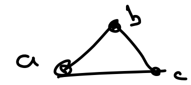
Def: A <u>triangle</u> in a graph $G=(V,E)$ is a triple $t=\{a,b,c\}\subseteq V$ s.t. $\{a,b\},\{b,c\},\{a,c\}\in E$.



Def: If $G=(V,E,w)$ has edge-weights $w:E\to[-U,+U]$ the <u>weight</u> of a triangle is

$$w(t) := w(a,b)+w(b,c)+w(a,c).$$

Δ-detection: Given $G=(V,E)$ does it contain a triangle?

Min-Δ: Given $G=(V,E,w)$, $w:E\to[1,+U]$ return the min weight of any triangle in G.

* can assume $E=V\times V$ (non-edge $\equiv$ weight $+\infty$)

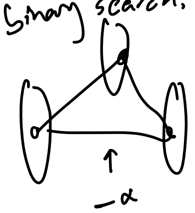
Neg-Δ: Given $G=(V,E,w)$: $w:E\to[-U,+U]$ does it contain a triangle t with $w(t)<0$?

Zero-Δ: same but ... with $w(t)=0$?

---

(0) min-$\Delta \to$ Neg-Δ

binary search:



neg-$\Delta \Leftrightarrow \exists\,\Delta$ with $w(t)<\alpha$

Easy: Δ-det - subcubic time

Hard: Neg-$\Delta \leftrightarrow$ APSP; Neg-$\Delta \leftrightarrow$ Min-Δ; Min-$\Delta \leftrightarrow$ APSP

Very hard: Min-$\Delta \xrightarrow{\oplus}$ Zero-$\Delta \xleftarrow{\oplus}$ Conv-3SUM $\leftrightarrow$ 3SUM

$\oplus$ Next week.

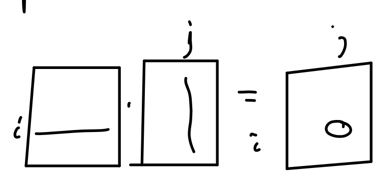
---

(1) **Thm:** Δ-det is in $O(n^{\omega})$ time.

Def: ω is the smallest number s.t. $n\times n$ Matrix Mult is in $O(n^{\omega})$ time.

Matrix Mult: given two $n\times n$ matrices A,B, compute $C=AB$ s.t.

$$C[i,j]=\sum_{k=1}^{n} A[i,k]\cdot B[k,j]$$



(2)

Thm [Strassen '69, ..., Alman-Vassilevska Williams SODA'21]

$$2 \le \omega < 2.37$$

Conj: $\omega < 2 + \varepsilon \quad \forall \varepsilon > 0.$

---

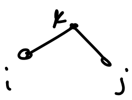
(1) **Alg:** - Def A to be the adjacency matrix of G

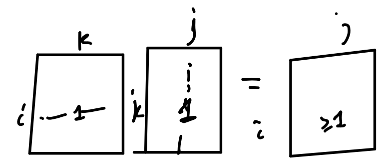
$$A[i,j] = \begin{cases} 1 & i \neq j, \{i,j\} \in E \\ 0 & \text{o.w.} \end{cases}$$

- Compute $A^2 = A \cdot A$
- return "yes" iff $\exists \{i,j\} \in E$ s.t. $A^2[i,j] \ge 1$.

Time: $O(n^2 + n^\omega)$. ✓

Correctness: - $A^2[i,j] = \sum_k A[i,k] \cdot A[k,j] = \left(\# \text{ nodes } k \text{ s.t. } \begin{matrix}\{i,k\} \\ \{k,j\}\end{matrix} \in E\right)$





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(2) Reminder: why $\omega < 3$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix}$$

8 multiplications...

Strassen's alg (see Wikipedia):

$M_1 = a \cdot (y - w)$

$M_2 = (a + b) \cdot w$

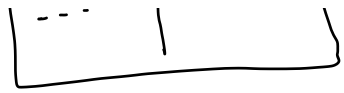
$M_3 = (a + d) \cdot (x + w)$

.

$$\begin{bmatrix} M_3 + M_4 - M_2 + M_7 & M_1 + M_2 \\ & \cdots \end{bmatrix}$$

$\vdots$

$M_7 = \cdots$

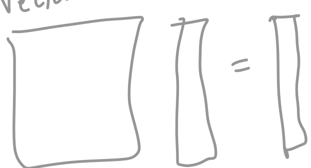


Only 7 multiplications!

Recursion: $T(n) = 7 \cdot T(n/2) + O(n^2)$

$$= O(7^{\log_2 n}) = O(n^{\log_2 7}) = O(n^{2.81})$$

Matrix-Vector



$\Theta(n^2)$

$n \times$ (Matrix-Vector) ... much less than $n \cdot n^2$!

* requires subtraction.

