Fine-Grained Complexity

Lecture 3: 3SUM and Geometric Problems

Amir Abboud, November 15th, 2021

Last time: Equivalence of many 3SUM variants…

3SUM Given $S \subseteq [-U, +U]$ are there $a, b, c \in S$ such that $a + b + c = 0$?

3SUM-Finding

Colored-3SUM

3SUM' Given $S \subseteq [0, +U]$ are there $a, b, c \in S$ such that $a + b = c$?

3SUM-with-Duplicates

Small-Integer-3SUM

Target-3SUM

Today: 3SUM Hardness results

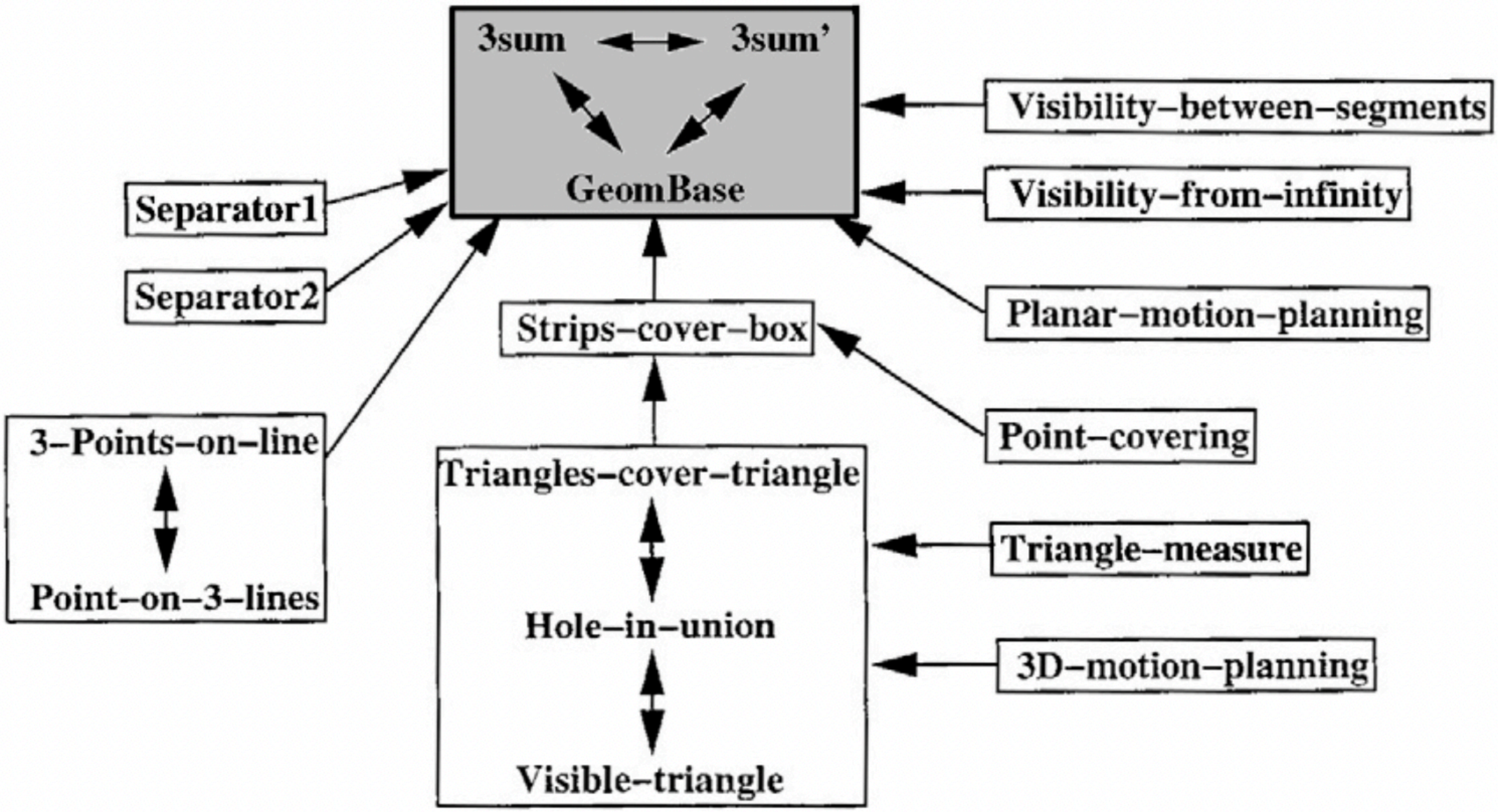

Fig. 11. Overview of the different relations.

On a class of $O(n^2)$ problems in computational geometry ☆

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An Equivalent Geometric Formulation of 3SUM

Geom-Base

Given n points in 2D with $y \in \{0,1,2\}$, does there exist a non-horizontal line that touches 3 points?

