Hardness of Approximation meets Parameterized Complexity

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Rutgers University

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Outline

Part 1: Handwaving Introduction

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Part 2: Dominating Set

Part 3: Hardness of Approximation

- Hardness of Approximation in NP
- Hardness of Approximation in Parameterized Complexity

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Part 4: Coding Theory

- Definition and Geometric Intuition
- Random Codes
- Algebraic Codes

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Optimization Problems

- Dominating Set/Set Cover
- Set Intersection
- Clique
- Vertex Cover
- Clustering
- Satisfiability

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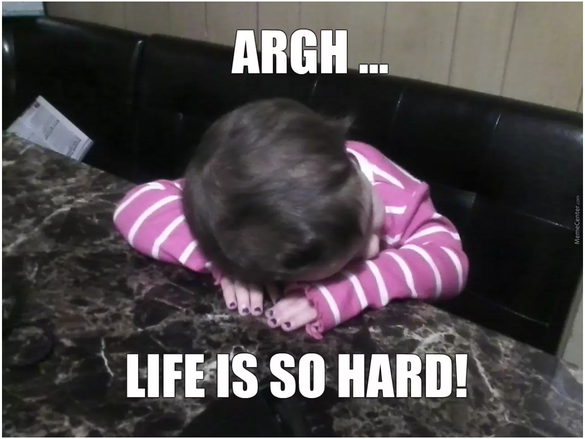
- Satisfiability

Many Optimization Problems are NP-hard!

Coping Mechanism



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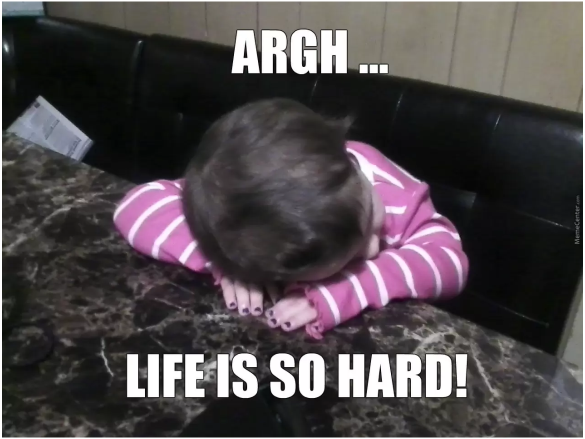
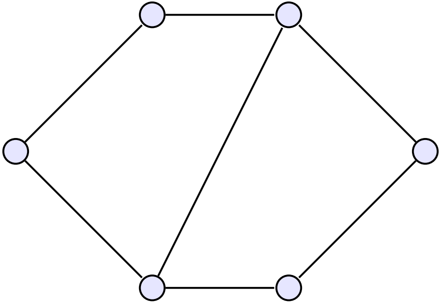


- Approximation Algorithms
- Fixed Parameter Tractability

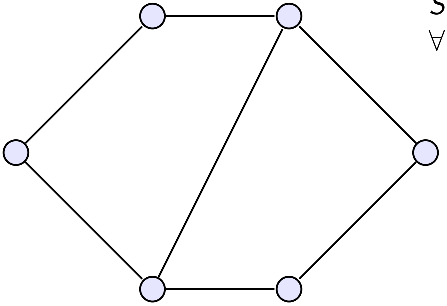
Outline

Dominating Set Problem



$G(V, E)$

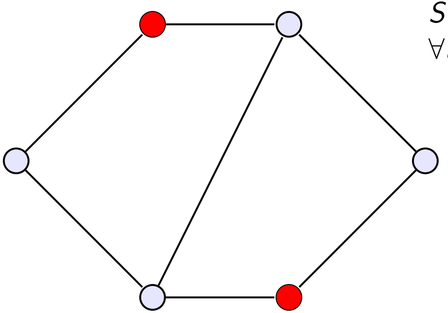
Dominating Set Problem



$S \subseteq V$ is a Dominating Set of G if $\forall u \in V$:

- $u \in S$, or
- $\exists\ v \in S$ such that $(u, v) \in E$

Dominating Set Problem



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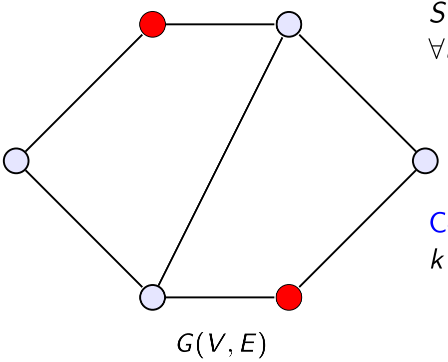

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Computational Problem: Given G and $k \in \mathbb{N}$, determine if $\exists\ S \subseteq V$:

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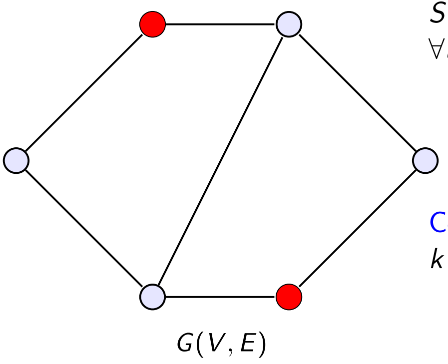
Dominating Set Problem



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$\longrightarrow$ NP-Complete [Karp'72]

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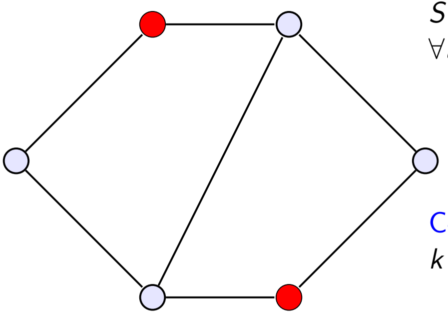

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$\longrightarrow$ $\ln |V|$ approximation is in P [Slavík'96]

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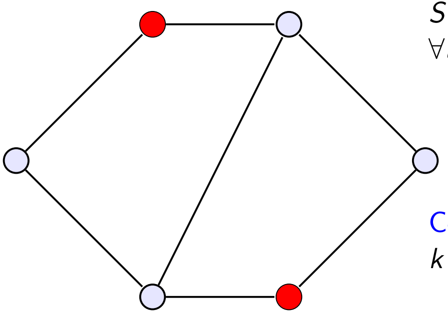

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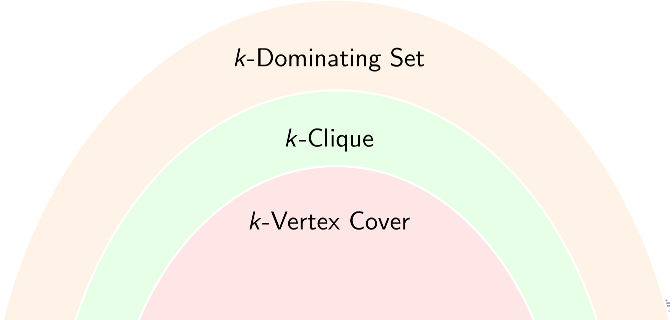

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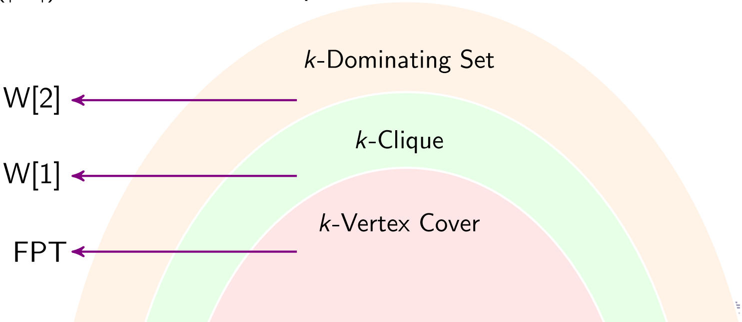

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Is there some computable function T for which the above problem is in FPT?