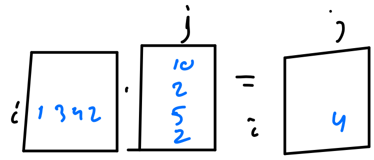
---

(3) Thm: Min-Δ is in $(\min,+)$-MM time.

$(\min,+)$-MM: given two $n \times n$ matrices A, B, compute $C = A \cdot B$ s.t.

$$C[i,j] = \min_{k=1}^{n} A[i,k] + B[k,j]$$

"Tropical Algebra"



Thm (Williams STOC'14): $(\min,+)$-MM is in $O\left(n^3 / 2^{\sqrt{\log n}}\right)$ time.

Conj: no $O(n^{3-\varepsilon})$ time alg for $(\min,+)$-MM. [$\equiv$ APSP Conj.]

---

(3) Alg: – Def A to be the weighted adjacency matrix of G

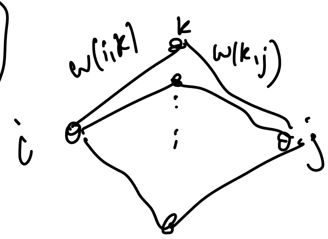
$$A[i,j] = \begin{cases} w(i,j) & i \neq j, \{i,j\} \in E \\ +\infty & \text{o.w.} \end{cases}$$

- Compute $A^2 = A \oplus A$
- return $\min_{i,j} w(i,j) + A^2[i,j]$

Time: $O(n^2 + \text{MinPlusMM}(n))$. ✓

Correctness: – $A^2[i,j] = \min_k A[i,k] + A[k,j] =$ min weight 2-path from i to j

APSP: Given $G = (V, E, w)$, $w : E \to [1, +U]$
return $\text{dist}[u,v] = $ (min weight $i \to j$ path)
for all $i, j \in V$.



(4) Thm: APSP is in $O(T_{(\min,+)\text{-MM}}(n) \cdot \log n)$ time.

Alg: - Def A s.t.

$$A[i,j] = \begin{cases} w(i,j) & i \neq j \ \{i,j\} \in E \\ 0 & i = j \\ +\infty & \text{o.w.} \end{cases}$$

- compute $A^n = \underbrace{A \oplus A \oplus \cdots \oplus A}_{n \text{ times}} = \underbrace{\left(\left(\left(A^2\right)^2\right)^{\cdots}\right)^2}_{\log n \text{ times}}$

- return $\text{dist}(i,j) = A^n[i,j]$

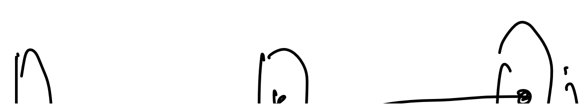
Time: $O(n^2 + T_{(\min,+)\text{-MM}}(n) \cdot \log n)$. ✓

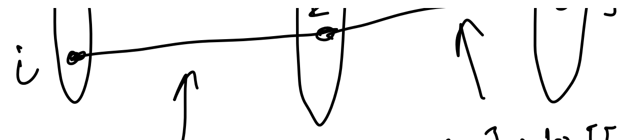
Correctness: (similar to before – omitted).

$$\text{APSP} \underset{(5)}{\overset{(4)}{\rightleftarrows}} (\min,+)\text{-MM}$$

(5) Thm: $(\min,+)$-MM is in $O(T_{APSP}(n))$ time.

pf: $A + 10 \cdot U$ $B + 10 \cdot U$





$$\text{dist}(i,j) = \min_k A[i,k] + B[k,j] + 20\cdot U$$

Thm: (min,+)-MM is in $O(U \cdot n^{\omega})$ time.

pf idea: encode x with $(n+1)^x \ldots$

$\cdot \rightarrow +$

$+ \rightarrow \sim \min$

issue: multiplying U-bit numbers takes $\tilde{O}(U)$ time, not $O(1)$.

Main Thm [Vassilevska - Williams FOCS'10]

If Neg-Δ is in $O(n^{3-\varepsilon})$ time then (min,+)-MM is in $\tilde{O}(n^{3-\varepsilon/3})$ time, and the APSP Conjecture is false.

(min,+)-MM $\rightarrow$ All-Pairs Neg-Δ $\rightarrow$ Neg-Δ

$\boxed{[U]}$ $\quad$ $\boxed{\{0,1\}}$ $\quad$ $\{0,1\}$

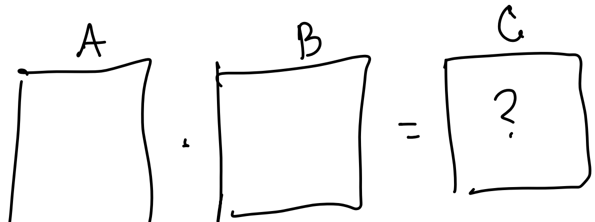
<u>All-Pairs Neg-Δ:</u> Given a tripartite $G=(V,E,w)$, $w: E \to [-U,+U]$