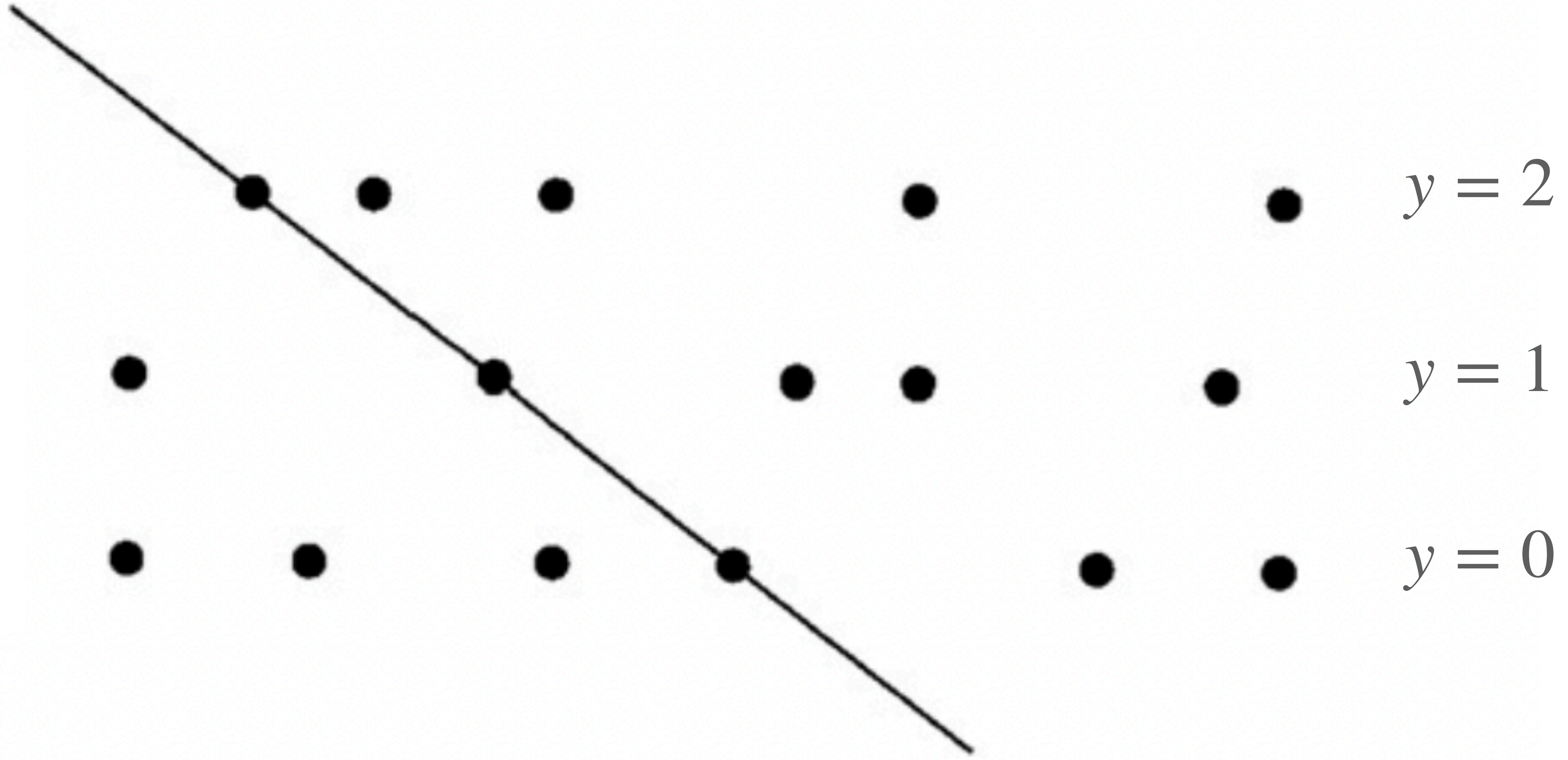


Fig. 1. An example of GEOMBASE.

Incidence Problems

3-Points-on-a-Line

Given n points in 2D ~~with~~ ~~$y \in \{0,1,2\}$~~, does there exist a ~~non-horizontal~~ line that touches 3 points?