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Part 5: Hardness of Approximating MaxCover

- MaxCover with Projection Property
- Gap Creation

Part 6: Hardness of Approximating Dominating Set

- MinLabel
- Gap Translation to Dominating Set

Hardness of Approximation

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PCP Theorem & Label Cover

PCP Theorem: Bedrock of NP-Hardness of Approximation

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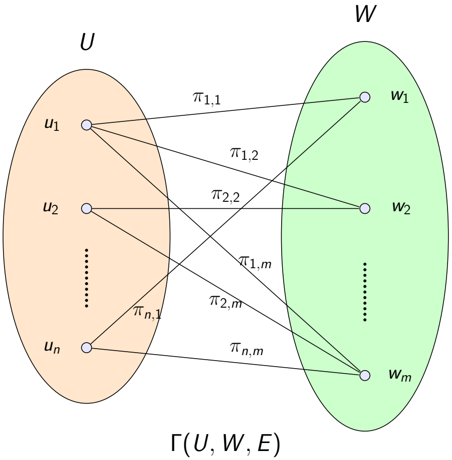

PCP Theorem: Bedrock of NP-Hardness of Approximation

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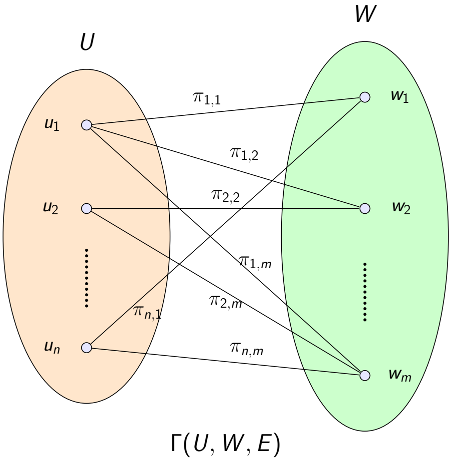

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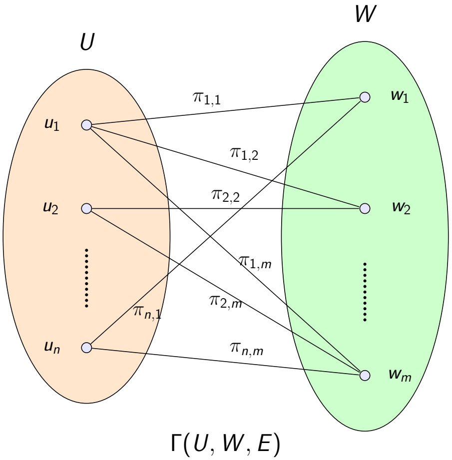

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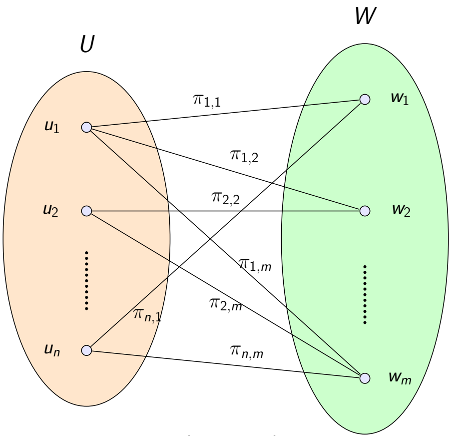

$\Gamma(U, W, E)$

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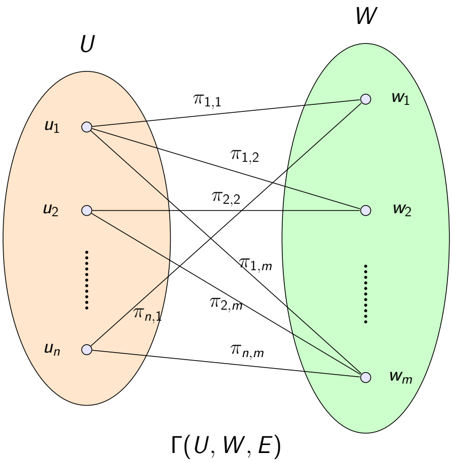

