Hardness of Approximation meets Parameterized Complexity

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Part 1: Handwaving Introduction
Outline

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Part 2: Dominating Set
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Part 3: Hardness of Approximation
   - Hardness of Approximation in NP
   - Hardness of Approximation in Parameterized Complexity
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Part 4: Coding Theory
  - Definition and Geometric Intuition
  - Random Codes
  - Algebraic Codes
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  - MaxCover with Projection Property
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Optimization Problems

- Dominating Set/Set Cover
- Set Intersection
- Clique
- Vertex Cover
- Clustering
- Satisfiability
Optimization Problems

- Dominating Set/Set Cover
- Set Intersection
- Clique
- Vertex Cover
- Clustering
- Satisfiability

Many Optimization Problems are NP-hard!
Coping Mechanism

ARGH ...

LIFE IS SO HARD!
Approximation Algorithms  

Fixed Parameter Tractability
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Dominating Set Problem

\[ G(V, E) \]

A set \( S \subseteq V \) is a Dominating Set of \( G \) if \( \forall u \in V : u \in S \), or \( \exists v \in S \) such that \((u, v) \in E\).

Computational Problem: Given \( G \) and \( k \in \mathbb{N} \), determine if \( \exists S \subseteq V : S \) is a Dominating Set of \( G \) with \( |S| \leq k \rightarrow NP-Complete \) [Karp'72]

Approximation is in \( P \) [Slav´ık'96]

Approximation is \( (1-\varepsilon) \ln |V| \) NP-Complete [DS'14]

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Dominating Set Problem

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Dominating Set Problem

$S \subseteq V$ is a Dominating Set of $G$ if \( \forall u \in V: \)
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Computational Problem: Given $G$ and \( k \in \mathbb{N} \), determine if \( \exists S \subseteq V: \)
- $S$ is a Dominating Set of $G$
- \( |S| \leq k \)
Dominating Set Problem

\( S \subseteq V \) is a Dominating Set of \( G \) if \( \forall u \in V: \)
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Computational Problem: Given \( G \) and \( k \in \mathbb{N} \), determine if \( \exists S \subseteq V:\)
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\( \rightarrow \) NP-Complete [Karp’72]
Dominating Set Problem

$S \subseteq V$ is a Dominating Set of $G$ if
\[ \forall u \in V: \]
- $u \in S$, or
- $\exists v \in S$ such that $(u, v) \in E$

Computational Problem: Given $G$ and $k \in \mathbb{N}$, determine if $\exists S \subseteq V$:
- $S$ is a Dominating Set of $G$
- $|S| \leq k$

\[ \ln |V| \text{ approximation is in } P \] [Slavík’96]

\[ \text{NP-Complete [Karp’72]} \]
Dominating Set Problem

Let \( G(V, E) \) be a graph.

A set \( S \subseteq V \) is a Dominating Set of \( G \) if for all \( u \in V \):

- \( u \in S \), or
- there exists \( v \in S \) such that \((u, v) \in E\).

**Computational Problem:**
Given \( G \) and \( k \in \mathbb{N} \), determine if there exists a Dominating Set \( S \subseteq V \) such that \( |S| \leq k \).

\( \rightarrow \) ln \( |V| \) approximation is in \( P \) [Slavík’96]

\( \rightarrow \) \((1 - \varepsilon)\ln |V|\) approximation is NP-Complete [DS’14]

\( \rightarrow \) NP-Complete [Karp’72]
Parameterized Dominating Set Problem

Computational Problem: Given $G$ and parameter $k \in \mathbb{N}$, determine if

$\exists \ S \subseteq V:$

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Fixed Parameter Tractability (**FPT**): The problem can be decided in $F(k) \cdot \text{poly}(|V|)$ time, for some computable function $F$. 
Parameterized Dominating Set Problem

Computational Problem: Given $G$ and parameter $k \in \mathbb{N}$, determine if
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Fixed Parameter Tractability (FPT): The problem can be decided in $F(k) \cdot \text{poly}(|V|)$ time, for some computable function $F$. 

$k$-Dominating Set

$k$-Clique

$k$-Vertex Cover
Parameterized Dominating Set Problem

Computational Problem: Given \( G \) and parameter \( k \in \mathbb{N} \), determine if there exists a \( S \subseteq V \):
- \( S \) is a Dominating Set of \( G \)
- \(|S| \leq k\)

Fixed Parameter Tractability (FPT): The problem can be decided in \( F(k) \cdot \text{poly}(|V|) \) time, for some computable function \( F \).

- \( k\)-Dominating Set
- \( k\)-Clique
- \( k\)-Vertex Cover

\([W[2] \rightleftharpoons W[1] \rightleftharpoons \text{FPT}]\)
FPT Approximability: The problem has a $T(k)$ approximation algorithm running in time $F(k) \cdot \text{poly}(N)$ time.
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Approximate Parameterized Dominating Set Problem: Given a graph $G$ and parameter $k$ distinguish between:

- $\exists$ a dominating set of size at most $k$
- There is no dominating set of size $T(k) \cdot k$
FPT Approximability: The problem has a $T(k)$ approximation algorithm running in time $F(k) \cdot \text{poly}(N)$ time.

