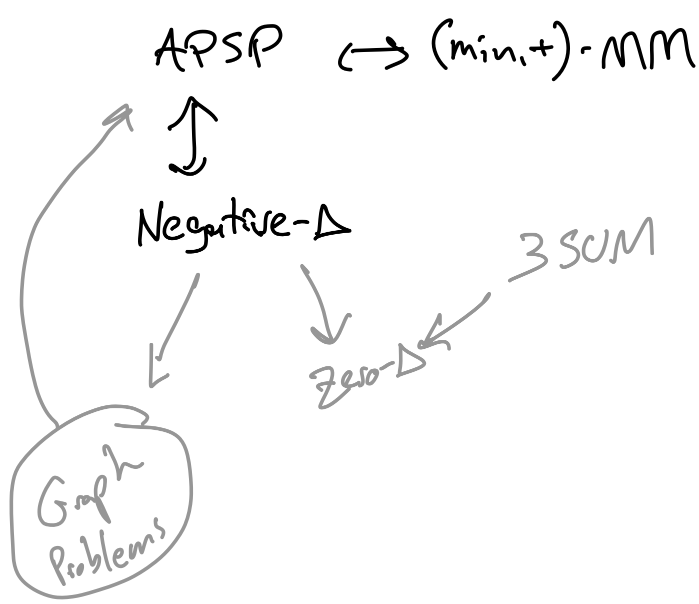
Last time:



Today:

Forgot to say last time:

⊛ main difference between (min,+)-MM and (+,·)-MM is that min has no inverse
$\Rightarrow$ we cannot subtract.

Given $G=(V,E,w)$, $w: E \to [1,U]$, $U = n^{O(1)}$.

APSP: Compute $d(s,t)$ $\forall s,t \in V$

n-Pair-SP: Given $P \subseteq V \times V$, $|P| = n$, compute $d(s,t)$ $\forall (s,t) \in P$.

Diameter: Compute $D = \max_{s,t} d(s,t)$.

Radius: Compute $R = \min_{c \in V} \max_{t \in V} d(c,t)$

$\equiv$ **Center**: find $c^* = \arg\min_{c \in V}$ —

1-Median: Compute $M = \min_{c \in V} \sum_{t \in V} d(c,t)$

Second-S.P.: Given s,t and a shortest s,t-path $P_{s,t}$ compute the length of the shortest s,t-path that differs from $P_{s,t}$.

Replacement Paths: - - — — — —

compute $d_{G \setminus \{e\}}(s,t) \quad \forall e \in P_{s,t}$.

All-Pair-Max-Flow: Compute $\text{MaxFlow}(s,t) \; \forall s,t \in V$

$\text{MaxFlow}(s,t) = \text{Min-Cut}(s,t)$

Which problems are in subcubic time?

Starting Point:

Neg-Δ: Given tripartite $G=(V,E,w)$, $w: E \to \{-U,\ldots,U\}$ $V = A \cup B \cup C$. Is there $a \in A, b \in B, c \in C$ s.t.

$$w(a,b) + w(b,c) + w(a,c) < 0 \;?$$

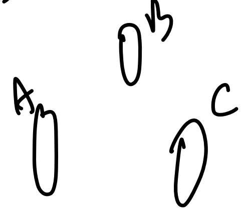
(*) may assume that $E = A \times B \cup B \times C \cup A \times C$

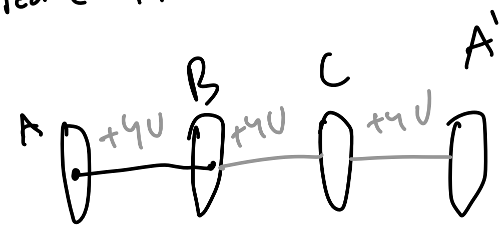
(by adding edges of weight $3U$ as needed.)

① APSP $\longleftrightarrow$ n-Pair-SP.

