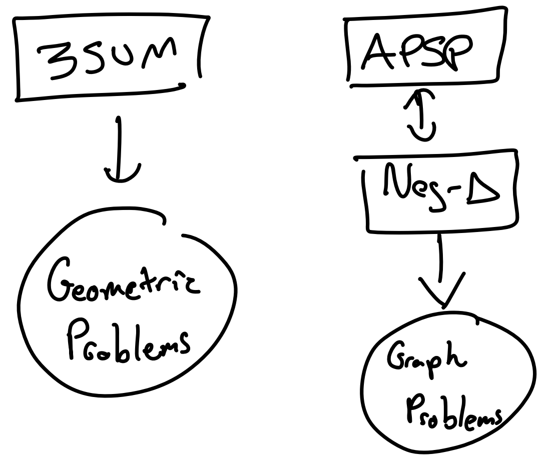
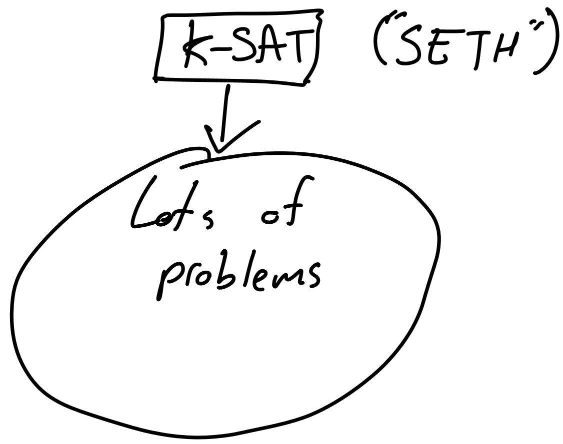
So far:



Next few weeks:



Today: Introducing SETH

Recall: k-CNF formula is a conjunction (AND) of clauses, where each clause is a disjunction (OR) of up to k literals, where a literal is a variable or its negation.

Example of 3-CNF on 5 vars $\{x_1, \ldots, x_5\}$:

$$(x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_4) \wedge (\bar{x}_4 \vee x_2 \vee x_3)$$
$$\wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_5) \wedge (x \vee x_4 \vee x_5)$$

k-SAT: Given a k-CNF formula on
- N vars
- M clauses

is it satisfiable?

- 2-SAT is in $O(N+M)$ time.
- NP-Hard for all $k \geq 3$.

- **Algorithms:**

	$k=3$	Large k
Trivial	2^N	2^N
⋮		
1985 [Monien & Speckenmeyer]	1.6181	$2^{(1-\frac{c}{2^k})\cdot N}$
· Backtracking		
idea: branch on $< 2^k$ options.		
⋮		
1999 [Schöning, PPSZ]	1.334	$2^{(1-\frac{c}{k})\cdot N}$
· Local Search via Random Walk		
⋮		
2011 [PPSZ+Hertli]	1.3071	
⋮ Random Assignments with simplification steps		

2019 [HKZZ]

Def: For any $k \geq 3$:

$$s_k = \inf\{\delta \mid k\text{-SAT has an } O(2^{\delta N})\text{-time alg}\}$$

ETH: $s_3 > 0$ i.e. $\exists\, \delta > 0$ s.t. 3-SAT is not in $O(2^{\delta N})$.

i.e. $2^{\Omega(n)}$ for 3-SAT.

SETH: $\lim_{k \to \infty} s_k = 1$ i.e. $\forall \varepsilon > 0\ \exists\, k \geq 3$ s.t. k-SAT is not in $O(2^{(1-\varepsilon) \cdot N})$.

Thm: ETH $\Rightarrow$ SETH.

Consequences? Let's look at Dom-Set.

Dominating-Set: Given a graph $G = (V, E)$, and a parameter q, is there a set $S \subseteq V$ of size $|S| = q$ s.t. $\forall v \in V$ either $v \in S$ or there is a node $u \in S$ and $\{u, v\} \in E$.

Alg: $\binom{n}{q} \cdot qn$ exponential time for large q.