By duality: **Point-on-3-Lines**

Also: **Point-on-3-(anything)**

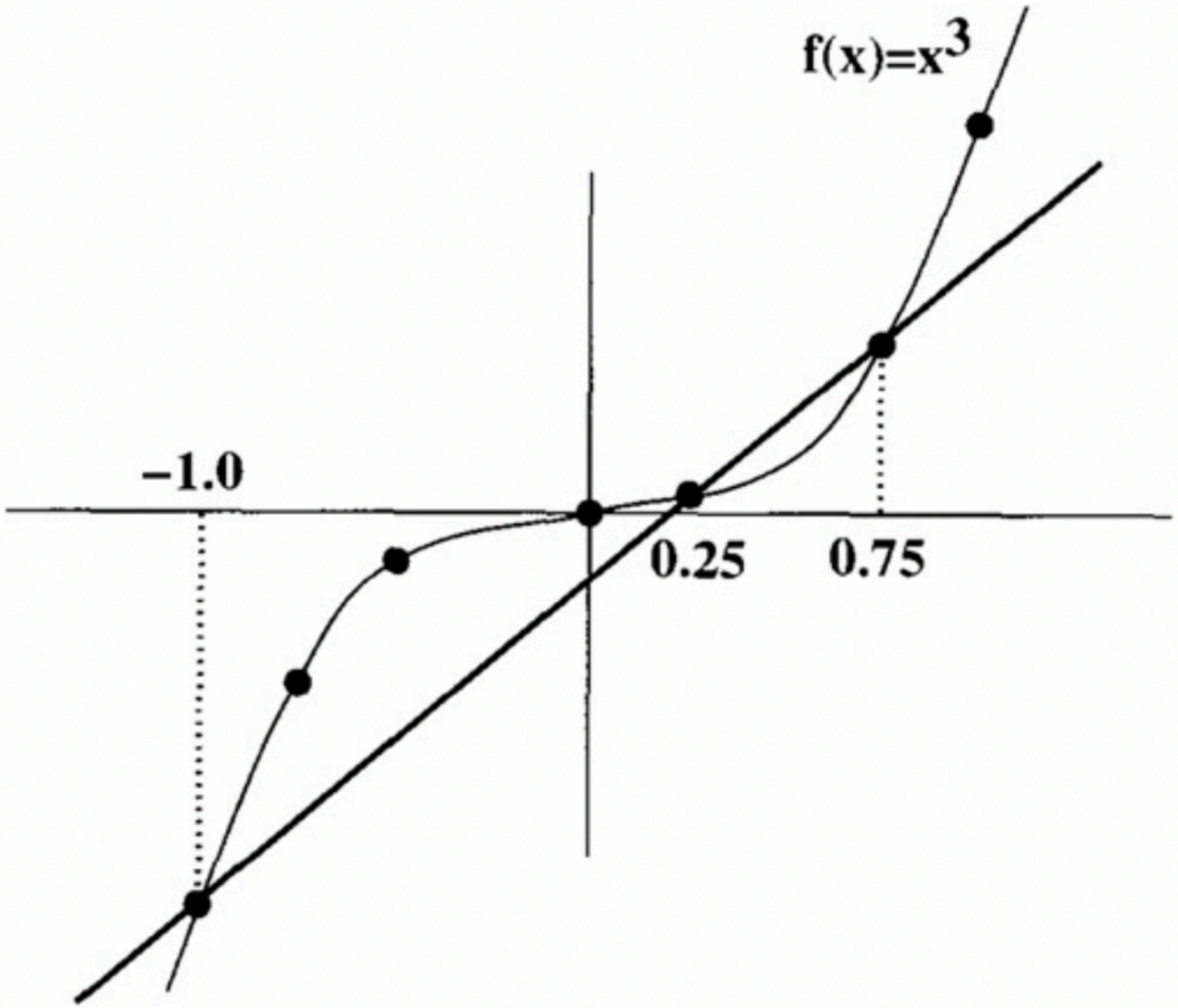


Fig. 2. Transforming 3SUM to 3-POINTS-ON-LINE.

$$\frac{b^3-a^3}{b-a} = \frac{c^3-a^3}{c-a} \iff b^2+ba+a^2 = c^2+ca+a^2 \iff (b-c)(b+c+a) = 0 \iff b+c+a=0,$$

Visibility Problems

Point-to-Point-Visibility

$O(n)$

Given two points p,q in 2D and n opaque segments, can p see q?

Point-to-Segment-Visibility

$O(n \log n)$

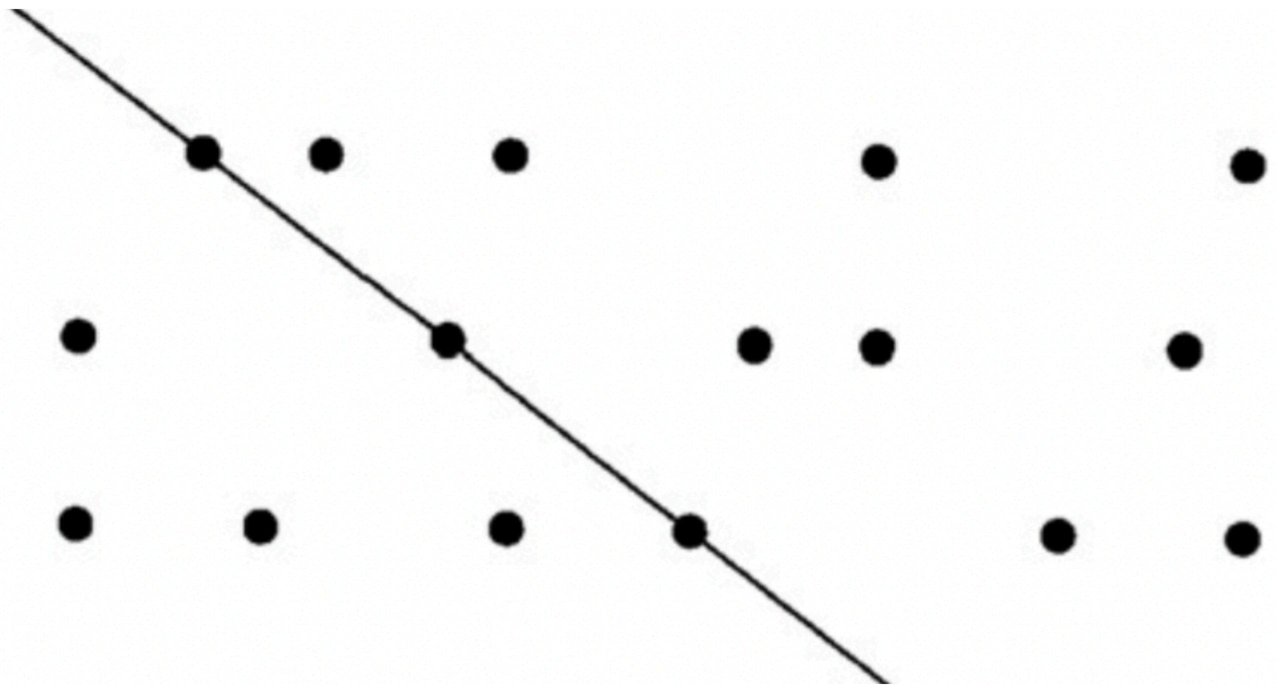
Given a point p and a segment Q in 2D and n opaque segments, can p see any point on Q?

Visibility-Between-Segments

3SUM-hard!

Given two segments P and Q in 2D and n opaque segments, can any point on P see any point on Q?

Visibility Problems



Visibility-Between-Segments

Given two segments P and Q in 2D and n opaque segments, can any point on P see any point on Q?