Approximate Parameterized Dominating Set Problem: Given a graph $G$ and parameter $k$ distinguish between:

- $\exists$ a dominating set of size at most $k$
- There is no dominating set of size $T(k) \cdot k$

Is there some computable function $T$ for which the above problem is in FPT?
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Many important optimization problems are not tractable. A typical way to cope with the intractability of optimization problems is to design algorithms that find solutions whose cost or value is close to the optimum. In several interesting cases, it is possible to prove that even finding good approximate solutions is as hard as finding optimal solutions. The area which studies such inapproximability results is called hardness of approximation.
PCP Theorem: Bedrock of NP-Hardness of Approximation
**PCP Theorem & Label Cover**

**PCP Theorem:** Bedrock of NP-Hardness of Approximation

\[ \pi_{i,j} \subseteq \Sigma_U \times \Sigma_W \]

\[ \Gamma(U, W, E) \]
PCP Theorem & Label Cover

PCP Theorem: Bedrock of NP-Hardness of Approximation

\[ \pi_{i,j} \subseteq \Sigma_U \times \Sigma_W \]

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\[ \sigma_W : W \rightarrow \Sigma_W \text{ is a labeling of } W \]
PCP Theorem: Bedrock of NP-Hardness of Approximation

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\( \sigma_W : W \rightarrow \Sigma_W \) is a labeling of \( W \)

\((u_i, w_j) \in E\) is satisfied by \((\sigma_U, \sigma_W)\)
if \((\sigma_U(u_i), \sigma_W(w_j)) \in \pi_{i,j}\)
PCP Theorem: Bedrock of NP-Hardness of Approximation

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$$(u_i, w_j) \in E$$ is satisfied by $$(\sigma_U, \sigma_W)$$ if $$(\sigma_U(u_i), \sigma_W(w_j)) \in \pi_{i,j}$$

$$\text{VAL}(\Gamma, \sigma_U, \sigma_W) = \text{Fraction of edges satisfied by } (\sigma_U, \sigma_W)$$
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\[ \text{VAL}(\Gamma) = \max_{\sigma_U, \sigma_W} \text{VAL}(\Gamma, \sigma_U, \sigma_W) \]
PCP Theorem & Label Cover

**PCP Theorem**: Bedrock of NP-Hardness of Approximation

\[ \pi_{i,j} \subseteq \Sigma_U \times \Sigma_W \]

\( \sigma_U : U \to \Sigma_U \) is a labeling of \( U \)

\( \sigma_W : W \to \Sigma_W \) is a labeling of \( W \)

\((u_i, w_j) \in E\) is satisfied by \((\sigma_U, \sigma_W)\)

if \((\sigma_U(u_i), \sigma_W(w_j)) \in \pi_{i,j}\)

\[ \text{VAL}(\Gamma) = \max_{\sigma_U, \sigma_W} \text{VAL}(\Gamma, \sigma_U, \sigma_W) \]

Determining if \( \text{VAL}(\Gamma) = 1 \) or

if \( \text{VAL}(\Gamma) \leq 0.99 \) is NP-Hard
Extended Label Cover

\[ \Gamma_{\text{ext}}(U_{\text{ext}}, W_{\text{ext}}, E_{\text{ext}}) \]

- \( n \cdot |\Sigma_U| \) nodes in \( U \)
- \( m \cdot |\Sigma_W| \) nodes in \( W \)
- \((u_i, \alpha), (w_j, \beta) \in E_{\text{ext}} \) iff \((u_i, w_j) \in E\) and \((\alpha, \beta) \in \pi_{i,j}\)
Extended Label Cover

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\( S \subseteq W \) is a labeling of \( W \) if 
\[ \forall i \in [k], |S \cap W_i| = 1 \]

\( T \subseteq U \) is a labeling of \( U \) if 
\[ \forall i \in [k], |T \cap U_i| = 1 \]
Extended Label Cover

\[ \Gamma_{\text{ext}}(U_{\text{ext}}, W_{\text{ext}}, E_{\text{ext}}) \]

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MaxCover

Each $W_i$ is a Right Super Node
Each $U_i$ is a Left Super Node

$S \subseteq W$ is a labeling of $W$ if $
\forall i \in [k], |S \cap W_i| = 1$ 

$S$ covers $U_i$ if $\exists u \in U_i, \forall v \in S, (u, v) \in E$

MaxCover($\Gamma$, $S$) = Fraction of $U_i$'s covered by $S$

MaxCover($\Gamma$) = max $\{ $MaxCover($\Gamma$, $S$) $\}$
Determine if $\text{MaxCover}(\Gamma) = 1$ or $\text{MaxCover}(\Gamma) \leq s$.

Each $W_i$ is a Right Super Node.
Each $U_i$ is a Left Super Node.

$S \subseteq W$ is a labeling of $W$ if $\forall i \in [k], |S \cap W_i| = 1$.

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$\text{MaxCover}(\Gamma, S) = \text{Fraction of } U_i's$ covered by $S$

$\text{MaxCover}(\Gamma) = \max_S \text{MaxCover}(\Gamma, S)$
MaxCover

Each $W_i$ is a **Right Super Node**
Each $U_i$ is a **Left Super Node**

$S \subseteq W$ is a **labeling** of $W$ if

$\forall i \in [k], |S \cap W_i| = 1$

$\Gamma(U, W, E)$

Determines if $\text{MaxCover}(\Gamma) = 1$ or $\text{MaxCover}(\Gamma) \leq s$

Each $W_i$ is a **Right Super Node**
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$S \subseteq W$ is a **labeling** of $W$ if

$\forall i \in [k], |S \cap W_i| = 1$
Each \( W_i \) is a Right Super Node
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\[
\Gamma(U, W, E)
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MaxCover

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MaxCover($\Gamma, S$) = Fraction of $U_i$’s covered by $S$
Determine if $\text{MaxCover}(\Gamma) = 1$ or $\text{MaxCover}(\Gamma) \leq s$

Each $W_i$ is a **Right Super Node**
Each $U_i$ is a **Left Super Node**

$S \subseteq W$ is a labeling of $W$ if
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$\text{MaxCover}(\Gamma, S) =$ Fraction of $U_i$'s covered by $S$

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Determine if $\text{MaxCover}(\Gamma) = 1$
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$\text{MaxCover}(\Gamma, S) =$ Fraction of $U_i$’s covered by $S$

$\text{MaxCover}(\Gamma) = \max_S \text{MaxCover}(\Gamma, S)$
$k$-Clique as MaxCover

Input of $k$-Clique problem:

$\Gamma(U, W, E)$

Determine if MaxCover($\Gamma$) = 1 or MaxCover($\Gamma$) ≤ $1 - 1/k$. 

