Today: SETH $\xrightarrow{}$ lots of problems.
$\uparrow$
simple reductions.

Next weeks: more complicated reductions...

SETH: $\forall \varepsilon > 0 \;\; \exists k \geq 3$ s.t. k-SAT is not in $2^{(1-\varepsilon)\cdot N}$ time.

(*) Can assume only $M = O(N)$ clauses.

SAT $\to$ OV $\to$ Problems in P

OV:
Given two sets $A, B \subseteq \{0,1\}^d$ of binary vectors where $|A| = |B| = n$, is there $a \in A, b \in B$ s.t.

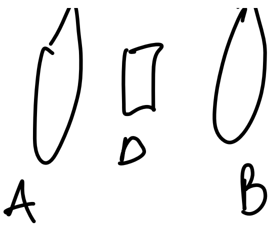
$$\langle a, b \rangle = 0 \quad \text{i.e.} \quad \forall j: \left(a[j] = 0 \;\text{ OR }\; b[j] = 0 \right)$$

ex.

0010	1010
1100	0110
✓	✗

Graph-OV:

Given graph $V = A \cup B \cup D$, $E \subseteq A \times D \cup B \times D$
where $|A| = |B| = n$, $|D| = d$,
is there $a \in A, b \in B$, s.t. $d(a,b) > 2$?



- $O(n^2 d)$ trivial
- $O(2^d \cdot n)$ subquadratic for $d < \log n$
- $n^{2-\frac{d}{\Theta(\log n)}}$ subquadratic for $d = c \log n$ for all c [A-Williams-Yu '15]

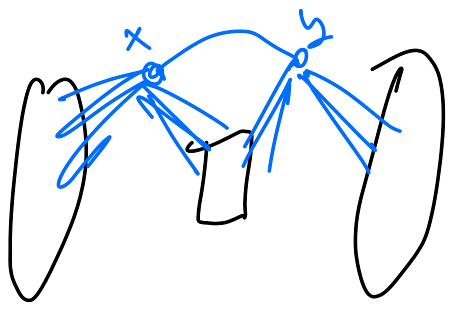
Thm: SETH $\Rightarrow$ "$\Omega(n^{2-\varepsilon})$ for OV even when $d = O(\log n)$"
i.e. $\forall \varepsilon > 0$, $\exists c > 0$ s.t. OV with $d = c \log n$
cannot be solved in $O(n^{2-\varepsilon})$ time.

pf: Given a k-CNF formula φ.

(See LaTeX notes)

Diameter: compute $\max_{u,v} d(u,v)$.

Thm: SETH $\Rightarrow$ $\Omega(n^{2-\varepsilon})$ for Diameter when $m = \tilde{O}(n)$.

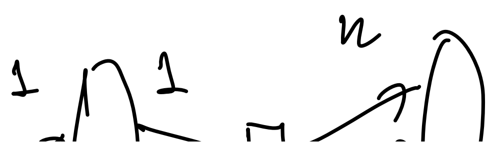


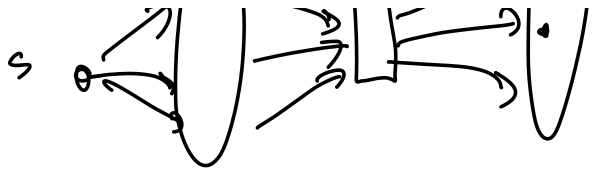
- $1.5-\delta$ apx $\Omega(n^{2-\varepsilon})$
- exact: $O(n^2)$
- 2-apx $\tilde{O}(n)$
- 1.5-apx $\tilde{O}(n^{1.5})$ — best possible in subq. time
- Very recently: [DLV'21]
  No $(2-\delta)$-apx in $\tilde{O}(n)$ time.

U.B.	L.B.
$n^{1+\frac{1}{k}}$ time	$n^{1+\frac{1}{k}}$ time
$\sim 2-1/2^k$ apx	$\sim 2-\frac{1}{k}$ apx

Max Flow from s to t is the largest number of paths from s to t s.t. each edge is used only $c(e)$ times.

Thm: SETH $\Rightarrow$ $\Omega(n^{2-\varepsilon})$ for Single-Source Max Flow when $m=\tilde{O}(n)$.