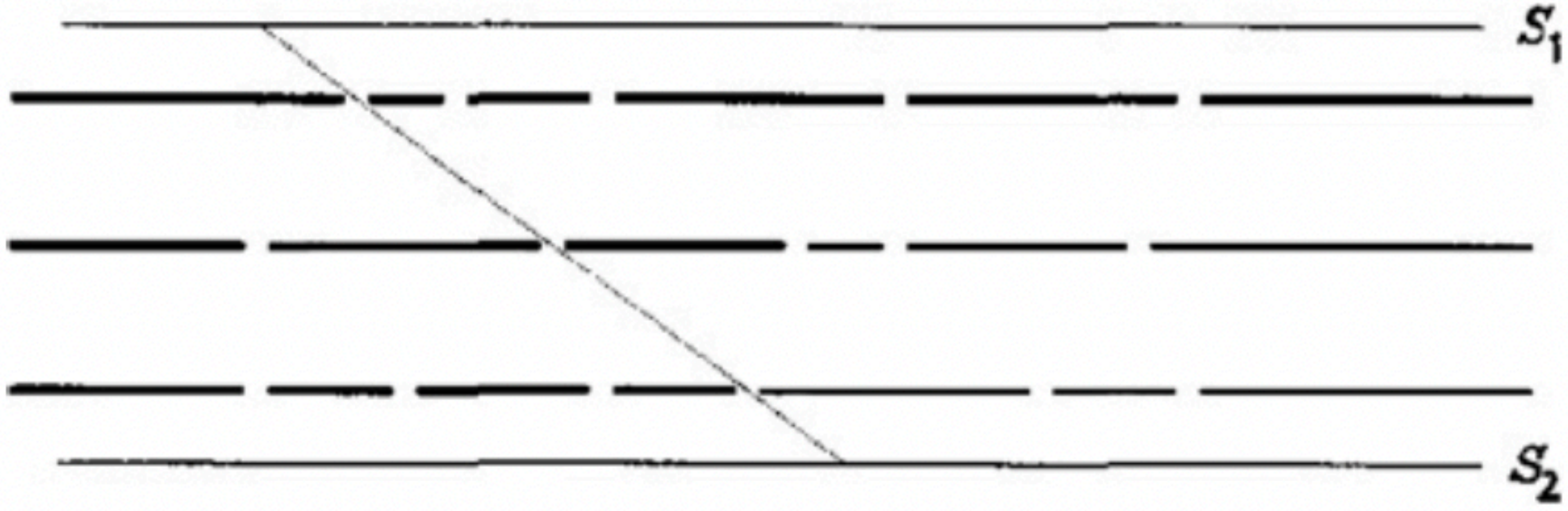


Fig. 7. Transformation from GEOMBASE to VISIBILITY-BETWEEN-SEGMENTS.

<u>Also:</u>

Visibility-From-Infinity

Given one segment P in 2D and n opaque segments, is there any infinite ray starting at P that does not intersect any segments?

Motion Planning

Planar-Motion-Planning

Given a set of n obstacles (axis-parallel segments) in 2D, and a robot (a segment), decide if it can be moved from a given starting position to a given destination via translation and rotation.

