Fine-Grained Complexity
Week 13

- Recap
- Quantum?
- Algorithmic success stories

24.1.22

Fine-Grained Complexity
or: Hardness in P

Take a problem X in P, say in ***$O(n^2)$*** time.

And prove that:

" X *probably* *cannot* be solved in $O(n^{2-\varepsilon})$ time. "

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Why care about n vs. n^2 vs. n^3...?

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Here's one example where it matters…

Local Alignment

Input: two (DNA) sequences of length n and a scoring matrix.

AGCCCGTCTACGTGCAACCGGGGAAAGTATA
AAACGTGACGAGAGAGAGAACCCATTACGAA

Output: The optimal alignment of two substrings.

C C G – T C T A C G
C C C A T – T A C G
+1 +1 -0.5 -1 +1 -1 +1 +1 +1 +1 = +4.5

	A	C	G	T	–
A	+1	-1.4	-1.8	-0.7	-1
C	-1.4	+1	-0.5	-1	-1
G	-1.8	-0.5	+1	-1.9	-1
T	-0.7	-1	-1.9	+1	-1
-	-1	-1	-1	-1	$-\infty$

Typically: ***$n \gg 10^6$***

[Smith-Waterman '81] *$O(n^2)$* with dynamic programming - too slow!

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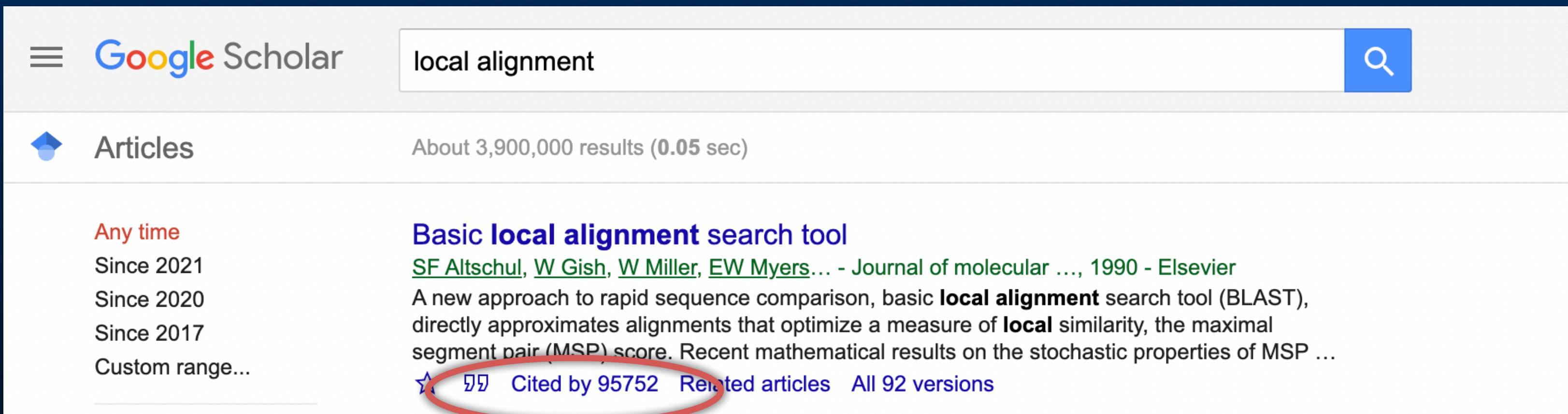
Here's one example where it matters…

BLAST: A *heuristic, linear time* algorithm for **Local Alignment**.


Google Scholar | local alignment

Articles | About 3,900,000 results (**0.05** sec)

Any time
Since 2021
Since 2020
Since 2017
Custom range...

Basic **local alignment** search tool
SF Altschul, W Gish, W Miller, EW Myers… - Journal of molecular …, 1990 - Elsevier
A new approach to rapid sequence comparison, basic **local alignment** search tool (BLAST), directly approximates alignments that optimize a measure of **local** similarity, the maximal segment pair (MSP) score. Recent mathematical results on the stochastic properties of MSP …
Cited by 95752 Related articles All 92 versions

95k citations!