Each $W_j$ is a copy of $E_0$. 

Each $U_i$ is a copy of $[n]$. 

For distinct $i, j, j'$, introduce all edges between $W_j, j'$ and $U_i$. 

$\Gamma(U, W, E)$
$k$-Clique as MaxCover

Input of $k$-Clique problem: $G([n], E_0)$
**$k$-Clique as MaxCover**

Input of $k$-Clique problem: $G([n], E_0)$

- Each $W_j$ is a copy of $E_0$
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$\Gamma(U, W, E)$
$k$-Clique as MaxCover

Input of $k$-Clique problem:
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Each $W_j$ is a copy of $E_0$
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For distinct $i, j, j'$, introduce all edges between $W_{j,j'}$ and $U_i$
$k$-Clique as MaxCover

Input of $k$-Clique problem:
$G([n], E_0)$

Each $W_j$ is a copy of $E_0$
Each $U_i$ is a copy of $[n]$

For distinct $i, j, j'$, introduce all edges between $W_{j,j'}$ and $U_i$

Determine if $\text{MaxCover}(\Gamma) = 1$
or $\text{MaxCover}(\Gamma) \leq 1 - 1/k$
MaxCover: Results

- $W[1]$-Complete if there are $F(k)$ left super nodes
MaxCover: Results

- W[1]-Complete if there are $F(k)$ left super nodes
- 1 vs. $k/n^{1/\sqrt{k}}$ is W[1]-Hard
MaxCover: Results

- $W[1]$-Complete if there are $F(k)$ left super nodes

- $1$ vs. $k/n^{1/\sqrt{k}}$ is $W[1]$-Hard

- Central problem to understand parameterized inapproximability of Dominating Set and Clique
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Consider all strings/points in $\{0, 1\}^n$. 

What is the largest subset of $\{0, 1\}^n$ of even Hamming weight?

What is the largest subset of $\{0, 1\}^n$ whose all pairwise Hamming distances is at least 3?

What is the largest subset of $\{0, 1\}^n$ whose all pairwise Hamming distances is at least 9?

What is the largest subset of $\{0, 1\}^n$ whose all pairwise Hamming distances is at least 49?
Consider all strings/points in $\{0, 1\}^n$

Consider subset of $\{0, 1\}^n$ of even Hamming weight
Consider all strings/points in \( \{0, 1\}^n \)

- Consider subset of \( \{0, 1\}^n \) of even Hamming weight
- What is the largest subset of \( \{0, 1\}^n \) whose all pairwise Hamming distances is at least 3?
Consider all strings/points in \( \{0, 1\}^n \)
- Consider subset of \( \{0, 1\}^n \) of even Hamming weight
- What is the largest subset of \( \{0, 1\}^n \) whose all pairwise Hamming distances is at least 3?
- What is the largest subset of \( \{0, 1\}^n \) whose all pairwise Hamming distances is at least 0.9n?
Consider all strings/points in $\{0, 1\}^n$

Consider subset of $\{0, 1\}^n$ of even Hamming weight

What is the largest subset of $\{0, 1\}^n$ whose all pairwise Hamming distances is at least 3?

What is the largest subset of $\{0, 1\}^n$ whose all pairwise Hamming distances is at least $0.9n$?

What is the largest subset of $\{0, 1\}^n$ whose all pairwise Hamming distances is at least $0.5n$?
Consider all strings/points in \( \{0, 1\}^n \)

- Consider subset of \( \{0, 1\}^n \) of even Hamming weight
- What is the largest subset of \( \{0, 1\}^n \) whose all pairwise Hamming distances is at least 3?
- What is the largest subset of \( \{0, 1\}^n \) whose all pairwise Hamming distances is at least 0.9n?
- What is the largest subset of \( \{0, 1\}^n \) whose all pairwise Hamming distances is at least 0.5n?
- What is the largest subset of \( \{0, 1\}^n \) whose all pairwise Hamming distances is at least 0.49n?
Coding Theory: Definitions

- $C \subseteq \{0, 1\}^L$
Coding Theory: Definitions

- $C \subseteq \{0, 1\}^L$

- **Distance of** $C$:

  $$\Delta(C) := \min_{x, y \in C} \|x - y\|_0$$
Coding Theory: Definitions

- $C \subseteq \{0, 1\}^L$

- **Distance of** $C$:
  \[ \Delta(C) := \min_{x, y \in C} \|x - y\|_0 \]

A good code: for $\rho, \delta > 0$, $|C| = 2^{\rho L}$, $\Delta(C) = \delta L$. 
Random Strings are Good Codes

For some small $\rho > 0$, if we pick $2^{\rho L}$ random strings uniformly and independently then they form a code with distance at least $1/4$ (whp).
Random Strings are Good Codes

For some small $\rho > 0$, if we pick $2^{\rho L}$ random strings uniformly and independently then they form a code with distance at least $1/4$ (whp).

- $\mathbb{E}[\|x - y\|_0] = L/2$
- **Chernoff:** $\Pr[\|x - y\|_0 \leq L/4] = e^{-L/100}$
Random Codes

Random Strings are Good Codes

For some small $\rho > 0$, if we pick $2^{\rho L}$ random strings uniformly and independently then they form a code with distance at least $1/4$ (whp).