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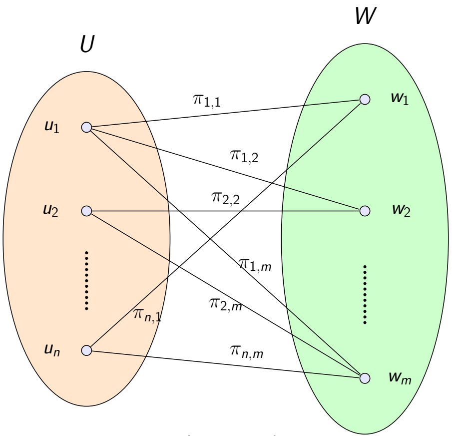

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Determining if $\text{VAL}(\Gamma) = 1$ or if $\text{VAL}(\Gamma) \leq 0.99$ is NP-Hard

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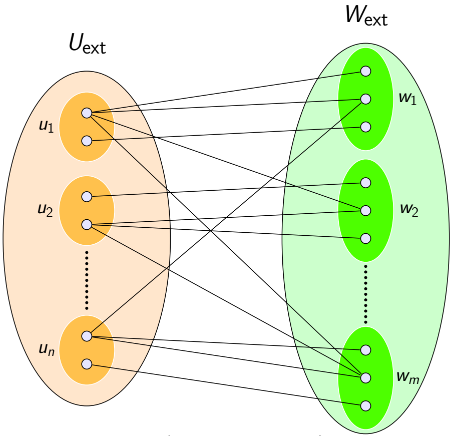
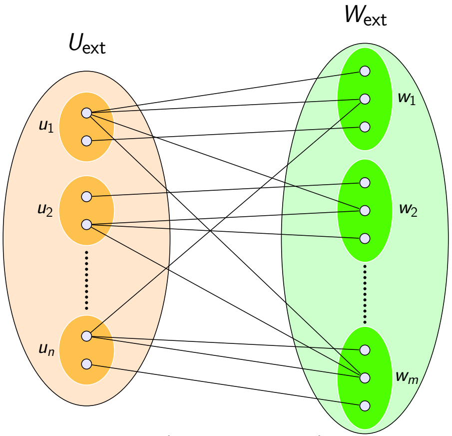


$\Gamma_{\text{ext}}(U_{\text{ext}}, W_{\text{ext}}, E_{\text{ext}})$

Extended Label Cover



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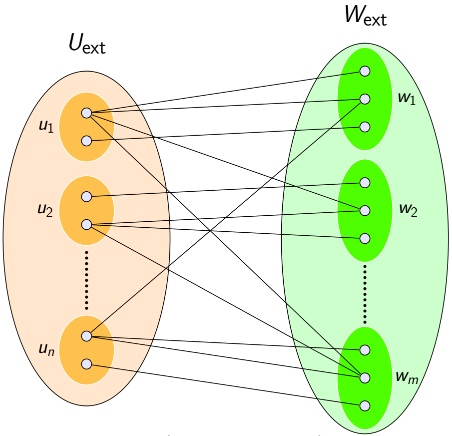

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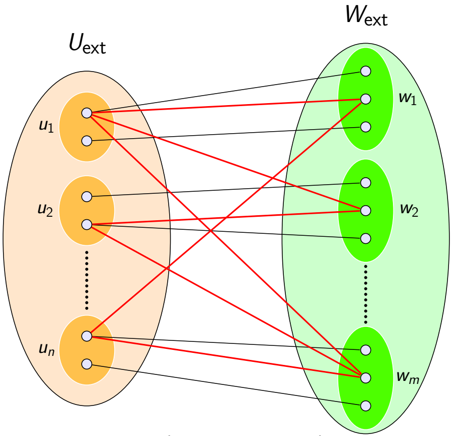

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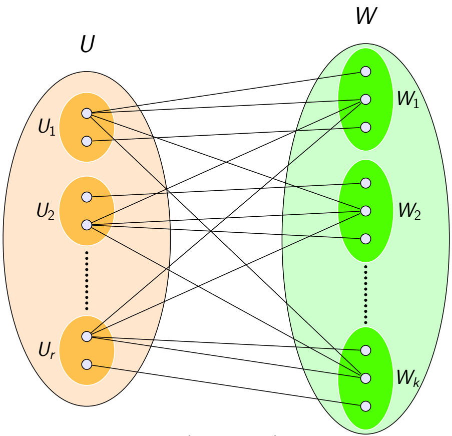
MaxCover