<u>Thm:</u> Neg-Δ $\longrightarrow$ n-Pair-SP

given G $\longrightarrow$ create H





- $V_H = A \cup B \cup C \cup A'$
- $E_H = \{(a,b) \in A \times B \mid (a,b) \in E_G\}$ $\leftarrow$ $w_H(a,b) = w_G(a,b) + 4U$

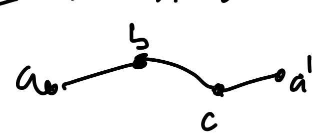
  $\cup \{(b,c) \in B \times C \mid (b,c) \in E_G\}$ $\leftarrow$ same

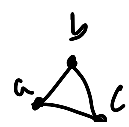
  $\cup \{(c,a') \in C \times A' \mid (c,a') \in E_G\}$ $\leftarrow$ $w_H(c,a') = w_G(a,c) + 4U$

<u>Time</u> ✓.

<u>Correctness:</u> <u>Claim:</u> $d_H(a,a') < 12U$ iff a is in neg-Δ in G.

<u>pf:</u> shortest a,a'-path looks like



and corresponds to a Δ  in G

of weight exactly $d_H(a,a')-12U$.

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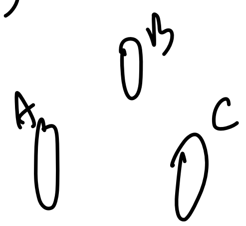
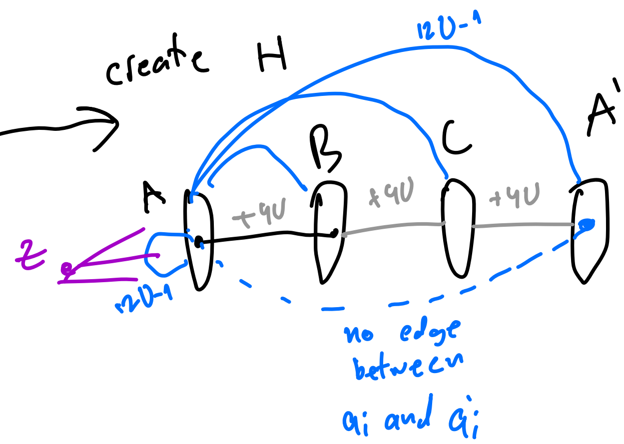
(2) APSP $\longleftrightarrow$ Radius

<u>Thm:</u> Neg-Δ $\longrightarrow$ Radius

given G $\longrightarrow$ create H





- $V_H = A \cup B \cup C \cup A' \cup \{z\}$
- $E_H = \{(a,b) \in A\times B \mid (a,b) \in E_G\}$ ← $\omega_H(a,b) = \omega_G(a,b) + 4U$

  $\cup \{(b,c) \in B\times C \mid (b,c) \in E_G\}$ ← same

  $\cup \{(c,a') \in C\times A' \mid (c,a') \in E_G\}$ ← same

  $\cup \{(a_i,x) \in A\times V \mid x \neq a_i'\}$ ← $\omega_H(a_i,x) = 12U-1$

  including $x = z$.

want: a is a "good center" in H
iff a is in neg-Δ in G.

solution: add direct edges to everyone except a'.

Claim: $\exists a \in A$ s.t. $\max_{t \in V_H} d_H(a,t) < 12U$ iff $\exists$ neg-Δ in G.

pf: for $t \neq a'$ take a direct blue edge. for a' use the above claim.

potential issue: what if there is $b \in B$ s.t. $\max_{t \in V_H} d_H(b,t) < 12U$?

fix: force the center to be in A.

Final Claim: $R = \min_{c \in V_H} \max_{t \in V_H} d_H(c,t) < 12U$

iff $\exists$ neg-Δ in G.

Famous Open Question: APSP $\rightarrow$ Diameter ?