- $\mathbb{E}[\| x - y \|_0] = L/2$
- **Chernoff:** $\Pr[\| x - y \|_0 \leq L/4] = e^{-L/100}$
- **Union Bound:**

$$\Pr[\min_{x,y \in C} \{\| x - y \|_0 \} \leq L/4] = 2^{2\rho L} e^{-L/100} < 0.001$$
Random Codes

Random Strings are Good Codes

For some small $\rho > 0$, if we pick $2^{\rho L}$ random strings uniformly and independently, then they form a code with distance at least $1/4$ (whp).

- $\mathbb{E}[\|x - y\|_0] = L/2$
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- **Union Bound:**

  \[
  \Pr[\min_{x, y \in C} \{\|x - y\|_0\} \leq L/4] = 2^{2\rho L} e^{-L/100} < 0.001
  \]

Many Efficient Deterministic Good Codes Exist!
\( C \subseteq [q]^L \)
- $C \subseteq [q]^L$

- **Distance** of $C$:

  \[ \Delta(C) := \min_{x,y \in C} \|x - y\|_0 \]
\( C \subseteq [q]^L \)

**Distance of \( C \):**

\[ \Delta(C) := \min_{x,y \in C} \| x - y \|_0 \]

**Singleton Bound:** \( |C| \leq q^{L - \Delta(C) + 1} \)
Coding Theory: Reed Solomon Codes

- $C \subseteq [q]^L$

- **Distance** of $C$:

  $$\Delta(C) := \min_{x, y \in C} \|x - y\|_0$$

- **Singleton Bound**: $|C| \leq q^{L-\Delta(C)+1}$

- Reed Solomon Codes: All degree $d$ univariate polynomials over $\mathbb{F}_q$
Coding Theory: Reed Solomon Codes

- \( C \subseteq [q]^L \)

- **Distance** of \( C \):
  \[
  \Delta(C) \defeq \min_{x,y \in C} \| x - y \|_0
  \]

- **Singleton Bound**: \( |C| \leq q^{L-\Delta(C)+1} \)

- Reed Solomon Codes: All degree \( d \) univariate polynomials over \( \mathbb{F}_q \)

- \( |RS| = q^{d+1} \)
Coding Theory: Reed Solomon Codes

- $C \subseteq [q]^L$

- **Distance** of $C$:
  \[
  \Delta(C) := \min_{x, y \in C} \|x - y\|_0
  \]

- **Singleton Bound**: $|C| \leq q^{L-\Delta(C)+1}$

- Reed Solomon Codes: All degree $d$ univariate polynomials over $\mathbb{F}_q$
  - $|RS| = q^{d+1}$
  - $\Delta(RS) = q - d$ (because any degree $d$ univariate polynomial can have at most $d$ roots)
Coding Theory: Reed Solomon Codes

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- Reed Solomon Codes meet the Singleton bound!
Outline

Part 1: Handwaving Introduction
Part 2: Dominating Set
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  - Hardness of Approximation in NP
  - Hardness of Approximation in Parameterized Complexity
Part 4: Coding Theory
  - Definition and Geometric Intuition
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  - MaxCover with Projection Property
  - Gap Creation
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  - MinLabel
  - Gap Translation to Dominating Set
MaxCover: Projection Property

\[ \Gamma(\mathcal{U}, \mathcal{W}, E) \]

\( \mathcal{U} \) and \( \mathcal{W} \) are sets.

- \( \mathcal{U} = \{ U_1, U_2, \ldots, U_r \} \)
- \( \mathcal{W} = \{ W_1, W_2, \ldots, W_k \} \)

\( \Gamma \) has projection property:

For every \( U_i \) and \( W_j \), the induced subgraph of \( (U_i, W_j) \) is:

- complete bipartite graph (i.e., irrelevant), or,
- \( \forall w \in W_j \), \( \deg(w) = 1 \) (i.e., projection)
MaxCover: Projection Property

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MaxCover with Projection Property is $W[1]$-Hard

\[ \Gamma(U, W, E) \]

Input: $G([n], E_0)$
MaxCover with Projection Property is $\text{W}[1]$-Hard

Input: $G([n], E_0)$

$U_i = [n]$ and $W_{j,j'} = E_0$
MaxCover with Projection Property is $W[1]$-Hard

Input: $G([n], E_0)$

$U_i = [n]$ and $W_{j,j'} = E_0$

$W_{j,j'}$ has projection to $U_j$ and $U_{j'}$
MaxCover: Gap Creation

Inapproximability of MaxCover

There is a FPT reduction from MaxCover instance $\Gamma_0 = \left( U_0 = \bigcup_{j=1}^{r} U_j^0, W = \bigcup_{j=1}^{k} W_i, E_0 \right)$ with projection property to a MaxCover instance $\Gamma = \left( U = \bigcup_{j=1}^{O(\log |U_0|)} U_j, W = \bigcup_{j=1}^{k} W_i, E \right)$ such that

- If $\text{MaxCover}(\Gamma_0) = 1$ then $\text{MaxCover}(\Gamma) = 1$
- If $\text{MaxCover}(\Gamma_0) < 1$ then $\text{MaxCover}(\Gamma) \leq 0.75 |\Gamma| = \tilde{O}(2^r \cdot |W| \cdot \log |U_0|)$

The reduction runs in time $2^{O(r)} \cdot \text{poly}(|\Gamma_0|)$. 

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Inapproximability of MaxCover

There is a FPT reduction from MaxCover instance \( \Gamma_0 = (U_0 = \bigcup_{j=1}^{r} U_j^0, W = \bigcup_{j=1}^{k} W_i, E_0) \) with projection property to a MaxCover instance \( \Gamma = (U = \bigcup_{j=1}^{O(\log |U_0|)} U_j, W = \bigcup_{j=1}^{k} W_i, E) \) such that

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- $|\Gamma| = \tilde{O}(2^r \cdot |W| \cdot \log |U_0|)$
- The reduction runs in time $2^{O(r)} \cdot \text{poly}(|\Gamma_0|)$. 
Threshold Graph Construction

Let $A_t = \{0, 1\}^r$ and $U_i = C$.