$\Gamma(U, W, E)$

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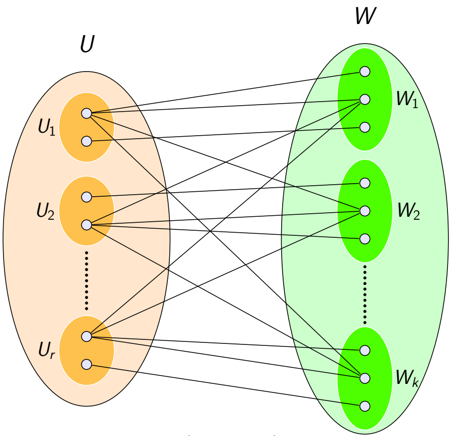
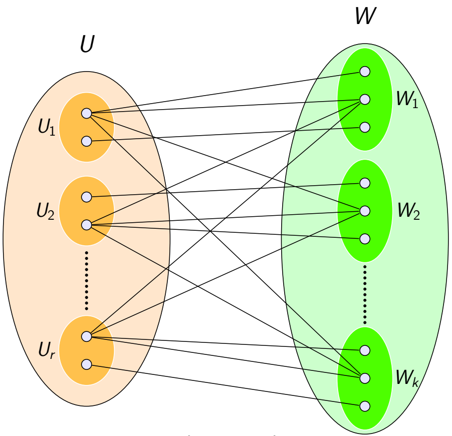


$\Gamma(U, W, E)$

Each W_i is a Right Super Node
Each U_i is a Left Super Node

MaxCover



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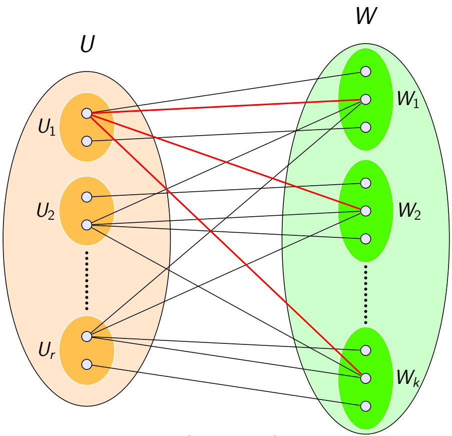

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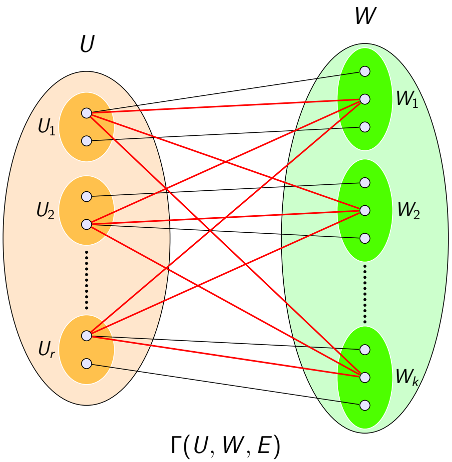
S covers U_i if
$\exists u \in U_i,\ \forall v \in S, (u, v) \in E$

MaxCover



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MaxCover(Γ, S) = Fraction of
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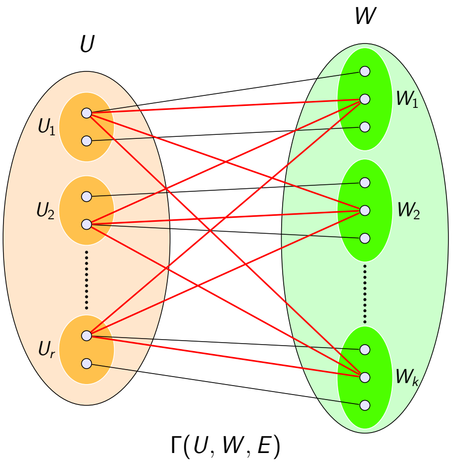
MaxCover



