Omega(n \log n)$ for sorting in the comparisons-only model.

Not general enough, and only gives partial answers.
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Unconditional polynomial lower bounds?

- “Any Turing Machine has to spend $\Omega(n^2)$ time...”

Time Hierarchy Thm (1965): Some (artificial) problems require $\Omega(n^2)$ time.
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Time Hierarchy Thm (1965): Some (artificial) problems require $\Omega(n^2)$ time.

But $\Omega(n^2)$ for natural problems, even for SAT, is far out of reach of current techniques. Best lower bound is $3.1n$. 
An $O(n^{1.9})$ algorithm for problem Y (surprisingly fast)  

Approach: Reductions!  

*Imitate NP-hardness, but make it more refined.*

To prove “lower bounds”, reduce famous problems to your problem.

Unexpected breakthroughs in different areas of CS
An $O(n^{1.9})$ algorithm for problem Y (surprisingly fast)

Unexpected breakthroughs in different areas of CS
(some conjecture is refuted)

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Imitate NP-hardness, but make it more refined.

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Unexpected breakthroughs in different areas of CS
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**Conclusion:** “Probably no $O(n^{1.9})$ algorithm for Y”
Weeks 2+3: 3SUM

* Equivalences among variants

* Reductions to geometric problems
Weeks 4+5: APSP

* Equivalence with $(\min,+)$-Matrix-Mult and Negative-Triangle

* Reductions from Neg-$\Delta$ to graph problems
Weeks 6+7+8: SETH

- SAT, ETH, sparsification Lemma, SETH \(2^{o(n)}\) \((2^{\pm\epsilon})^n\)

- Reductions to many \(n^2, n^3\) problems

- Reduction to sequence similarity problems (LCS)

- Circuit-SETH, circuit lower bounds, shaving log factors.
The Old P

- k-clique
- Radius
- RNA folding
- Maximum Matching
- Diameter
- LCS
- 3SUM
- Edit-Distance
- Orthogonal Vectors
- All Pairs Shortest Paths
- Local Alignment
- CFG Parsing
- Polygon containment
- Dynamic reachability
- Frechet distance
- ...
The New P

- **k-SAT**: Diameter, Closest Pair, Local Alignment, Dynamic Reachability, Single-Source Max-Flow, Subtree Isomorphism, Stable Matching, Edit-Distance, Frechet, LCS, ...

- **3SUM**: Colinearity, Polygon Containment, Strips Cover Rectangle, Triangle Enumeration, Compressed Inner Product, Dynamic Max Matching, Set Intersection, ...

- **APSP**: Radius, Dynamic Max Matching, Stochastic Context-Free Grammar Parsing, Negative Triangle, Dynamic Max Flow, Replacement Paths, Median, ...

...
Week 9: The NSETH Barrier

* Non-reducibility: \( \text{SAT} \rightarrow \text{3SUM} \) or \( \text{APSP} \) refutes NSETH.
Week 10: Dynamic Graph Problems

- Simple but strong lower bounds for basic problems. (reachability, SCC, diameter, matching)

- The Online Matrix Vector Multi-Conjecture.
Week 11: Hardness of Approximation and Parameterized Complexity

- Gap amplification via fools from coding theory
Week 12: k-Clique

* lower bounds for RNA folding and more...

* Triangle is hard even on compressible inputs.
The New P

k-SAT
- Diameter
- k-Dominating-Set
- Dynamic reachability
- Stable Matching
- Local Alignment
- Edit-Distance
- Frechet
- LCS
- ...

3SUM
- Polygon Containment
  - 3 points on a line
  - Dynamic Max Matching
- Listing Triangles
  ...

APSP
- Radius
  - Dynamic Max Matching
  - Dynamic Planar APSP
  - Negative Triangle
  - Median
  ...

Min-k-Clique
- Stochastic CFG Parsing
- Shortest Cycle
- Tree Edit Dist.
- Max Rectangle
- Viterbi
- ...