**Diagram:**

- Set $A_t = \{0, 1\}^r$ is connected to $U_i = C$ via edges.
- The connection structure is illustrated with nodes $U_i$ and $A_t$.
- Additional elements $U_1^0, U_2^0, \ldots, U_r^0$ are connected to $A_t$.

**Mathematical Expression:**

$U_i^0 = C$
Threshold Graph Construction

$A_t = \{0, 1\}^r$

$U_i^0 = C$

$(u, (q_1, \ldots, q_r)) \in U_i^0 \times A_t$ is an edge $\iff u_t = q_i$
Threshold Graph Properties

Completeness

For every \((u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0\) and every \(A_t\)
there exists a unique common neighbor of \((u^1, \ldots, u^r)\) in \(A_t\)
Threshold Graph Properties

Completeness

For every \((u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0\) and every \(A_t\), there exists a unique common neighbor of \((u^1, \ldots, u^r)\) in \(A_t\).

Soundness

For every \(u, u' \in U_i^0\), there are at most \(L - \Delta(C)\) many supernodes in \(A\) which have a common neighbor of \(u\) and \(u'\).
Threshold Graph Composition

\[ A_t = \{0, 1\}^r \]

\[ U_i^0 = C \]
Threshold Graph Composition

\[ A_t = \{0, 1\}^r \]

\[ U_i^0 = C \]

\((w, (q_1, \ldots, q_r)) \in W_j \times A_t\) is an edge \(\iff\) \(\exists (u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0\) such that \(\forall i \in [k], (w, u^i)\) and \((u^i, (q_1, \ldots, q_r))\) are both edges.
Completeness of Reduction

- Let \((w_1, \ldots, w_k) \in W_1 \times \cdots \times W_k\) be optimal labeling of \(\Gamma_0\).

- Let \((u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0\) be common neighbors of \((w_1, \ldots, w_k)\) in \(\Gamma_0\).
Completeness of Reduction

- Let \((w_1, \ldots, w_k) \in W_1 \times \cdots \times W_k\) be optimal labeling of \(\Gamma_0\)
- Let \((u^1, \ldots, u^r) \in U^0_1 \times \cdots \times U^0_r\) be common neighbors of \((w_1, \ldots, w_k)\) in \(\Gamma_0\)

Completeness of Threshold Graph

For every \((u^1, \ldots, u^r) \in U^0_1 \times \cdots \times U^0_r\) and every \(A_t\) there exists a unique common neighbor of \((u^1, \ldots, u^r)\) in \(A_t\)
Soundness of Reduction

- Fix $(w_1, \ldots, w_k) \in W_1 \times \cdots \times W_k$
- There exists $U_i^0$ not covered by $(w_1, \ldots, w_k)$
Soundness of Reduction

- Fix \((w_1, \ldots, w_k) \in W_1 \times \cdots \times W_k\)
- There exists \(U_i^0\) not covered by \((w_1, \ldots, w_k)\)
- There exists \(w_j\) and \(w_j'\) with neighbors \(u\) and \(u'\) resp. in \(U_i^0\) \((u \neq u')\)
Soundness of Reduction

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- There exists \(w_j\) and \(w_j'\) with neighbors \(u\) and \(u'\) resp. in \(U_i^0\) \((u \neq u')\)
- If \(a \in A\) is common neighbor of \(w_j\) and \(w_j'\) in \(\Gamma\) then \(u\) and \(u'\) are common neighbors of \(a\) in Threshold graph
Soundness of Reduction

- Fix \((w_1, \ldots, w_k) \in W_1 \times \cdots \times W_k\)
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Soundness of Threshold Graph

For every \(u, u' \in U_i^0\), there are at most \(L - \Delta(C)\) many supernodes in \(A\) which have a common neighbor of \(u\) and \(u'\)
There is a FPT reduction from MaxCover instance $\Gamma_0 = \left( U_0 = \bigcup_{j=1}^{r} U_j^0, W = \bigcup_{j=1}^{k} W_i, E_0 \right)$ with projection property to a MaxCover instance $\Gamma = \left( U = \bigcup_{j=1}^{O(\log |U_0|)} U_j, W = \bigcup_{j=1}^{k} W_i, E \right)$ such that

- If $\text{MaxCover}(\Gamma_0) = 1$ then $\text{MaxCover}(\Gamma) = 1$
- If $\text{MaxCover}(\Gamma_0) < 1$ then $\text{MaxCover}(\Gamma) \leq 0.75$
- $|\Gamma| = \tilde{O}(2^r \cdot |W| \cdot \log |U_0|)$
- The reduction runs in time $2^{O(r)} \cdot \text{poly}(|\Gamma_0|)$. 
Threshold Graph Composition with Reed Solomon Codes

\[ A_t = [q]^r \]

\[(w, (q_1, \ldots, q_r)) \in W_j \times A_t \text{ is an edge } \iff \exists (u^1, \ldots, u^r) \in U_1^0 \times \cdots U_r^0 \text{ such that } \forall i \in [k], (w, u^i) \text{ and } (u^i, (q_1, \ldots, q_r)) \text{ are both edges} \]
Threshold Graph Properties

Completeness

For every \((u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0\) and every \(A_t\) there exists a unique common neighbor of \((u^1, \ldots, u^r)\) in \(A_t\).