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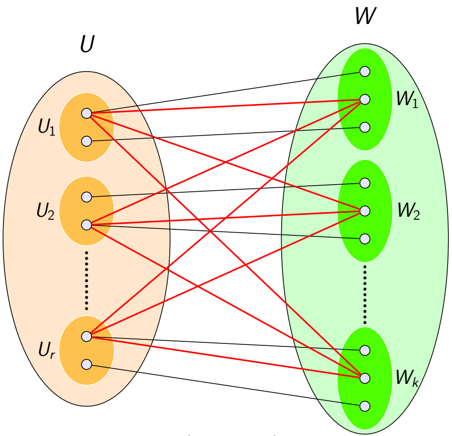
MaxCover



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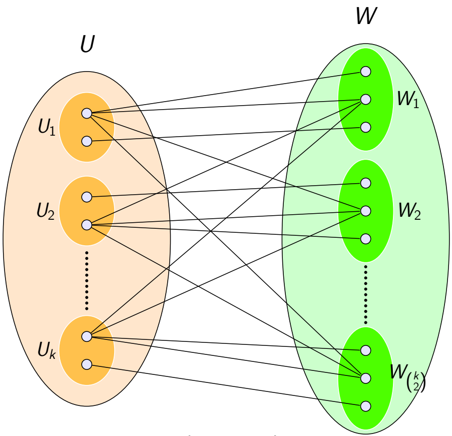
$\text{MaxCover}(\Gamma, S) =$ Fraction of U_i's covered by S

$$\text{MaxCover}(\Gamma) = \max_S \text{MaxCover}(\Gamma, S)$$

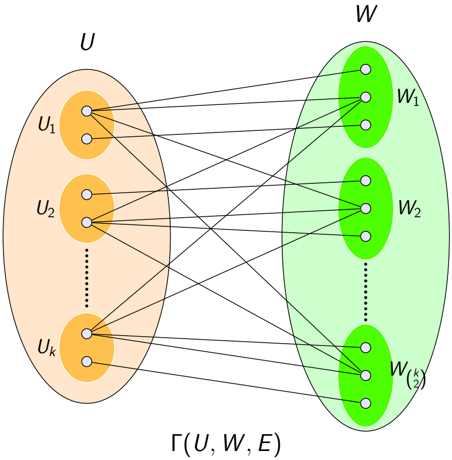
Determine if $\text{MaxCover}(\Gamma) = 1$
or $\text{MaxCover}(\Gamma) \leq s$

k-Clique as MaxCover



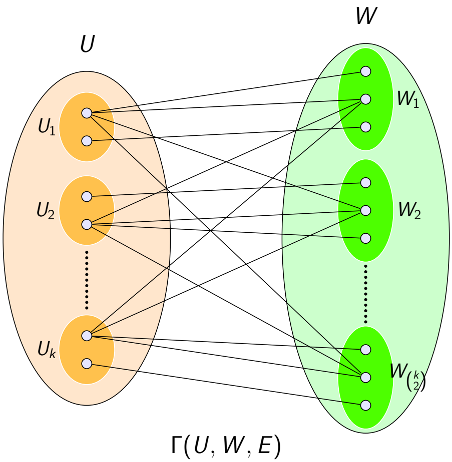
$\Gamma(U, W, E)$

k-Clique as MaxCover



Input of k-Clique problem:
$G([n], E_0)$

k-Clique as MaxCover



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Each W_j is a copy of E_0
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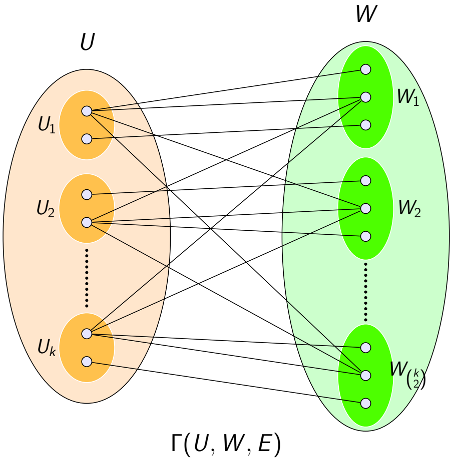
k-Clique as MaxCover