Claim:

$\forall b \in B$: $\text{MaxFlow}(s,b) = n$ iff $\forall a \in A$ $\langle a,b \rangle \neq 0$.

Next: higher $\Omega(n^3)$ l.b. for All-Pairs Max Flow.

3-OV: Given sets $A, B, C \subseteq \{0,1\}^d$ of n binary vectors, is there $a \in A$, $b \in B$, $c \in C$ s.t. $\langle a,b,c \rangle = \sum_{j=1}^{d} a[j] \cdot b[j] \cdot c[j] = 0$?

ex.

$$\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{matrix} \qquad\qquad \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{matrix}$$

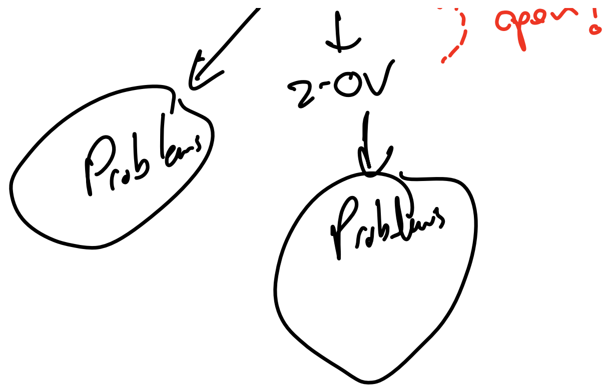
$\checkmark$ $\qquad\qquad\qquad$ $\times$

Thm: SETH $\Rightarrow$ no $O(n^{3-\varepsilon})$ alg when $d = \Omega(\lg n)$.
(same proof, just split the vars into 3 sets of size $\frac{N}{3}$.)