Soundness

For every \(u, u' \in U_i^0\), there are at most \(\log_q |U_0|\) many supernodes in \(A\) which have a common neighbor of \(u\) and \(u'\).
Inapproximability of MaxCover using Reed Solomon Codes

There is a FPT reduction from MaxCover instance $\Gamma_0 = \left( U_0 = \bigcup_{j=1}^{r} U_j^0, W = \bigcup_{j=1}^{k} W_i, E_0 \right)$ with projection property to a MaxCover instance $\Gamma = \left( U = \bigcup_{j=1}^{q} U_j, W = \bigcup_{j=1}^{k} W_i, E \right)$ such that

- If $\text{MaxCover}(\Gamma_0) = 1$ then $\text{MaxCover}(\Gamma) = 1$
- If $\text{MaxCover}(\Gamma_0) < 1$ then $\text{MaxCover}(\Gamma) \leq \frac{\log_q |U_0|}{q}$
- $|\Gamma| = \tilde{O}(q^r \cdot |W| \cdot \log |U_0|)$
- The reduction runs in time $q^r \cdot \text{poly}(|\Gamma_0|)$. 

This reduction leads to an inapproximability result for MaxCover, as the size of the solution in the reduced instance $\Gamma$ is tightly connected to the solution in the original instance $\Gamma_0$. The reduction time is $q^r \cdot \text{poly}(|\Gamma_0|)$, indicating a polynomial blow-up in the size of the input, which is typical for FPT reductions.
Part 1: Handwaving Introduction ✓

Part 2: Dominating Set ✓

Part 3: Hardness of Approximation ✓
  - Hardness of Approximation in NP ✓
  - Hardness of Approximation in Parameterized Complexity ✓

Part 4: Coding Theory ✓
  - Definition and Geometric Intuition ✓
  - Random Codes ✓
  - Algebraic Codes ✓

Part 5: Hardness of Approximating MaxCover ✓
  - MaxCover with Projection Property ✓
  - Gap Creation ✓

Part 6: Hardness of Approximating Dominating Set
  - MinLabel
  - Gap Translation to Dominating Set
Determine if $\text{MinLabel}(\Gamma) = k$ or $\text{MinLabel}(\Gamma) \geq s \cdot k$

Each $W_i$ is a **Right Super Node**
Each $U_i$ is a **Left Super Node**

$S \subseteq W$ is a labeling of $W$ if
$$\forall i \in [k], |S \cap W_i| = 1$$

$S$ covers $U_i$ if
$$\exists u \in U_i, \forall v \in S, (u, v) \in E$$
MinLabel

Each $W_i$ is a **Right Super Node**
Each $U_i$ is a **Left Super Node**

$S \subseteq W$ is a **labeling** of $W$ if
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$S$ covers $U_i$ if
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$$\text{MinLabel}(\Gamma) = \text{smallest } X \subseteq W:$$
$\forall i \in [r], \exists \text{labeling } S \subseteq X,$
$S$ covers $U_i$
Each $W_i$ is a Right Super Node
Each $U_i$ is a Left Super Node

$S \subseteq W$ is a labeling of $W$ if
\[ \forall i \in [k], |S \cap W_i| = 1 \]

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$\text{MinLabel}(\Gamma) = \text{smallest } X \subseteq W$:
\[ \forall i \in [r], \exists \text{labeling } S \subseteq X, S \text{ covers } U_i \]

Determine if $\text{MinLabel}(\Gamma) = k$
or $\text{MinLabel}(\Gamma) \geq s \cdot k$
Maxcover to MinLabel

Reduction from MaxCover to MinLabel

Given a MaxCover instance $\Gamma = \left( U = \bigcup_{j=1}^{r} U_j, W = \bigcup_{j=1}^{k} W_i, E \right)$,
Maxcover to MinLabel

Reduction from MaxCover to MinLabel

Given a MaxCover instance $\Gamma = \left( U = \bigcup_{j=1}^{r} U_j, W = \bigcup_{j=1}^{k} W_i, E \right)$,

- **Completeness**: If MaxCover($\Gamma$) = 1, then MinLabel($\Gamma$) = $k$

- In Soundness case, if $X \subseteq W$ is a MinLabel solution then every labeling $S \subseteq X$ covers at most $\epsilon$ fraction of the left supernodes.

There are at most $\left( \frac{|X|}{k} \right)$ distinct labeling of $W$ in $X$.

$\left( \frac{|X|}{k} \right) \cdot \epsilon \geq 1$
Maxcover to MinLabel

Reduction from MaxCover to MinLabel

Given a MaxCover instance $\Gamma = \left( U = \bigcup_{j=1}^{r} U_j, W = \bigcup_{j=1}^{k} W_i, E \right)$,

- **Completeness**: If $\text{MaxCover}(\Gamma) = 1$, then $\text{MinLabel}(\Gamma) = k$
- **Soundness**: If $\text{MaxCover}(\Gamma) \leq \varepsilon$, then $\text{MinLabel}(\Gamma) \geq \left(\frac{1}{\varepsilon}\right)^{1/k} \cdot k$

Completeness is obvious. In the Soundness case, if $X \subseteq W$ is a MinLabel solution, then every labeling $S \subseteq X$ covers at most $\varepsilon$ fraction of the left supernodes. There are at most $\left|\frac{|X|}{k}\right|$ distinct labelings of $W$ in $X$. Hence, $\left|\frac{|X|}{k}\right| \cdot \varepsilon \geq 1$. Therefore, $\text{MinLabel}(\Gamma) \geq \left(\frac{1}{\varepsilon}\right)^{1/k} \cdot k$. 
Reduction from MaxCover to MinLabel

Given a MaxCover instance $\Gamma = \left( U = \bigcup_{j=1}^{r} U_j, W = \bigcup_{j=1}^{k} W_i, E \right)$,