2 degrees of freedom => $O(n \log n)$

3 degrees of freedom => **3SUM-hard!**

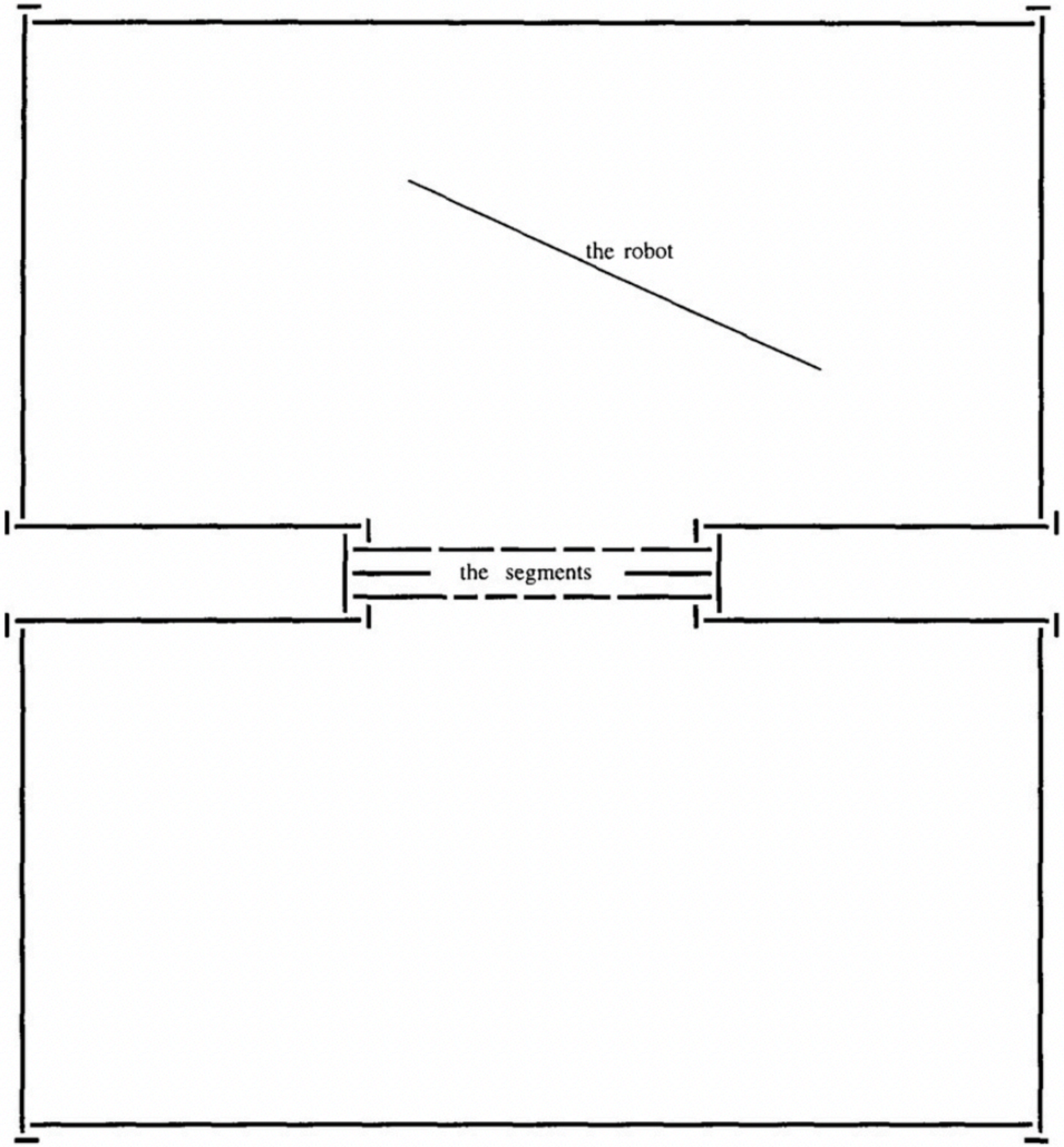


Fig. 9. Reduction from GEOMBASE to PLANAR-MOTION-PLANNING.

3D Motion Planning

Even translation-only is hard in 3D…

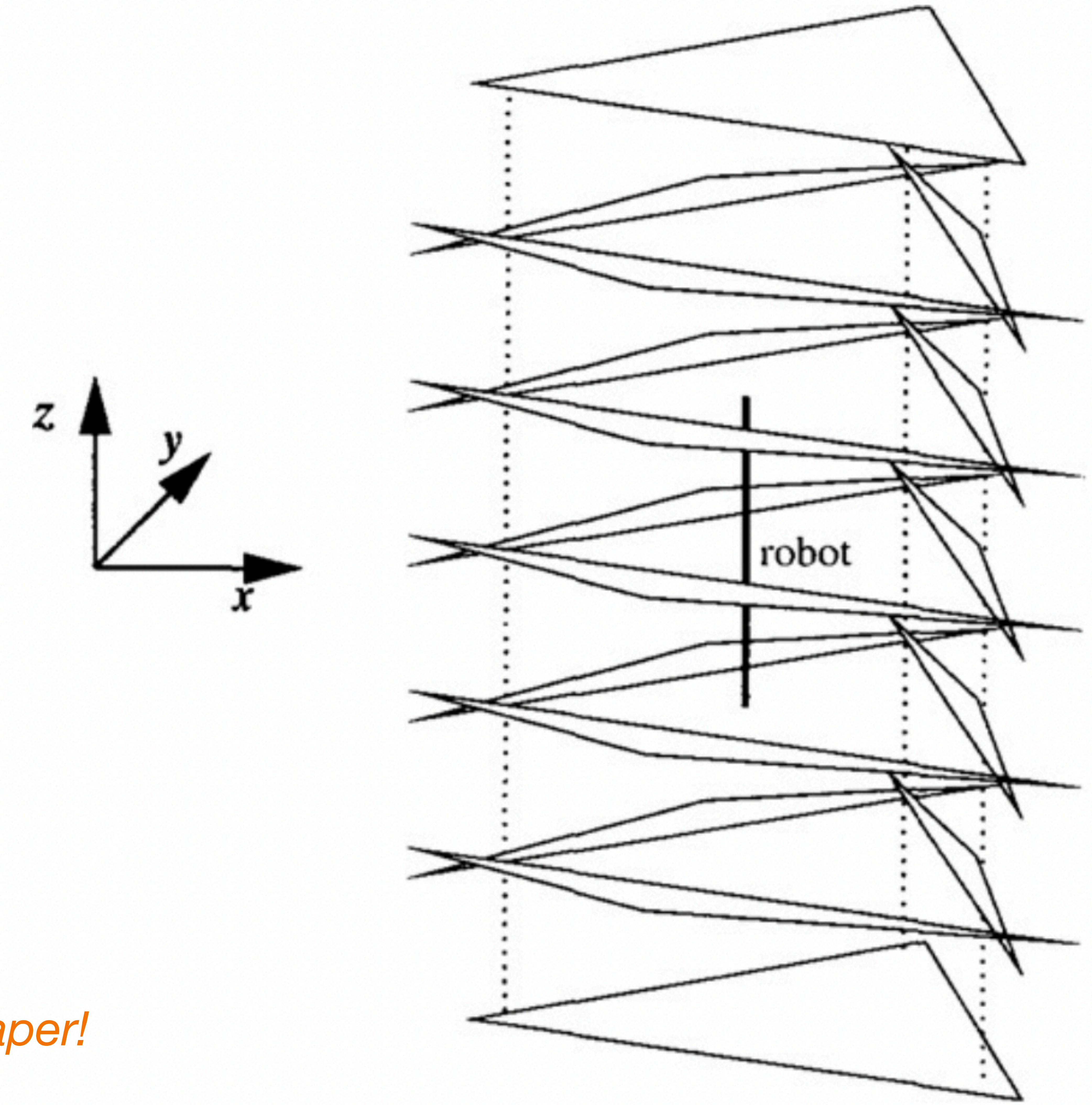
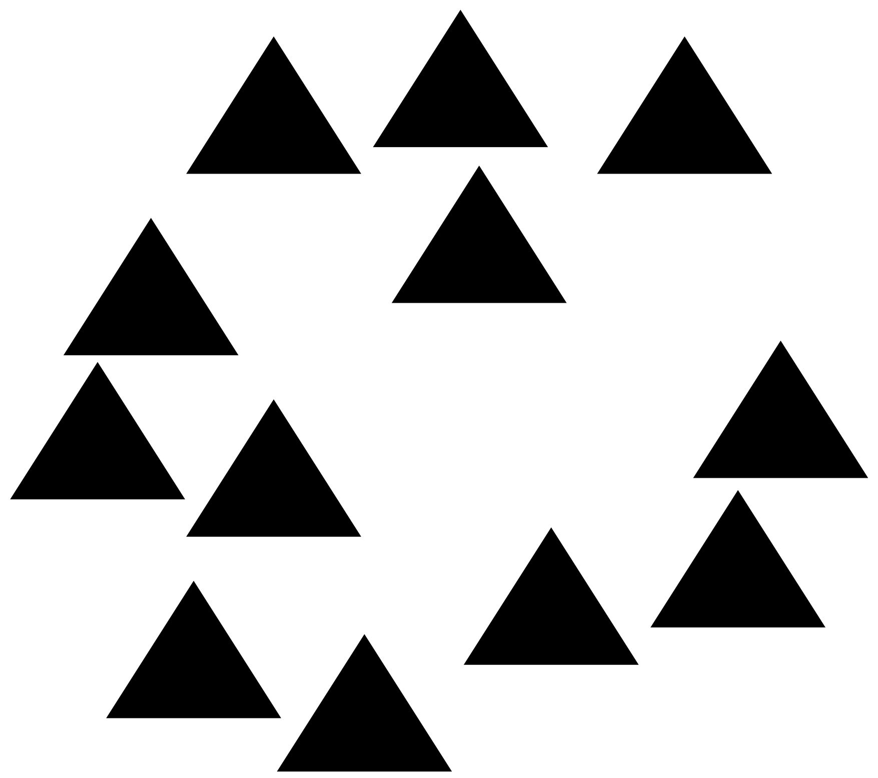