...
The New P

- k-SAT
- OV[d=n^{\text{eps}}]

- 3SUM
- Poly Containment
- 3 points on a line
- Dynamic Max Matching
- Listing Triangles
- Dynamic Max Matching
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- Dynamic Planar APSP
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- APSP
- Diameter
- Local Alignment
- Dynamic reachability
- Stable Matching
- Frechet
- Frechet
- LCS
- LCS

- Min-3Clique
- Stochastic CFG Parsing
- Shortest Cycle
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- Max Rectangle
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- Min-k-Clique
Quantum FG-Complexity

What happens when we build quantum computers?
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Grover's “Unstructured” Search:
Square-root-speedup for many problems.

Quantum Minimum Finding [Dürr-Høyer 96]:
Let $a_1, \ldots, a_n$ be integers accessed by a procedure $P$. There exists a quantum algorithm that finds $\min(a_1, \ldots, a_n)$ with success probability at least 0.99 using $O(\sqrt{n})$ applications of $P$. 
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Great for “parallelizable” problems with exhaustive search algs...
But doesn't always help with dynamic programming.
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But doesn't always help with dynamic programming.

Big open question:
Edit distance and LCS in subquadratic quantum time?

\[ n^2 \] sequential steps, \( O(1) \) time each.
Quantum FG-Complexity

Many recent works...


\[ n \leq 2^{(1-\epsilon)n} \quad \text{for } k\text{-SAT} \quad \Rightarrow \quad \mathcal{O}(n) \quad \text{for } \epsilon > 0 \]

Stronger Quantum SETH [Buhrman et al. ’19]

\[ \mathcal{O}(n^{1.5}) \quad \text{for } \epsilon > 0 \]

Quantum Speedups for Set Cover and TSP [Ambainis et al. ’19]

\[ \begin{align*}
\# \text{sets} & \quad 2 \\
\# \text{elements} & \quad 2 \\
\Rightarrow & \quad 1.8 
\end{align*} \]

Quantum \( O(1) \)-approx for Edit Distance [Boroujeni et al. ’18]
Fine-Grained Complexity

- Few Core Problems
  - Conditional Lower Bounds
  - Understanding the Structure
  - Identifying Solvable Problems

Lots of problems
Algorithmic Success Stories

[Boroujeni et al. ’18]
Quantum $O(n^{2-\varepsilon})$ time $O(1)$-approx for Edit Distance

[Das - Chakraborty - Goldenberg - Koucky - Saks]
(FOCS ’18 best paper)

$O(n^{2-\varepsilon})$ time $O(1)$ approx for Edit Distance!
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Led to many breakthrough approximation algorithms, e.g. LCS, Tree Edit Distance, Language Edit Distance, ...

[Andoni-Nosatzki ’20]:
$O(n^{1+\epsilon})$ time $O_\epsilon(1)$ approx for Edit Distance.
Algorithmic Success Stories

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[Andoni-Nosatzki ’20]:
$O(n^{1+\varepsilon})$ time $O_\varepsilon(1)$ approx for **Edit Distance**.

**Still open:** any 1.00001 lower bound.
Algorithmic Success Stories

* RNA Folding

* Max Flow $O(m)$ $O(m + n^{1.5})$

* All Pairs Max Flow
Algorithmic Success Stories

[Cabello] (SODA ’17 best paper)

\( O(n^{2-\varepsilon}) \) time Diameter in planar graphs!
Algorithmic Success Stories

[Cabello] (SODA ’17 best paper)

$O(n^{2-\varepsilon})$ time Diameter in planar graphs!

Led to many breakthroughs, e.g. distance oracles, dynamic shortest paths, edit distance oracles, ...

[Gawrychowski-Kaplan-Mozes-Sharir-Weimann ’18]:

$O(n^{1+2/3})$ time Diameter in planar graphs.
Algorithmic Success Stories

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[Gawrychowski-Kaplan-Mozes-Sharir-Weimann ’18]:
\(O(n^{1+2/3})\) time Diameter in planar graphs.

Still open: any \(\Omega(n^{1.01})\) lower bound.
Fine-Grained Complexity

- Few Core Problems
  - Conditional Lower Bounds
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  - Identifying Solvable Problems

Lots of problems

That's it for now!

*Please fill out the feedback form that's on my webpage!