k-SAT

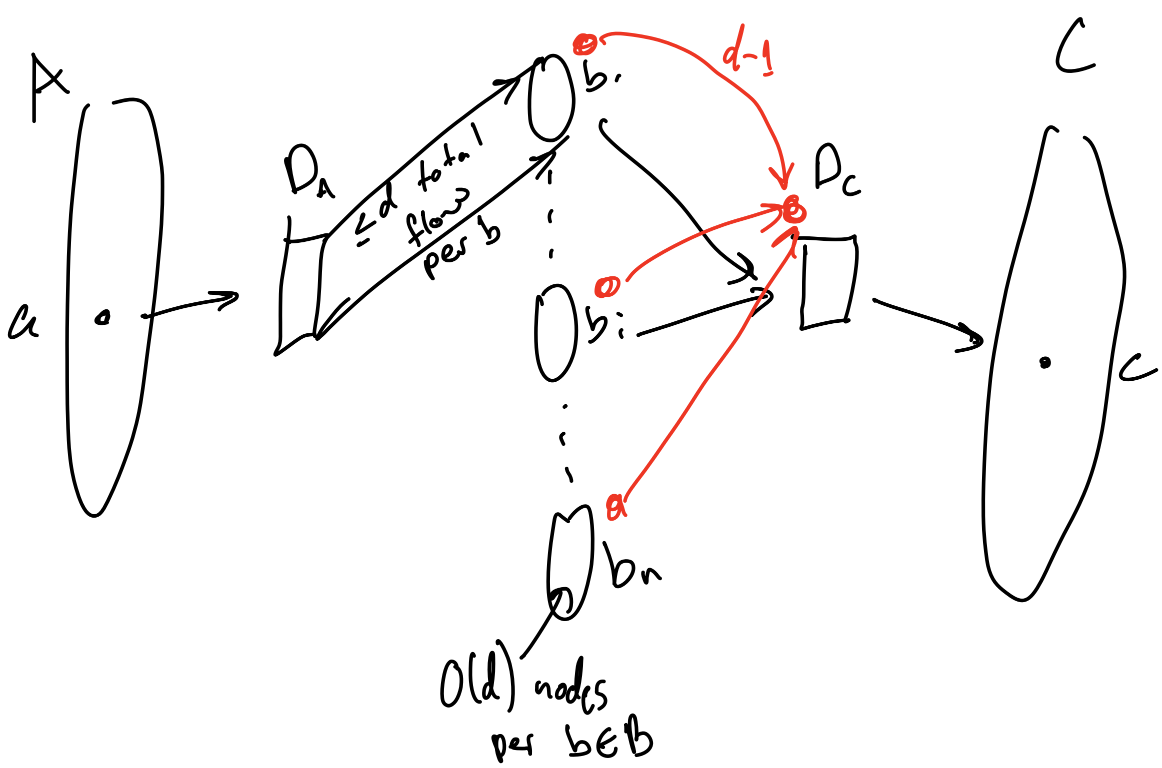
$\downarrow$

3-OV



$3\text{-OV} \longrightarrow \text{APMF}$

<u>high level structure of the reduction:</u>



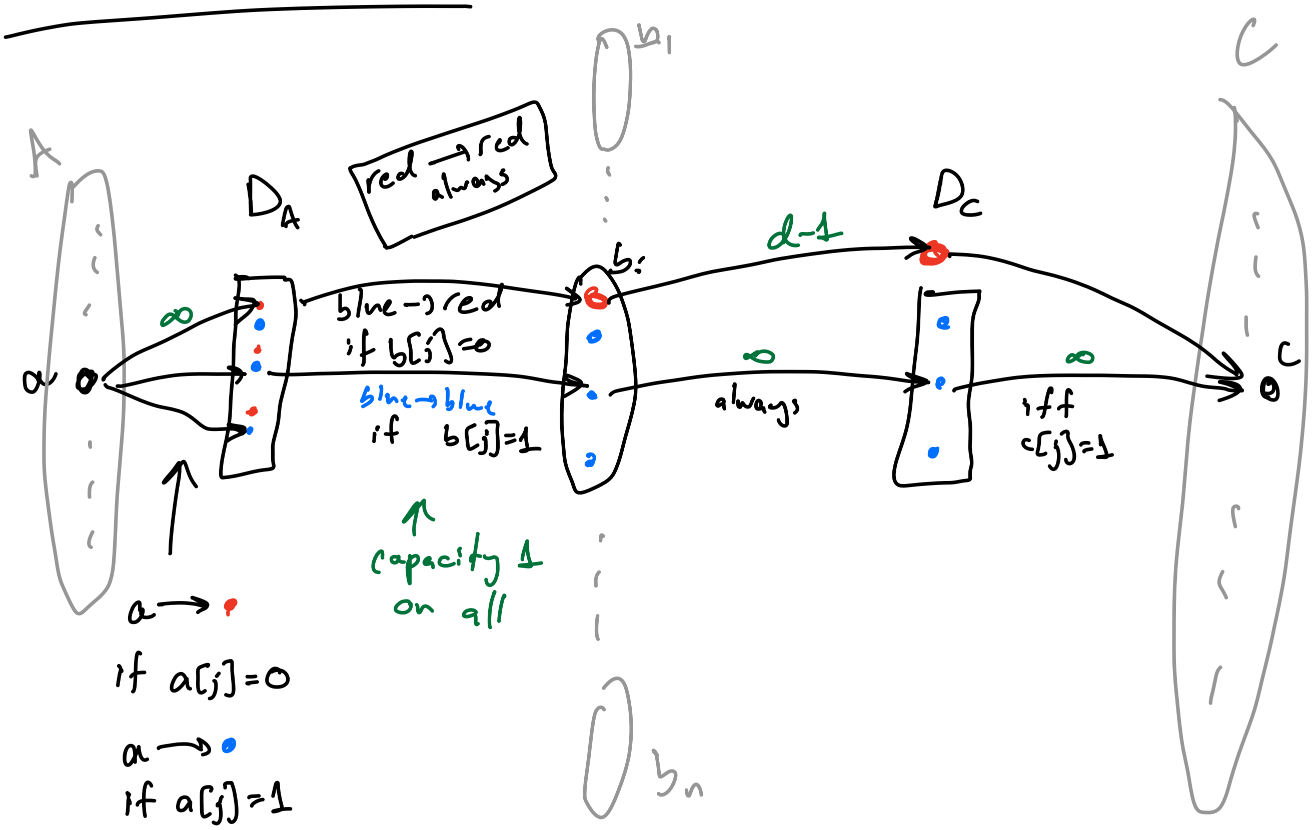
<u>Goal:</u> check if $\forall a,c: \forall b$ there is a path via b.