(3) APSP $\longleftrightarrow$ Median

Issue with previous reduction:

Cannot control $d(a_i, a_j')$.

Nor $d(a,b), d(a,c)$ - but these we will handle.

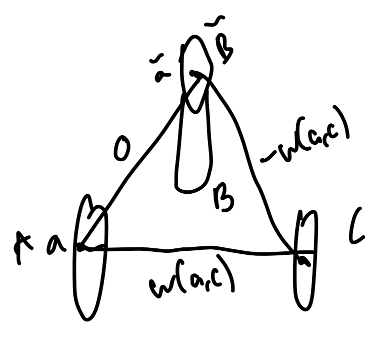
Neg-Δ $\to$ Median:

- Preliminary about Neg-Δ:

We can assume that $\forall a \in A, c \in C$: $\exists b \in B$ s.t. $w(a,b,c) = 0$.

pf: add $\tilde{a}$ to B and set $w(a,\tilde{a}) = 0$

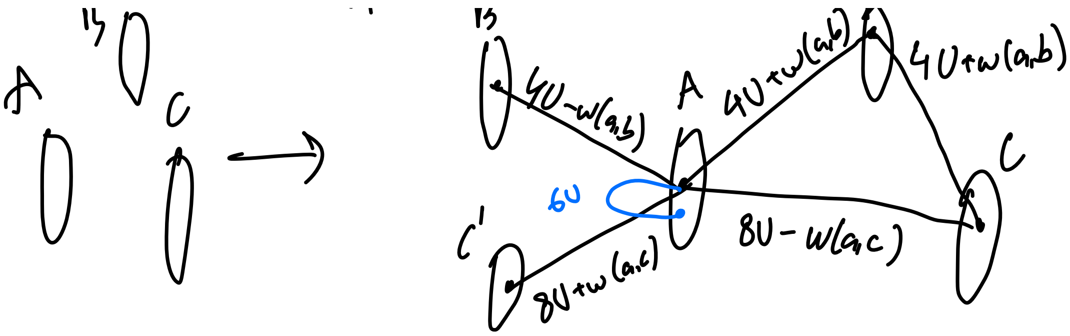
$w(\tilde{a},c) = -w(a,c)$.



- $\exists$ zero-Δ for all (a,c).
- no neg-Δ added: $w(a,\tilde{a},c) = 0$ $\forall a,c \ldots$

G

H a' B



$$d_H(a,c) = \min \begin{cases} 8U - w(a,c) \\ \min_b \; 8U + w(a,b) + w(b,c) \end{cases}$$

$$8U - w(a,c) > w(a,b) + w(b,c) + 8U \iff w(a,b,c) < 0$$

$\Rightarrow d_H(a,c) = 8U - w(a,c)$ <u>unless</u> (a,c) is in neg-Δ in G.

<u>issue:</u> what if a' has smaller $\sum_c -w(a',c)$, even though a is in a neg-Δ... (<u>fix:</u> add C')

<u>For any $a \in A$:</u>

$$\sum_{v \in V_H} d_H(a,v) = \sum_{b \in B} d(a,b) + \sum_{c \in C} d(a,c) + \underbrace{\sum_{\substack{\hat{a} \in A \\ a \neq \hat{a}}} d(a,\hat{a})}_{6 \cdot U \cdot (n-1)}$$