NP-Hardness proof:

3-SAT $\longrightarrow$ Dom Set

formula φ graph G

N vars
M clauses

$O(N+M)$ time

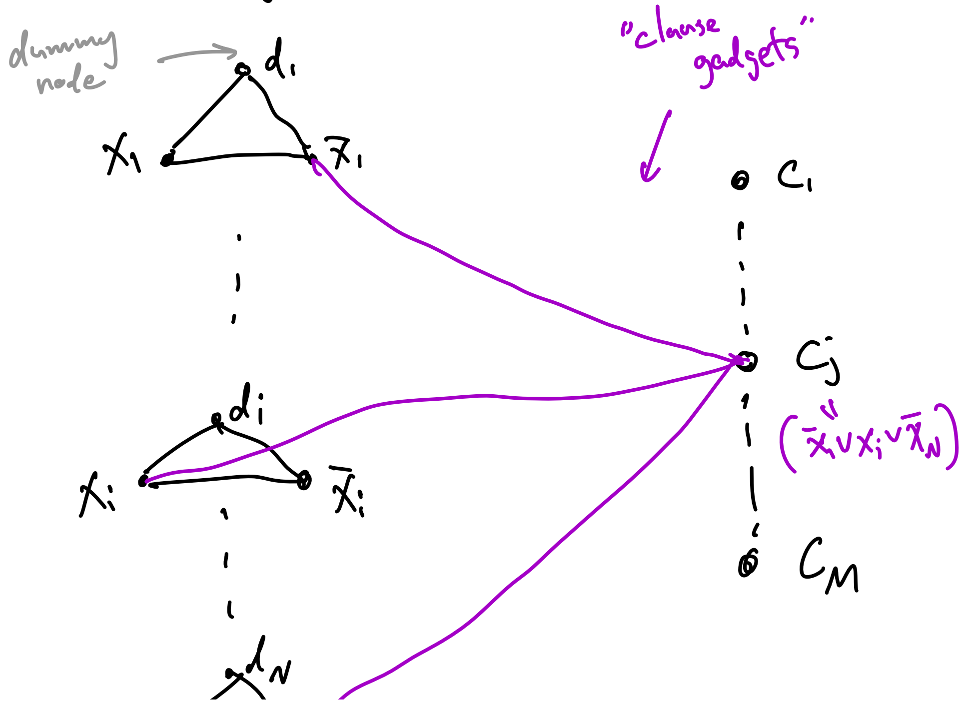
$n = 3N + M$ nodes
$q = N$

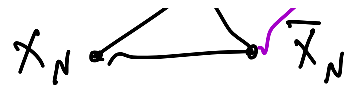
$P = NP \Leftarrow N^{O(1)} \Leftarrow M^{O(1)} \Leftarrow n^{O(1)}$

(<u>Claim:</u> φ is satisfiable iff G has a dom-set of size q.)

<u>Construction:</u>

"variable gadgets"





Thm (weak): ETH $\Rightarrow$ $2^{\Omega(n^{1/3})}$ for Dom-Set.

pf: assume $\forall \delta > 0$ Dom-Set is in $O(2^{\delta n^{1/3}})$.
Let $\delta' > 0$. To solve 3-SAT in $O(2^{\delta' N})$ time, use the reduction:

$$n = 3N + M \leq 3N + \binom{N}{3} \cdot 2^3 = 11 N^3$$

and get an alg: $O\left(2^{\delta (11 N^3)^{1/3}}\right) = O\left(2^{\delta' \cdot N}\right)$
for $\delta = \delta' / \sqrt[3]{11}$.

Want: $2^{\Omega(n)}$ for Dom-Set.
How? Can we assume that $M = O(N)$? Yes!

The Sparsification Lemma (Impagliazzo - Paturi - Zane 2001):

$\forall k \geq 3$ and $\varepsilon > 0$ there is a constant $C = C(k, \varepsilon)$ and an alg s.t.

1. Given a k-CNF φ on N vars, alg computes $\varphi_1, \dots, \varphi_t$.
2. φ is satisfiable $\iff$ $\exists i$ s.t. φ_i is satisfiable.
3. $t \leq 2^{\varepsilon N}$ and alg takes $O(2^{\varepsilon N} \mathrm{poly}(N))$ time.
4. Each φ_i is k-CNF on N vars and $M_i \leq C \cdot N$ clauses.

$$\varphi \rightarrow \mathrm{OR}(\varphi_1, \dots, \varphi_t)$$

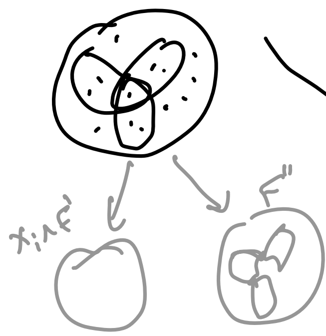
only $M = O(n)$ clauses

$t = 2^{o(n)}$

$C(k,\varepsilon) = \left(\frac{k}{\varepsilon}\right)^{O(k)}$

High level idea of proof:

find the var (or more generally, a sub-clause) that appears most often.



set it to true
$\Downarrow$
remove all clauses that contain it

set it to false
$\Downarrow$
all these clauses become smaller

— repeat εN times...

Thm: ETH $\Rightarrow$ $2^{\Omega(n)}$ for Dom-Set.

pf: Suppose $\forall \delta' > 0$, Dom-Set is in $O(2^{\delta' N})$.