<u>Claim:</u> $\forall a,c:\ \text{MaxFlow}(a,c) = n\cdot d$ iff $\forall b: \langle a,b,c\rangle > 0$.

how to go from 1 path to flow d?

<u>idea:</u> the red nodes guarantee $d-1$ flow per b.

The construction:



<u>idea:</u> blue means 1 so far (at coordinate j)
red means already 0.

<u>Correctness:</u> only $\leq d$ per b_j and $= d$ iff $\langle a,b,c \rangle \neq 0$.

<u>efficiency:</u> $O(n)$ nodes, $O(nd)$ edges.

$\Rightarrow$ <u>Thm:</u> APMF in $O(n^{3-\varepsilon})$ time when $m = \tilde{O}(n)$

refutes SETH.

Next: OV $\longrightarrow$ "Closest Pair."

Euclidean CP: Given $A, B \subseteq \mathbb{R}^d$, $|A| = |B| = n$

find $\min_{\substack{a \in A \\ b \in B}} \|a-b\|_2$

Hamming CP: Given $A, B \subseteq \{0,1\}^d$, $|A| = |B| = n$

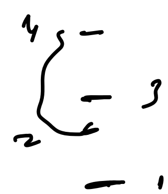
find $\min_{\substack{a \in A \\ b \in B}} \text{Hamming Dist}(a,b)$

Reductions:

embed orthogonality into these metrics.

Euclidean:

$a \in A \to \hat{a}$ $\qquad\qquad$ $b \in B \to \hat{b}$



$$\begin{array}{lcc|cc} \text{Replace:} & 0 & 1 & 0 & 1 \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ \text{with:} & 3 & 1 & 2 & 4 \end{array}$$

Claim: $\langle a,b\rangle = 0 \implies \|\hat{a}-\hat{b}\|_2 = \sqrt{\sum_{j=1}^{d} 1^2}$

$\langle a,b\rangle \neq 0 \implies \|\hat{a}-\hat{b}\|_2 > \sqrt{d}$

Hamming:

Hamming:

Replace: $0 \to 0011$, $1 \to 0000$

With: $0 \to 0101$, $1 \to 1111$

Claim: $\langle a,b\rangle = 0 \implies \text{HammDist}(\hat{a},\hat{b}) = 2\cdot d$

$\langle a,b\rangle \neq 0 \implies \text{HammDist}(\hat{a},\hat{b}) > 2d$

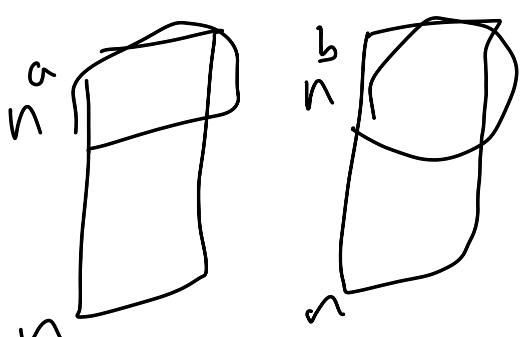
(a,b)-OV: Given sets $A,B \subseteq \{0,1\}^d$ with $|A| = n^a$, $|B| = n^b$, is there $a \in A$, $b \in B$ s.t. $\langle a,b\rangle = 0$?

Thm: $\forall a,b > 0$:

(a,b)-OV in $O(n^{a+b-\varepsilon})$ time, for some $\varepsilon > 0$.

$\implies$ OV is in $O(n^{2-\delta})$ time, for some $\delta > 0$.

pf:



$\to n^{1-a}\cdot n^{1-b}\cdot(n^{a+b-\varepsilon})$

$= n^{2-\varepsilon}$

Thm: SETH $\Rightarrow$ cannot preprocess a set of n points in d dimensions in time $O(n^{100})$ and and answer Nearest Neighbor queries in $O(n^{0.99})$ time.

using any of the metrics we discussed (for $d = \Omega(\log n)$)

pf: reduction from $(\frac{1}{100}, 1)$-OV:
prep A in time

$$\left(n^{1/100}\right)^{100} = n$$

Then check each $b \in B$ in time

$$n \cdot \left(n^{1/100}\right)^{0.99}$$

$\Rightarrow$ total time is $n^{1 + \frac{0.99}{100}}$ $\Rightarrow$ OV in $O(n^{2-\epsilon})$ time