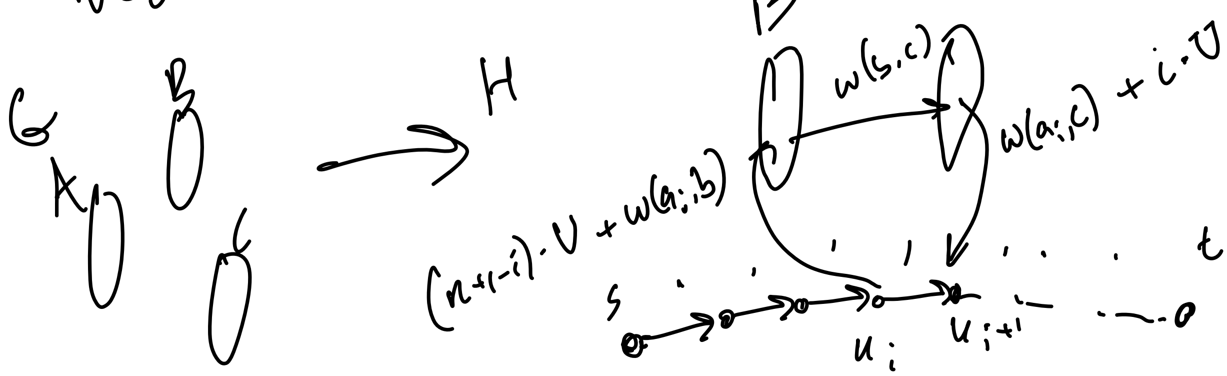
$$+ \underbrace{\sum_{b' \in B'} d(a,b')}_{= \sum_b 4U + w(a,b) + 4U - w(a,b) \;=\; 8U \cdot n} + \underbrace{\sum_{c' \in C'} d(a,c')}_{= \sum_c 8U - w(a,c) + 8U + w(a,c) \;=\; 16 \cdot nU}$$

unless a is in neg-Δ.

<u>Claim:</u> $M = \min_{c \in V_H} \sum_{v \in V_H} d_H(c,v) < 24nU + 6(n-1) \cdot U$

iff $\exists$ neg-Δ in G.

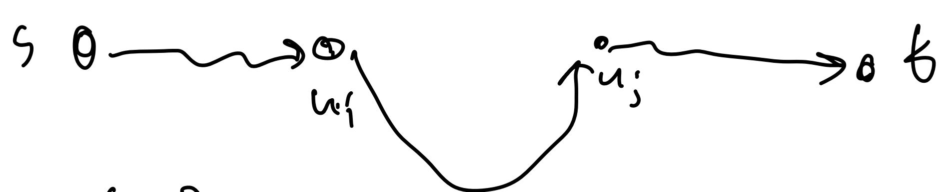
Neg-Δ $\to$ Replacement Paths in directed graphs



* same proof: Neg-Δ $\to$ Second-Shortest-Path.

<u>The idea for showing that</u> Replacement Paths $\to$ APSP

1. APSP in G
2. APSP in $G \setminus P_{s,t}$



$d_{G \setminus \{e\}}(s,t) =$

$$\min_{\substack{i<j \text{ s.t.} \\ e \text{ is between} \\ u_i \text{ and } u_j \text{ on } P_{s,t}}} d_G(s,u_i) + d_{G \setminus P_{s,t}}(u_i,u_j) + d_G(u_j,t)$$

(*) Replacement Paths in undirected graphs is in $\hat{O}(m)$ time.

(*) All-Pairs Max-Flow is in $\hat{O}(n^2)$ time.

Next: APSP $\rightarrow$ Zero-Δ $\leftarrow$ 3SUM

Neg-Δ

$\updownarrow$

Positive-Δ $\rightarrow$ Zero-Δ.

idea: $> \;\rightarrow\; =$.

intuition: $x > t$

x: $\overbrace{011}^{i}0101110$

t: $\overbrace{011}^{i}1 \cdots$

Lemma: $x > t$ iff $\exists i$ s.t. $\left\lfloor \frac{x}{2^i} \right\rfloor = \left\lfloor \frac{t}{2^i} \right\rfloor + 1$

Lemma a': $x+y+z > 0$ iff $\exists i$ s.t.

$$\left\lfloor \frac{x}{2^i} \right\rfloor + \left\lfloor \frac{y}{2^i} \right\rfloor + \left\lfloor \frac{z}{2^i} \right\rfloor \in \{1,2,3,\dots,7\}.$$

proof: $(\Leftarrow)$ obvious: $x+y+z \geq \cdots > 0$.