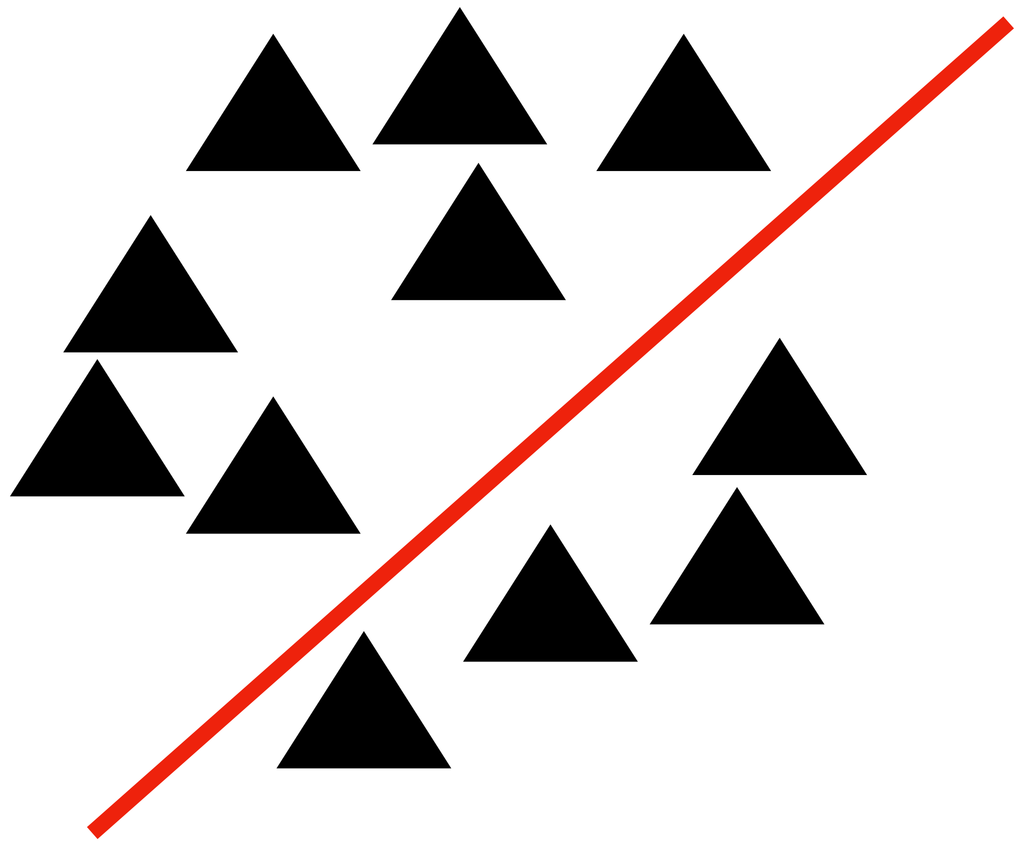
Fig. 10. A cage from which the robot cannot escape.

For more details see the paper!

Separator Problems

Definition 5.1 Given a set S of n objects in the plane, we call a line l a separator of S if l does not intersect any object in S and both halfplanes bounded by l contain a non-empty subset of the objects in S.





Separator Problems

Spearator1

Given a set of n possibly infinite line segments, is there a non-horizontal separator?