- Completeness: If $\text{MaxCover}(\Gamma) = 1$, then $\text{MinLabel}(\Gamma) = k$
- Soundness: If $\text{MaxCover}(\Gamma) \leq \varepsilon$, then $\text{MinLabel}(\Gamma) \geq (1/\varepsilon)^{1/k} \cdot k$

Completeness is obvious.
Maxcover to MinLabel

Reduction from MaxCover to MinLabel

Given a MaxCover instance \( \Gamma = \left( U = \bigcup_{j=1}^{r} U_j, W = \bigcup_{j=1}^{k} W_i, E \right) \),

- **Completeness:** If MaxCover(\( \Gamma \)) = 1, then MinLabel(\( \Gamma \)) = \( k \)
- **Soundness:** If MaxCover(\( \Gamma \)) \( \leq \) \( \varepsilon \), then MinLabel(\( \Gamma \)) \( \geq \) \( \left( \frac{1}{\varepsilon} \right)^{1/k} \cdot k \)

Completeness is obvious. In Soundness case, if \( X \subseteq W \) is a MinLabel solution
Maxcover to MinLabel

Reduction from MaxCover to MinLabel

Given a MaxCover instance $\Gamma = \left( U = \bigcup_{j=1}^{r} U_j, W = \bigcup_{j=1}^{k} W_i, E \right)$,

- Completeness: If $\text{MaxCover}(\Gamma) = 1$, then $\text{MinLabel}(\Gamma) = k$
- Soundness: If $\text{MaxCover}(\Gamma) \leq \varepsilon$, then $\text{MinLabel}(\Gamma) \geq \left( \frac{1}{\varepsilon} \right)^{1/k} \cdot k$

Completeness is obvious. In Soundness case, if $X \subseteq W$ is a MinLabel solution then every labeling $S \subseteq X$ covers at most $\varepsilon$ fraction of the left supernodes.
Maxcover to MinLabel

Reduction from MaxCover to MinLabel

Given a MaxCover instance $\Gamma = \left( U = \bigcup_{j=1}^{r} U_j, W = \bigcup_{j=1}^{k} W_i, E \right)$,

- Completeness: If $\text{MaxCover}(\Gamma) = 1$, then $\text{MinLabel}(\Gamma) = k$
- Soundness: If $\text{MaxCover}(\Gamma) \leq \varepsilon$, then $\text{MinLabel}(\Gamma) \geq (1/\varepsilon)^{1/k} \cdot k$

Completeness is obvious. In Soundness case, if $X \subseteq W$ is a MinLabel solution then every labeling $S \subseteq X$ covers at most $\varepsilon$ fraction of the left supernodes. There are at most $\binom{|X|/k}{k}$ distinct labeling of $W$ in $X$. 
Reduction from MaxCover to MinLabel

Given a MaxCover instance \( \Gamma = \left( U = \bigcup_{j=1}^{r} U_j, W = \bigcup_{j=1}^{k} W_i, E \right) \),

- **Completeness**: If \( \text{MaxCover}(\Gamma) = 1 \), then \( \text{MinLabel}(\Gamma) = k \)
- **Soundness**: If \( \text{MaxCover}(\Gamma) \leq \varepsilon \), then \( \text{MinLabel}(\Gamma) \geq \left( \frac{1}{\varepsilon} \right)^{1/k} \cdot k \)

Completeness is obvious. In Soundness case, if \( X \subseteq W \) is a MinLabel solution then every labeling \( S \subseteq X \) covers at most \( \varepsilon \) fraction of the left supernodes. There are at most \( \binom{|X|/k}{k} \) distinct labeling of \( W \) in \( X \).

\[
\binom{|X|/k}{k} \cdot \varepsilon \geq 1
\]
MinLabel to Dominating Set

Let $U = \{U_1, U_2, \ldots, U_r\}$ and $W = \{W_1, W_2, \ldots, W_k\}$ be sets.

Let $|U_i| = \ell$ for all $i$.

Let $\Gamma(U, W, E)$ be the graph defined by the edges.

The edges between $f \in F_i$ and $w \in W_j$ are defined as:

- There exists $u \in U_i$ such that $(u, w) \in \Gamma$ and $f(u) = j$.
MinLabel to Dominating Set

\[ F_i = \{ f : [\ell] \rightarrow [k] \} \]

\[ |U_i| = \ell \quad \Gamma(U, W, E) \]
MinLabel to Dominating Set

\[ F_i = \{ f : [\ell] \to [k] \} \]

\[ |U_i| = \ell \quad \Gamma(U, W, E) \]

Edge between \( f \in F_i \) and \( w \in W_j \) ⇔

\[ \exists u \in U_i \text{ such that } (u, w) \in \Gamma \text{ and } f(u) = j \]
MinLabel to Dominating Set

\[ |F_i| = k^{\ell} \quad H(F \cup W, E') \]

Determine if $\text{DomSet}(H) = k$ or $\text{DomSet}(H) \geq s \cdot k$ is hard!