$(\Rightarrow)$ let i be s.t. $2^{i+2} \leq x+y+z < 2^{i+3}$

then:

$$1 = 2^2 - 3 \leq \frac{x+y+z}{2^i} - 3 \leq \left\lfloor \frac{x}{2^i} \right\rfloor + \left\lfloor \frac{y}{2^i} \right\rfloor + \left\lfloor \frac{z}{2^i} \right\rfloor \leq \frac{x+y+z}{2^i} < 2^3 = 8$$

The Reduction: Positive-Δ $\to$ Zero-Δ.

- for $i = 0$ to $\lg U$:
  - define $w^i(e) = \left\lfloor \frac{w(e)}{2^i} \right\rfloor$
  - for $k \in \{1,\dots,7\}$:
    - if $\exists$ triangle in (G, w^i) of weight k (note: this reduces to Zero-Δ)
      - return "yes".
- return "no".

$$3SUM \xrightarrow{} \text{Conv-3SUM} \xrightarrow{} \text{Zero-}\Delta$$

2 weeks ago — now

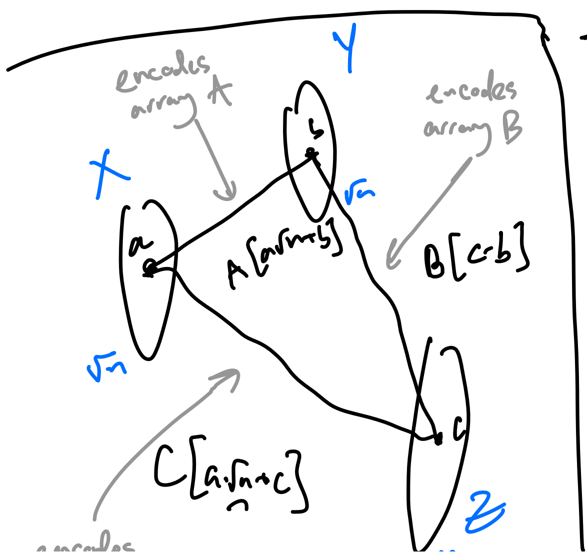
Recall:

Conv-3SUM: Given A, B, C arrays of n numbers are there i, j s.t. $A[i]+B[j-i]+C[j]=0$?

Thm: If Zero-Δ is in $O(n^{3-\epsilon})$ time, then Conv-3SUM is in $O(n^{2-\epsilon/2})$ time.

* Note: This is a reduction from an n^2 problem to an n^3 problem.

- given arrays A, B, C of length n, we define a tripartite graph $V = X \cup Y \cup Z$ where $|X| = |Y| = \sqrt{n}$, $|Z| = n$.



- we can then split Z into $\sqrt{n}$ subsets of size $\sqrt{n}$ and call the Zero-Δ alg on each of the resulting graphs.

$\Rightarrow$ The resulting time is:

$$\sqrt{n}\cdot(\sqrt{n})^{3-\epsilon} = n^{2-\epsilon/2}$$

array C n — # of calls — #nodes in the graph in each call

Correctness:

Claim: $\exists$ Zero-Δ iff $\exists$ Conv-3SUM.

$A[i]+B[j-i]+C[j]=0 \Longrightarrow \omega(a,b,c) = A[a\sqrt{n}+b]+B[c-b]+C[a\sqrt{n}+c] = A[i]+B[j-i]+C[j] = 0.$

pick a,b s.t. $a\sqrt{n}+b=i$
c s.t. $a\sqrt{n}+c=j$

$$w(a,b,c)=0 \Longrightarrow A[\underset{\substack{\uparrow \\ i}}{a\sqrt{n}+b}] + B[\underset{\substack{\uparrow \\ j-i}}{c-b}] + C[\underset{\substack{\uparrow \\ j}}{a\sqrt{n}+c}]=0$$