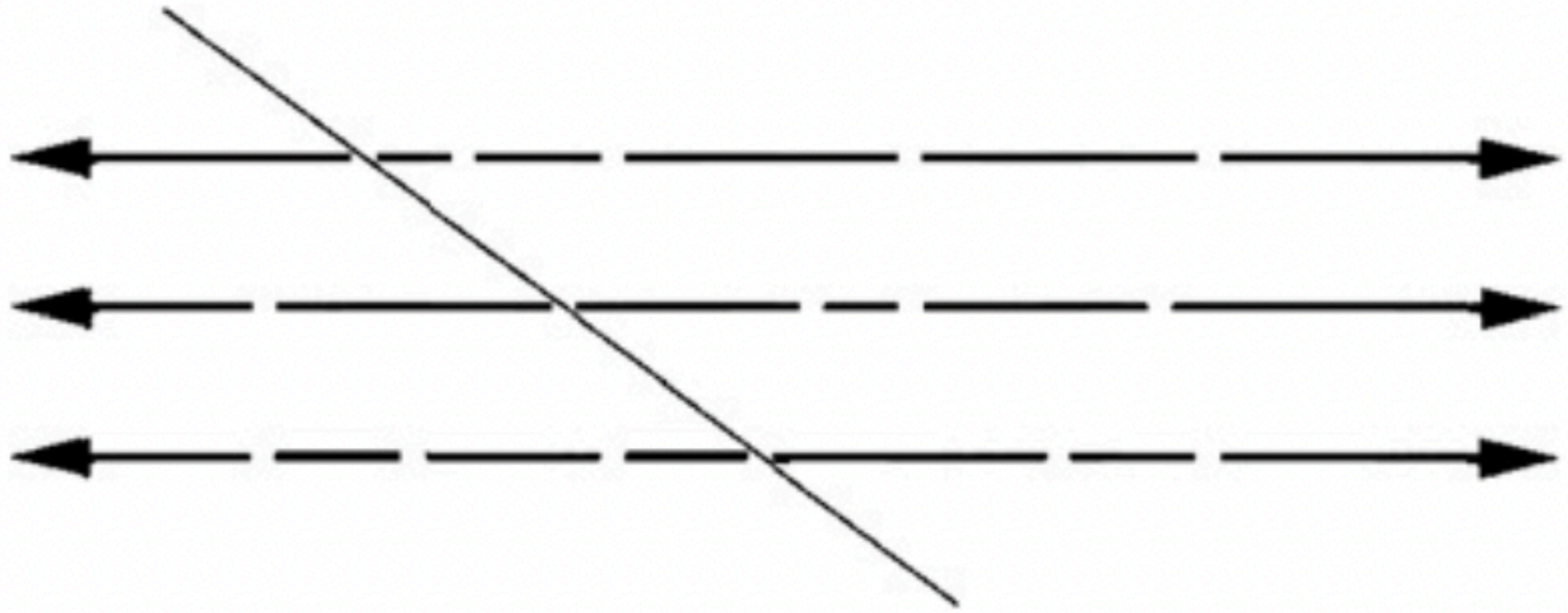


Fig. 3. Transforming GEOMBASE to SEPARATOR1.

Separator2

Given a set of n ~~possibly infinite~~ line segments, is there a ~~non-horizontal~~ separator?

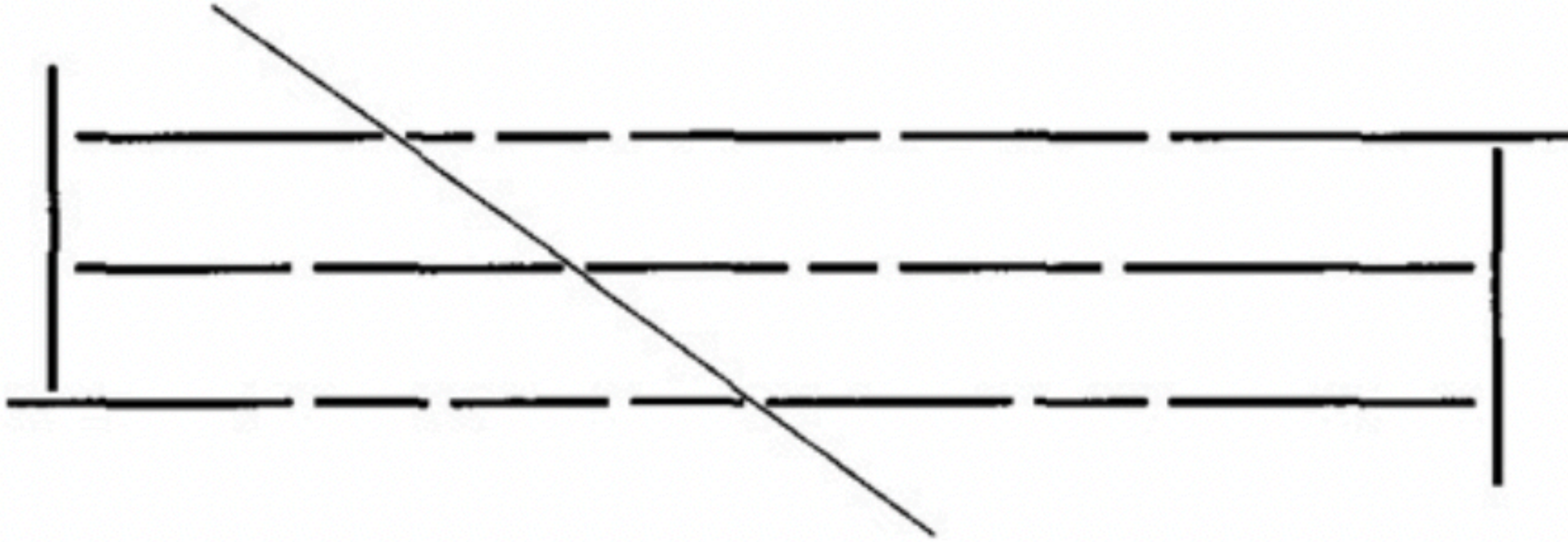


Fig. 4. Transforming GEOMBASE to SEPARATOR2.