Let $\delta > 0$, we will show: 3-SAT in $O(2^{\delta N})$.

Given φ on N vars:

- set $\varepsilon = \delta/2$ and use sparsification Lemma:
  $\varphi \longrightarrow \varphi_1, \dots, \varphi_t \qquad t \le 2^{\varepsilon N}, \; M_i \le C(3,\varepsilon)\cdot N.$

- set $\delta' = \frac{\delta}{2} \cdot \frac{1}{(3 + C(3,\varepsilon))}$.

- for $i = 1 \dots t$:
  - Reduce φ_i to G_i on $n \le 3N + M_i$ nodes
  - Solve Dom-Set on G_i in $O(2^{\delta' \cdot n})$.

$\Longrightarrow$ time per i:

$$2^{\delta' \cdot n} \le 2^{\delta' \cdot (3 + C(3,\varepsilon)) \cdot N} = 2^{\frac{\delta}{2} \cdot \frac{(3 + C(3,\varepsilon))}{(3 + C(3,\varepsilon))} \cdot N} = 2^{\frac{\delta}{2} \cdot N}$$

$\Longrightarrow$ for all i:

$$t \cdot 2^{\frac{\delta}{2} \cdot N} = 2^{\varepsilon N} \cdot 2^{\frac{\delta}{2} \cdot N} = 2^{\delta N}.$$

$\boxtimes$

Dom-Set
- $O(1.4969^n)$
- $\Omega(c^n)$ for some $c > 0$, assuming ETH.

Now let's get more fine-grained...

<u>q-Dom-Set:</u> Given G, is there a dom-set of size q?

[e.g. $q = 3$.]

- trivial: $O(n^q \cdot n) = O(n^{q+1})$
- using MM: $n^{q+o(1)}$ for $q \geq 7$.

<u>Q</u>: $n^{\sqrt{q}}$? $n^{q/10}$? n^{q-2}? $n^{q-\varepsilon}$?

Better reduction:

k-SAT $\longrightarrow$ q-Dom Set

φ: N vars
M clauses

$\xrightarrow[\text{exponential blowup}]{}$

G: $n = q \cdot 2^{N/q} + q + M$
$\approx 2^{N/q}$ nodes

$\boxed{\text{Claim: } \varphi \text{ is sat.} \iff G \text{ has } q \text{ dom-set.}}$

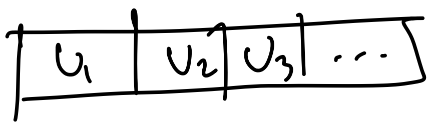
$\forall \delta > 0: O(2^{\delta N}) \Leftarrow 2^{N/\sqrt{q}} \Leftarrow n^{\sqrt{q}}$

$2^{(1-\varepsilon) \cdot N} \Leftarrow n^{(1-\varepsilon) \cdot q}$

The Reduction:

Prelim: Partition the vars of φ $U = \{x_1, \dots, x_N\}$ into q groups of size N/q.

$$U \to U_1, \dots, U_q \quad \text{s.t.} \quad U_i = \left\{ x_{i \cdot \frac{N}{q} + 1}, \dots, x_{(i+1) \cdot \frac{N}{q}} \right\}$$



Construction:

1. $\forall i$:
   - $\forall \alpha \in [2^{N/q}]$ partial assignment to U_i
     - create vertex v_α^i.
   - create dummy vertex d^i.
   - connect all v_α^i and d^i as a clique.