Are there fast algorithms with optimality guarantees?

Fine-Grained Complexity has the answers.

Why care about n vs. n^2 vs. n^3...?

MIT News

ON CAMPUS AND AROUND THE WORLD

How quickly do algorithms improve?

MIT scientists show how fast algorithms are improving across a broad range of examples, demonstrating their critical importance in advancing computing.

Rachel Gordon | MIT CSAIL
September 20, 2021

"For large computing problems, 43 percent of algorithm families had year-on-year improvements that were equal to or larger than the much-touted gains from Moore's Law."

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Lower bounds for restricted algorithms?

e.g. $\Omega(n \log n)$ for sorting in the comparisons-only model.

Not general enough, and only gives partial answers.

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Unconditional polynomial lower bounds?

"Any Turing Machine has to spend $\Omega(n^2)$ time…"

Time Hierarchy Thm (1965): Some (artificial) problems require $\Omega(n^2)$ time.

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Time Hierarchy Thm (1965): Some (artificial) problems require $\Omega(n^2)$ time.

But $\Omega(n^2)$ for natural problems, even for SAT, is far out of reach of current techniques. Best lower bound is $3.1n$.

Approach: Reductions!

Imitate NP-hardness, but make it more refined.

To prove "lower bounds", reduce famous problems to your problem.

An *$O(n^{1.9})$* algorithm for problem Y *(surprisingly fast)*	⇒	Unexpected breakthroughs in different areas of CS

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(some conjecture is refuted)

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***Conclusion**: "Probably no $O(n^{1.9})$ algorithm for Y"*

Weeks 2+3: 3SUM

* Equivalences among variants

* Reductions to geometric problems

Weeks 4+5: APSP

* Equivalence with (min,+)-Matrix-Mult and Negative-Triangle

* Reductions from Neg-Δ to graph problems

Weeks 6+7+8: SETH

- SAT, ETH, sparsification lemma, SETH

  $2^{o(n)}$ $(2-\epsilon)^n$

- Reductions to many n^2, n^3 problems

- Reduction to sequence similarity problems (LCS)

- Circuit-SETH, circuit lower bounds, shaving log factors.

The Old P

- k-clique
- Radius
- RNA folding
- Maximum Matching
- Diameter
- LCS
- 3SUM
- Linear Programming
- Orthogonal Vectors
- Edit-Distance
- All Pairs Shortest Paths
- Local Alignment
- CFG Parsing
- Polygon containment
- Dynamic reachability
- Frechet distance
- …

The New P

k-SAT

- Diameter
- Closest Pair
- Local Alignment
- Dynamic Reachability
- Single-Source Max-Flow
- Subtree Isomorphism
- Stable Matching
- Edit-Distance
- Frechet
- LCS
- …

3SUM

- Colinearity
- Polygon Containment
- Strips Cover Rectangle
- Triangle Enumeration
- Compressed Inner Product
- Dynamic Max Matching
- Set Intersection
- …

APSP

- Radius
- Dynamic Max Matching
- Stochastic Context-Free Grammar Parsing
- Negative Triangle
- Dynamic Max Flow
- Replacement Paths
- Median
- …

Week 9: The NSETH Barrier

* Nonreducibity: SAT → 3SUM or APSP refutes NSETH,

Week 10: Dynamic Graph Problems

* simple but strong lower bounds for basic problems.
  (reachability, SCC, diameter, matching)

* The Online Matrix Vector Mult. Conjecture.

Week 11:
Hardness of Approximation and Parameterized Complexity

* Gap amplification via tools from coding theory

Week 12:
k-Clique

* lowerbounds for RNA folding and more...

* Triangle is hard even on compressible inputs.