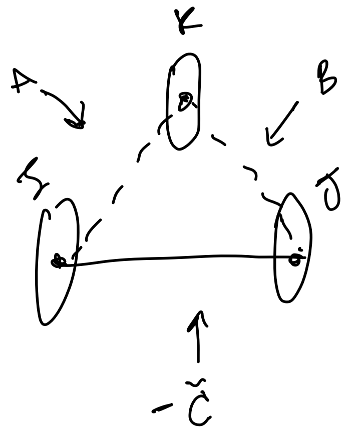
$V = I \cup J \cup K$, $E = (I \times J) \cup (I \times K) \cup (K \times J)$

return $\forall i \in I, j \in J$ $D[i,j] = \begin{cases} 1 & \exists k \text{ s.t. } w(i,j,k) < 0 \\ 0 & \text{o.w.} \end{cases}$

"is (i,j) in a neg-Δ?"

<u>$(\min,+)$-MM $\to$ All Pairs Neg-Δ:</u>





<u>idea:</u> simultaneous binary search $\forall i,j$.

- guess $\tilde{C}$
- use APNT to check each $\tilde{C}[i,j]$

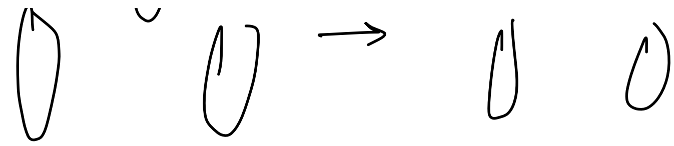
<u>Claim:</u> $D[i,j]=1$ iff $\tilde{C}[i,j] > \min_k A[i,k]+B[k,j]$

$\Rightarrow O(\log U)$ calls to APNT suffice.

All Pairs Neg-$\Delta \longrightarrow$ Neg-Δ

[<u>Prelim:</u> Neg-$\Delta \Leftrightarrow$ Neg-Δ-finding
- same as 3SUM $\Leftrightarrow$ 3SUM finding]

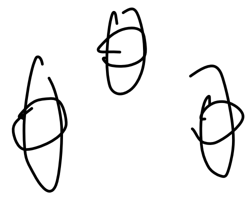
<u>1st idea:</u>



find 1 triangle at a time...

$n^3 \cdot n^{3-\varepsilon}$ ... too much!

2nd idea:



search in random subgraphs

$\left(\frac{n}{s}\right)^3 \cdot s^{3-\varepsilon}$ - each Δ is in ≥ 1 instance w.h.p...

need extra s^3 to find each Δ in instance.

3rd idea: find each triangle once across all instances.

$\Rightarrow$ extra $n^3 \cdot s^{3-\varepsilon}$, still too much

4th idea: find ≤ 1 triangle for each edge across all instances.

$\Rightarrow$ extra $n^2 \cdot s^{3-\varepsilon}$. good enough!

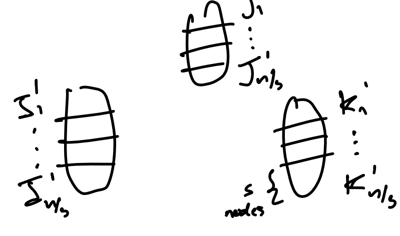
Reduction:

- Split I,J,K arbitrarily into s-size subsets

$I'_1, \ldots, I'_{n/s}$

$J'_1, \ldots, J'_{n/s}$

$K'_1, \ldots, K'_{n/s}$



- for all $\left(\frac{n}{s}\right)^3$ triples of subsets I', J', K' do (in arbitrary order):

(*) - If Find-Neg-Δ $(I', J', K') = (i, j, k)$
  - Set $D[i,j] = 1$, remove edge $\{i,j\}$ from G, i.e set $w(i,j) = +\infty$.
  - go to (*).
  - O.w. (Find-Neg-Δ $(I', J', K') = \phi$) continue to next triple.

Correctness: if $(i,j) \in I \times J$ is in a neg-Δ, it will be found...

Time: $\left[\left(\frac{n}{s}\right)^3 + k\right] \cdot O(s^{3-\varepsilon}) \leq \left[\left(\frac{n}{s}\right)^3 + n^2\right] \cdot O(s^{3-\varepsilon})$

$= O\left(\frac{n^3}{s^\varepsilon} + n^2 \cdot s^{3-\varepsilon}\right)$

$\overset{\text{set } s = n^{1/3}}{=} O(n^{3-\varepsilon/3})$.

k = # pairs (i,j) s.t. $D[i,j] = 1$ $\leq n^2$

$\Rightarrow$ Thm: If Neg-Δ is in $O(n^{3-\varepsilon})$ time then All-Pairs Neg-Δ is in $O(n^{3-\varepsilon/3})$ time.