$F = \{ (i, f) \mid i \subseteq [r], f : [\ell] \rightarrow [k] \}$

$(i, f), w \in H \iff \exists u \in U_i (u, w) \in \Gamma \text{ and } f(u) = j(w_1, \ldots, w_k) \text{ is labeling that covers every } U_i \Rightarrow (w_1, \ldots, w_k) \text{ dominates } H$}

$\forall (i, f) \in F, \exists u \in U_i, (u, w_j) \in \Gamma (\forall j \in [k])$
MinLabel to Dominating Set

\[ F = \{ (i, f) \mid i \in [r], f : [\ell] \rightarrow [k] \} \]

\[ |F_i| = k^\ell \quad H(F \cup W, E') \]
MinLabel to Dominating Set

\( F \)
\( W \)

|\( F_i \) | = \( k^\ell \)

\( H(F \cup W, E') \)

\( F = \{(i, f) \mid i \in [r], f : [\ell] \rightarrow [k]\} \)

\( ((i, f), w) \in H \iff \exists u \in U_i \) \( (u, w) \in \Gamma \) and \( f(u) = j \)
MinLabel to Dominating Set

\[ F = \{ (i, f) \mid i \in [r], f : [\ell] \to [k] \} \]

\[ ((i, f), w) \in H \iff \exists u \in U_i \]
\[ (u, w) \in \Gamma \text{ and } f(u) = j \]

\((w_1, \ldots, w_k)\) is labeling that covers every \(U_i\) \(\Rightarrow\)
\((w_1, \ldots, w_k)\) dominates \(H\)
MinLabel to Dominating Set

\[ F = \{ (i, f) \mid i \in [r], f : [\ell] \to [k] \} \]

\((i, f), w) \in H \iff \exists u \in U_i \]
\((u, w) \in \Gamma \text{ and } f(u) = j\)

\((w_1, \ldots, w_k)\) is labeling that covers every \(U_i \implies (w_1, \ldots, w_k)\) dominates \(H\)

\forall (i, f) \in F, \exists u \in U_i, \]
\((u, w_j) \in \Gamma (\forall j \in [k])\)
MinLabel to Dominating Set

$F = \{(i, f) \mid i \in [r], f : [\ell] \rightarrow [k]\}$

$((i, f), w) \in H \iff \exists u \in U_i$

$(u, w) \in \Gamma$ and $f(u) = j$

$(w_1, \ldots, w_k)$ is labeling that covers every $U_i \Rightarrow$

$(w_1, \ldots, w_k)$ dominates $H$

$\forall (i, f) \in F, \exists u \in U_i,$

$(u, w_j) \in \Gamma$ ($\forall j \in [k]$)

$|F_i| = k^\ell$

$H(F \cup W, E')$

Determine if DomSet$(H) = k$

or DomSet$(H) \geq s \cdot k$ is hard!
\[ F = \{(i, f) \mid i \in [r], f : [\ell] \rightarrow [k]\} \]

\[ ((i, f), w) \in \Gamma \iff \exists u \in U_i : (u, w) \in \Gamma \text{ and } f(u) = j \]
MinLabel to Dominating Set: Soundness Analysis

- \( F = \{(i, f) \mid i \in [r], f : [\ell] \rightarrow [k]\} \)
- \(((i, f), w) \in H \iff \exists u \in U_i : (u, w) \in \Gamma \text{ and } f(u) = j\)
- Suppose \( X \) is a Dominating Set of size \( sk - 1 \)
\[ F = \{(i, f) \mid i \in [r], f : [\ell] \to [k]\} \]

\[ ((i, f), w) \in H \Leftrightarrow \exists u \in U_i : (u, w) \in \Gamma \text{ and } f(u) = j \]

- Suppose \( X \) is a Dominating Set of size \( sk - 1 \)
- \( \exists U_i \) not covered by any labeling in \( X \)
MinLabel to Dominating Set: Soundness Analysis

- \( F = \{(i, f) \mid i \in [r], f : [\ell] \rightarrow [k]\} \)

- \(( (i, f), w) \in H \iff \exists u \in U_i : (u, w) \in \Gamma \text{ and } f(u) = j \)

- Suppose \( X \) is a Dominating Set of size \( sk - 1 \)

- \( \exists U_i \) not covered by any labeling in \( X \)

- For every \( u \in U_i \) there is some \( j \in [k] \) such that \( W_j \cap X \cap N(u) \) is empty
MinLabel to Dominating Set: Soundness Analysis

- \( F = \{(i, f) \mid i \in [r], f : [\ell] \rightarrow [k]\}\)
- \( ((i, f), w) \in H \Leftrightarrow \exists u \in U_i : (u, w) \in \Gamma \) and \( f(u) = j \)
- Suppose \( X \) is a Dominating Set of size \( sk - 1 \)
- \( \exists U_i \) not covered by any labeling in \( X \)
- For every \( u \in U_i \) there is some \( j \in [k] \) such that \( W_j \cap X \cap N(u) \) is empty
- Construct \( f \) using above \( u \)
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For every \( u \in U_i \) there is some \( j \in [k] \) such that \( W_j \cap X \cap N(u) \) is empty

Construct \( f \) using above \( u \)

\( (i, f) \) is not dominated by \( X \)
Parameterized Inapproximability of Dominating Set

Inapproximability of Dominating Set

There is a FPT reduction from $k$-clique instance $G([n], E)$ to a Dominating Set instance $H$ such that

- If $G$ has a $k$-clique then $\binom{k}{2}$ vertices in $H$ form a dominating set.
- If $G$ has no $k$-clique then $(\log n)^{1/k^2}$ vertices in $H$ are needed to form a dominating set.
- $|H| \leq \text{poly}(n)$
- The reduction runs in time $2^{\text{poly}(k)} \cdot \text{poly}(n)$.
Outline

Part 1: Handwaving Introduction ✓

Part 2: Dominating Set ✓

Part 3: Hardness of Approximation ✓
  - Hardness of Approximation in NP ✓
  - Hardness of Approximation in Parameterized Complexity ✓

Part 4: Coding Theory ✓
  - Definition and Geometric Intuition ✓
  - Random Codes ✓
  - Algebraic Codes ✓

Part 5: Hardness of Approximating MaxCover ✓
  - MaxCover with Projection Property ✓
  - Gap Creation ✓

Part 6: Hardness of Approximating Dominating Set ✓
  - MinLabel ✓
  - Gap Translation to Dominating Set ✓