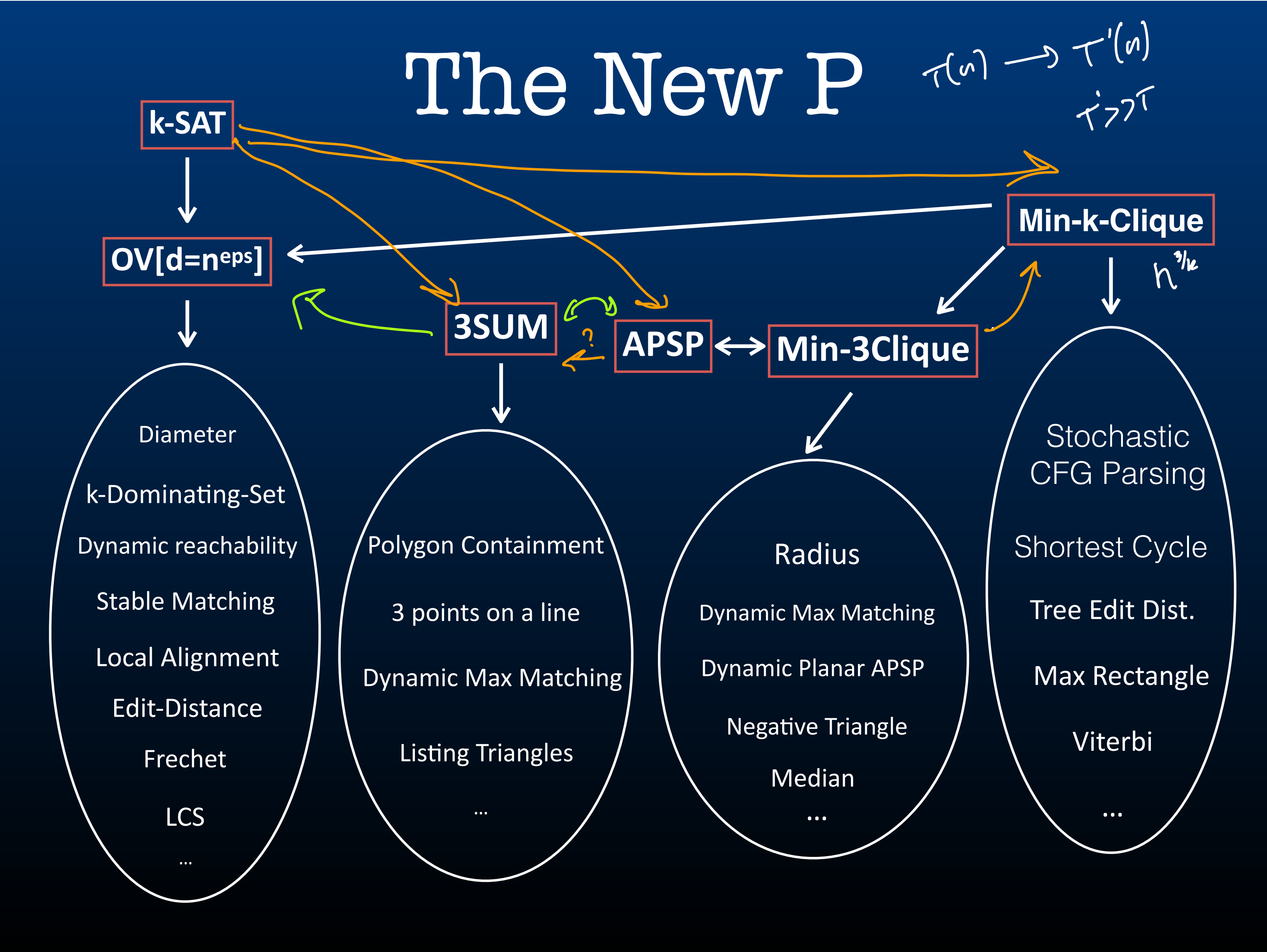
The New P

k-SAT →
- Diameter
- k-Dominating-Set
- Dynamic reachability
- Stable Matching
- Local Alignment
- Edit-Distance
- Frechet
- LCS
- …

3SUM →
- Polygon Containment
- 3 points on a line
- Dynamic Max Matching
- Listing Triangles
- …

APSP →
- Radius
- Dynamic Max Matching
- Dynamic Planar APSP
- Negative Triangle
- Median
- …