2. $\forall$ clause C_j create vertex u_j

3. Edges:

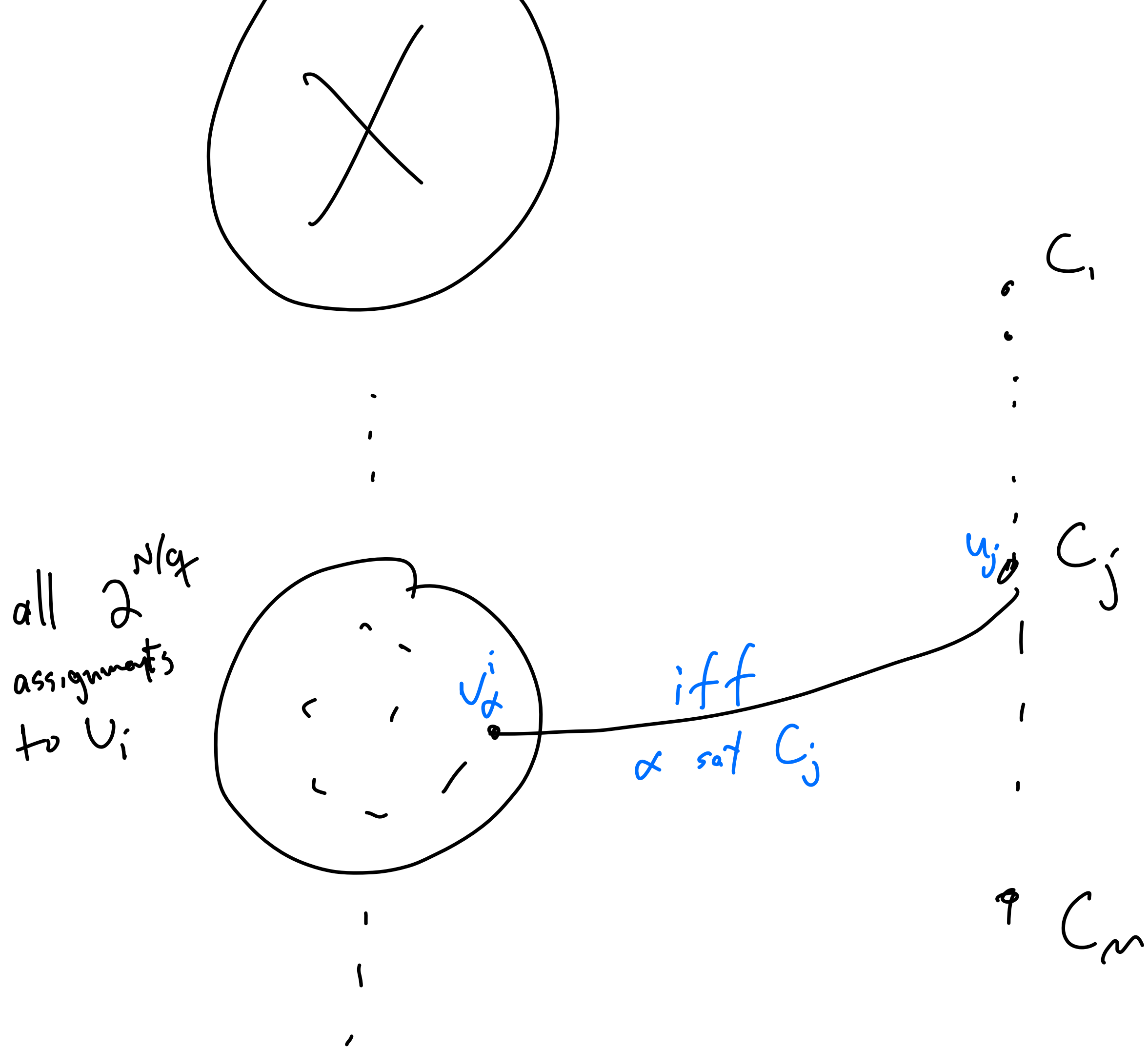
$$\{v_\alpha^i, u_j\} \in E \iff \begin{pmatrix} \exists x \in C_j : \alpha(x) = 1 \\ \text{OR} \\ \exists \bar{x} \in C_j : \alpha(x) = 0 \end{pmatrix}$$

* Examples: $\alpha : (x_7 = 1,\ x_8 = 0,\ x_9 = 1)$

α sat $(x_8 \vee x_9 \vee x_{10})$
α doesn't sat $(x_8 \vee x_{10})$
α doesn't sat $(x_{20} \vee x_{21})$

$n = q \cdot (2^{N/q} + 1) + M$

time for reduction $O(2^{2N/q})$



X

<u>Correctness</u>: just like in the simpler proof...

$\exists$ sat assignment $\Rightarrow$ choose the q nodes that represent it
$\Rightarrow$ all clause vertices are dominated
& all cliques are dominated.

$\exists$ dom-set of size $q \Rightarrow$ must pick one node per clique
$\Rightarrow$ define the assignment that is consistent with the q nodes.
$\forall$ clause j: u_j is dominated $\Rightarrow C_j$ is satisfied by the assignment.

<u>Thm:</u> SETH $\Rightarrow \forall q \geq 3$ and $\varepsilon > 0$: q-DomSet is <u>not</u> in $O(n^{q-\varepsilon})$ time.

<u>pf:</u> Suppose q-DomSet in $O(n^{q-\varepsilon})$:

$\forall k$: k-SAT in:

$$O(2^{\frac{2N}{q}}) + O\left((2^{N/q})^{q-\varepsilon}\right) = O(2^{(1-\varepsilon')\cdot n})$$

for the reduction time

for $\varepsilon' = \frac{\varepsilon}{9}$.

$\blacksquare$

$\Rightarrow \Omega(n^{2.99})$ for 3-Dom-Set.