Also: **Separate-(anything)**

Covering Problems

Strips-Cover-Box

Proof idea - duality:

$$(m, b) \to y = mx + b$$

$$y = ax + c \to (-a, c)$$

Also:

Triangles-Cover-Triangle

Hole-in-Union

Triangle-Measure

Point-Covering

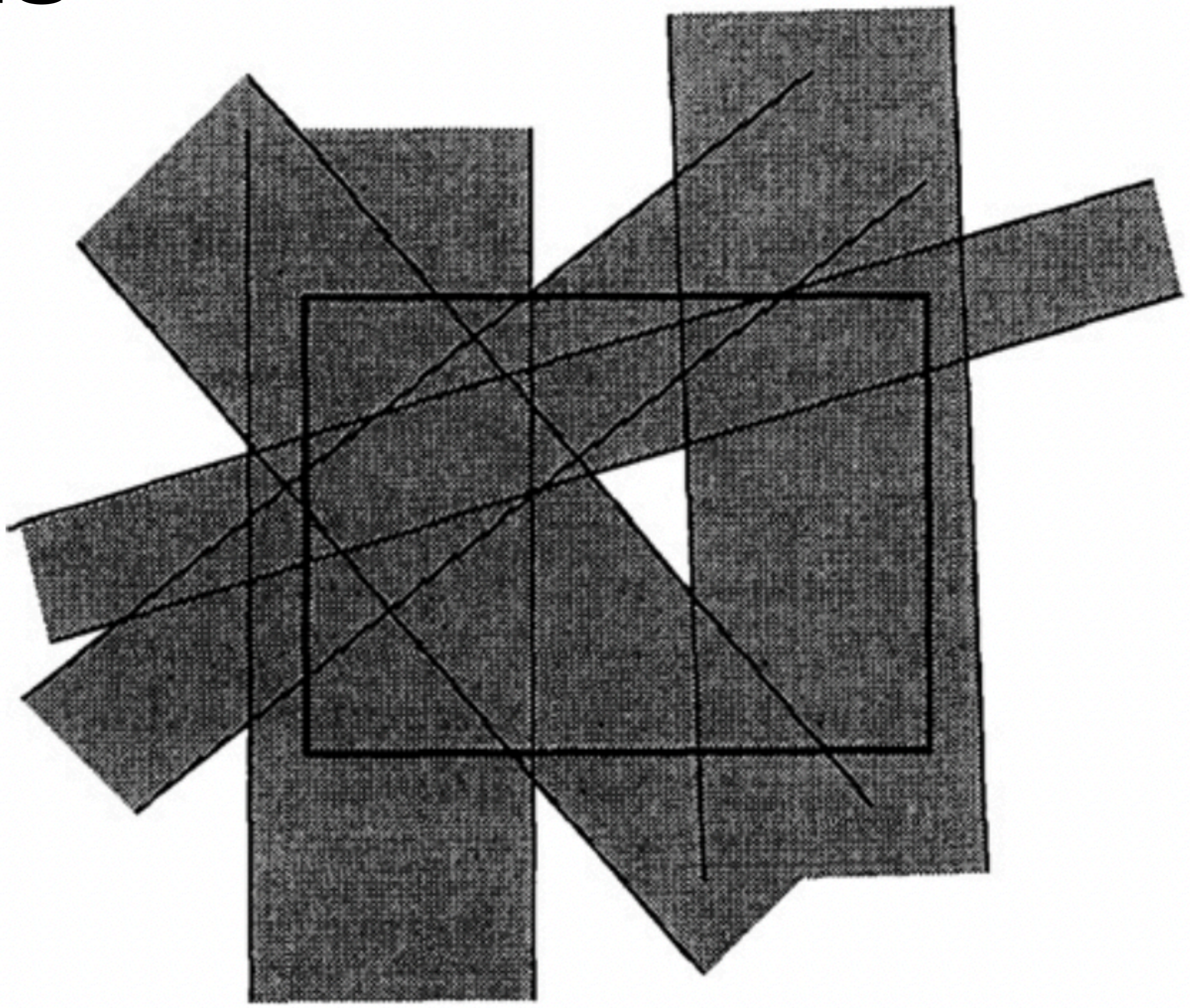
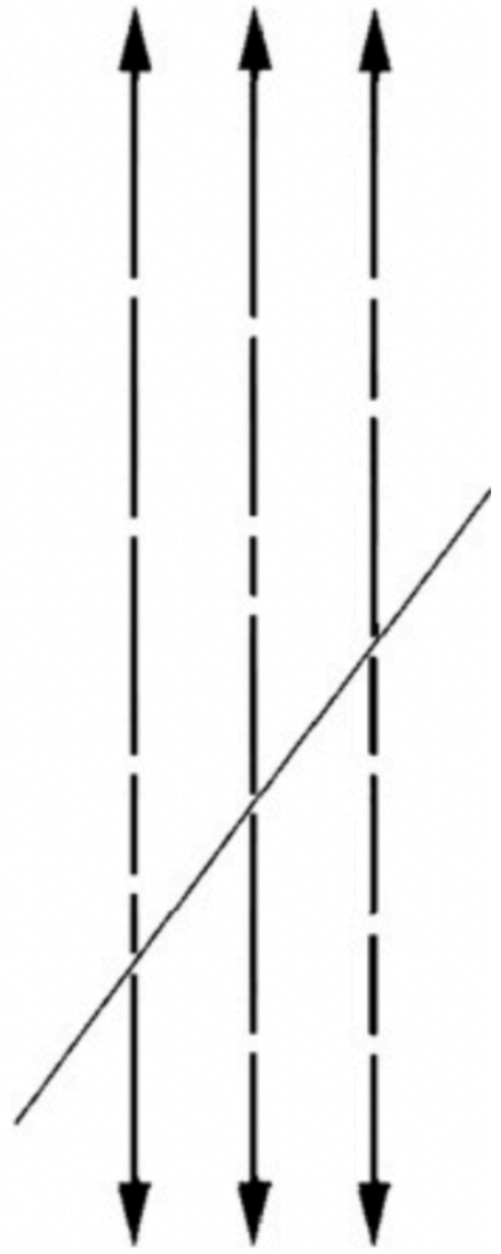


Fig. 5. The rectangle is not fully covered by the strips.



Polygon Containment

[Barequet and Har-Peled ’99]

Segments Containing Points

Given a set of points, and a set of segments in 1D (on the real line). Can the segments cover the points?

See proof in the handwritten notes.