Min-k-Clique →
- Stochastic CFG Parsing
- Shortest Cycle
- Tree Edit Dist.
- Max Rectangle
- Viterbi
- …


The New P
T(n) → T'(n)
T' >> T
k-SAT
OV[d=n^eps]
Min-k-Clique
n^{3/k}
3SUM
?
APSP
Min-3Clique
Diameter
k-Dominating-Set
Dynamic reachability
Stable Matching
Local Alignment
Edit-Distance
Frechet
LCS
…
Polygon Containment
3 points on a line
Dynamic Max Matching
Listing Triangles
…
Radius
Dynamic Max Matching
Dynamic Planar APSP
Negative Triangle
Median
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Stochastic CFG Parsing
Shortest Cycle
Tree Edit Dist.
Max Rectangle
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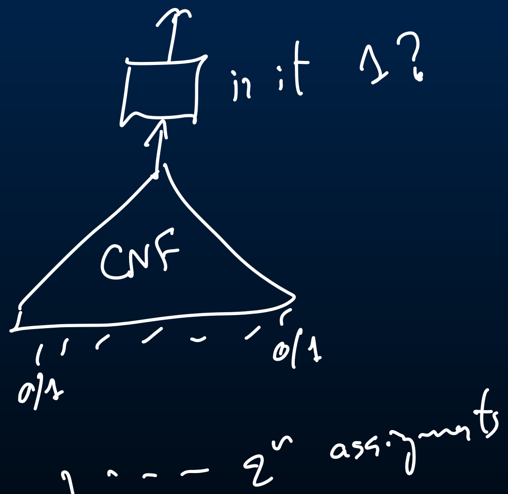
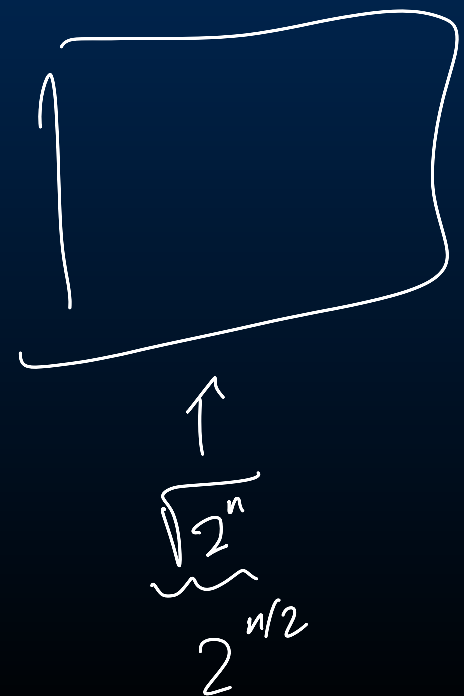
…

Quantum FG-Complexity

What happens when we build quantum computers?





Quantum FG-Complexity

What happens when we build quantum computers?

Grover's "Unstructured" Search:

Square-root-speedup for many problems.

<u>Quantum Minimum Finding [Dürr-Høyer 96]:</u>

Let $a_1, \ldots, a_n$ be integers accessed by a procedure P. There exists a quantum algorithm that finds $\min(a_1, \ldots, a_n)$ with success probability at least 0.99 using $O(\sqrt{n})$ applications of P.

k-SAT

2^n

- Diameter
- k-Dominating-Set
- Dynamic reachability
- Stable Matching
- Local Alignment
- Edit-Distance
- Frechet
- LCS
- ...

3SUM

n^2

- Polygon Containment
- 3 points on a line
- Dynamic Max Matching
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- ...

APSP

n^3

- Radius
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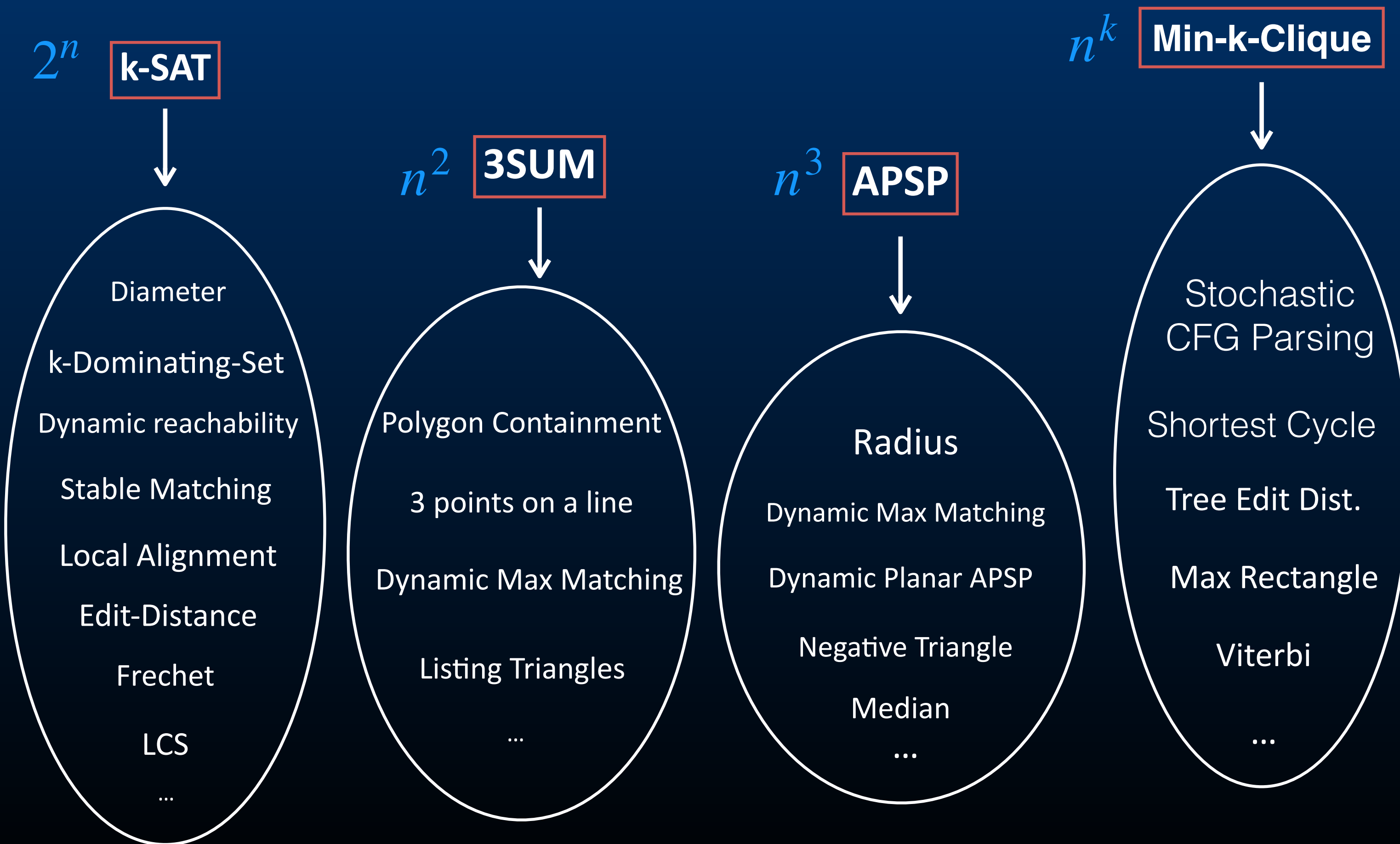
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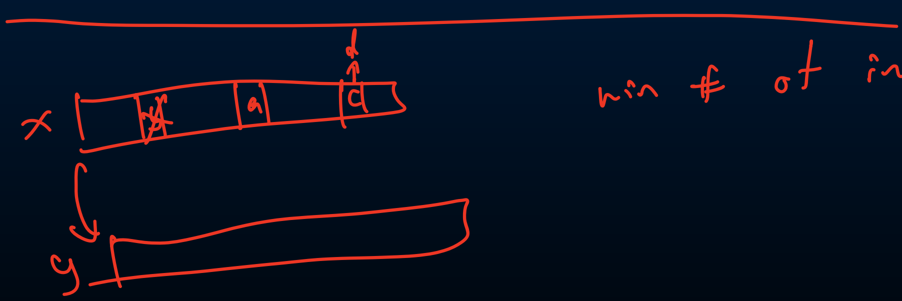
Great for "parallelizable" problems with exhaustive search algs…

But doesn't always help with dynamic programming.



min # of insertions, deletions, substit.

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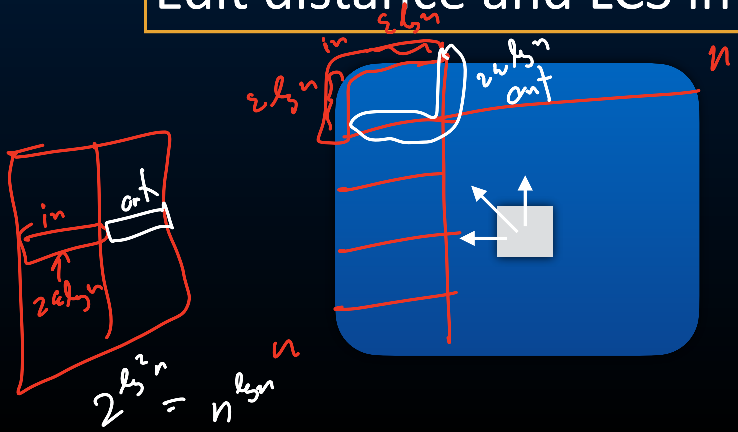
Big open question:

Edit distance and LCS in subquadratic quantum time?



n^2 sequential steps, O(1) time each.

Quantum FG-Complexity

Many recent works…

Quantum SETH [Aaronson et al. '19, Buhrman et al. '19]

"no $2^{(1-\epsilon)\cdot n}$ for k-SAT" $\rightarrow$ $\Omega(n)$ for ED

Stronger Quantum SETH [Buhrman et al. '19]

$\Omega(n^{1.5})$ for E.D.

Quantum Speedups for Set Cover and TSP [Ambainis et al. '19]

$2^{\#sets}$ $2^{\#elements}$ $\rightarrow$ $1.8^{\#elements}$

Quantum $O(1)$-approx for Edit Distance [Boroujeni et al. '18]

Fine-Grained Complexity

Few Core Problems

↓

Lots of problems

- ✓ Conditional Lower Bounds
- ✓ Understanding the Structure
- ✓ Identifying Solvable Problems

Algorithmic Success Stories

[Boroujeni et al. ’18]
Quantum $O(n^{2-\varepsilon})$ time $O(1)$-approx for Edit Distance

[Das - Chakraborty - Goldenberg - Koucky - Saks]
(FOCS ’18 best paper)
$O(n^{2-\varepsilon})$ time $O(1)$ approx for **Edit Distance!**

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[Andoni-Nosatzki '20]:
$O(n^{1+\varepsilon})$ time $O_{\varepsilon}(1)$ approx for **Edit Distance.**

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Still open: any 1.00001 lower bound.

Algorithmic Success Stories

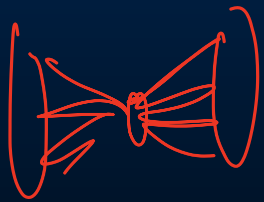
* RNA Folding

* Max Flow $\tilde{O}(m)$ $\tilde{O}(m+n^{1.5})$

* All Pairs Max Flow '61 $n \cdot MF$ → '21 $n^2 + MF$

Algorithmic Success Stories

[Cabello] (SODA '17 best paper)
$O(n^{2-\varepsilon})$ time Diameter in planar graphs!



Algorithmic Success Stories

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[Gawrychowski-Kaplan-Mozes-Sharir-Weimann '18]:
$O(n^{1+2/3})$ time Diameter in planar graphs.

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That's it for now!

*please fill out the feedback form that's on my webpage!