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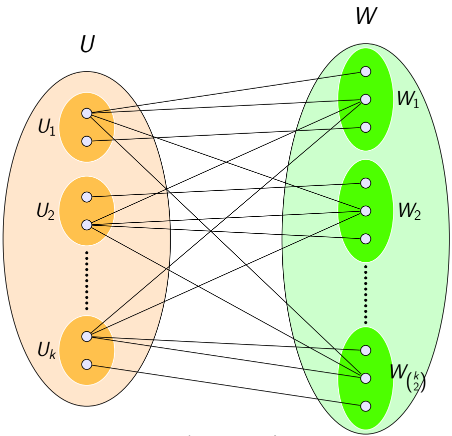
For distinct i, j, j', introduce all edges between $W_{j,j'}$ and U_i

k-Clique as MaxCover



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Determine if $\text{MaxCover}(\Gamma) = 1$ or $\text{MaxCover}(\Gamma) \leq 1 - 1/k$

MaxCover: Results

- W[1]-Complete if there are $F(k)$ left super nodes

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- W[1]-Complete if there are $F(k)$ left super nodes
- 1 vs. $k/n^{1/\sqrt{k}}$ is W[1]-Hard
- Central problem to understand parameterized inapproximability of Dominating Set and Clique

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Part 2: Dominating Set ✓

Part 3: Hardness of Approximation ✓

- Hardness of Approximation in NP ✓
- Hardness of Approximation in Parameterized Complexity ✓

Part 4: Coding Theory

- Definition and Geometric Intuition
- Random Codes
- Algebraic Codes

Part 5: Hardness of Approximating MaxCover

- MaxCover with Projection Property
- Gap Creation

Part 6: Hardness of Approximating Dominating Set

- MinLabel
- Gap Translation to Dominating Set

Coding Theory: Geometric Motivation

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Coding Theory:Definitions

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A good code: for $\rho, \delta > 0$, $|C| = 2^{\rho L}$, $\Delta(C) = \delta L$.

Random Codes

Random Strings are Good Codes

For some small $\rho > 0$, if we pick $2^{\rho L}$ random strings uniformly and independently then they form a code with distance at least $1/4$ (whp).

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Many Efficient Deterministic Good Codes Exist!

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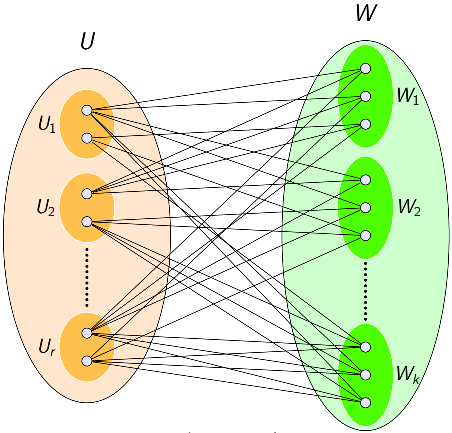
- Reed Solomon Codes meet the Singleton bound!

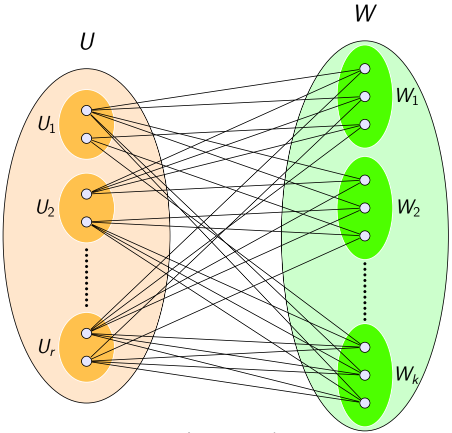
Outline

MaxCover: Projection Property



$\Gamma(U, W, E)$

MaxCover: Projection Property



$\Gamma(U, W, E)$

Γ has projection property:

For every U_i and W_j,

Induced subgraph of (U_i, W_j) is:

- complete bipartite graph (i.e., irrelevant), or,
- $\forall w \in W_j$, deg(w)=1 (i.e., projection)

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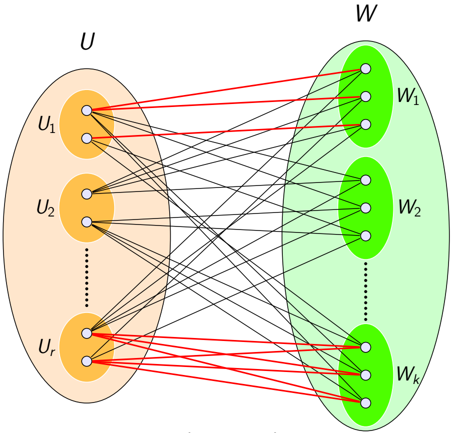

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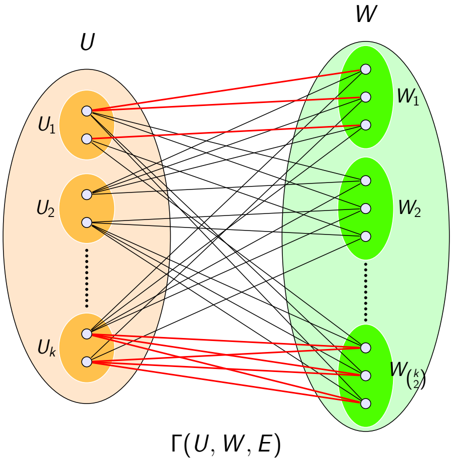
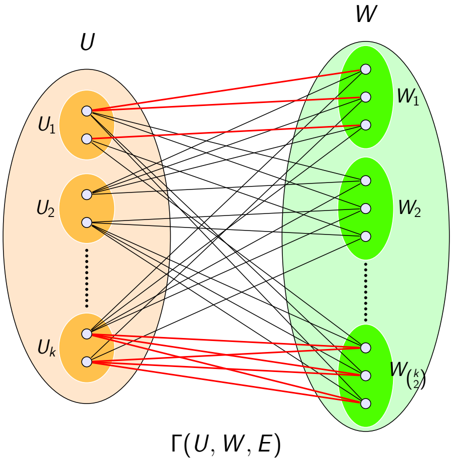
MaxCover with Projection Property is W[1]-Hard



Input: $G([n], E_0)$

MaxCover with Projection Property is W[1]-Hard



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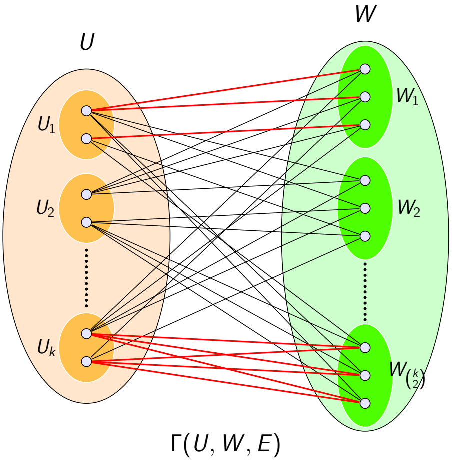
$U_i = [n]$ and $W_{j,j'} = E_0$



Input: $G([n], E_0)$

$U_i = [n]$ and $W_{j,j'} = E_0$

$W_{j,j'}$ has projection to U_j and $U_{j'}$

MaxCover: Gap Creation

Inapproximability of MaxCover

There is a FPT reduction from MaxCover instance $\Gamma_0 = \left(U_0 = \bigcup_{j=1}^{r} U_j^0, W = \bigcup_{j=1}^{k} W_i, E_0\right)$ with projection property to a MaxCover instance $\Gamma = \left(U = \bigcup_{j=1}^{O(\log|U_0|)} U_j, W = \bigcup_{j=1}^{k} W_i, E\right)$ such that

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- If $\text{MaxCover}(\Gamma_0) = 1$ then $\text{MaxCover}(\Gamma) = 1$
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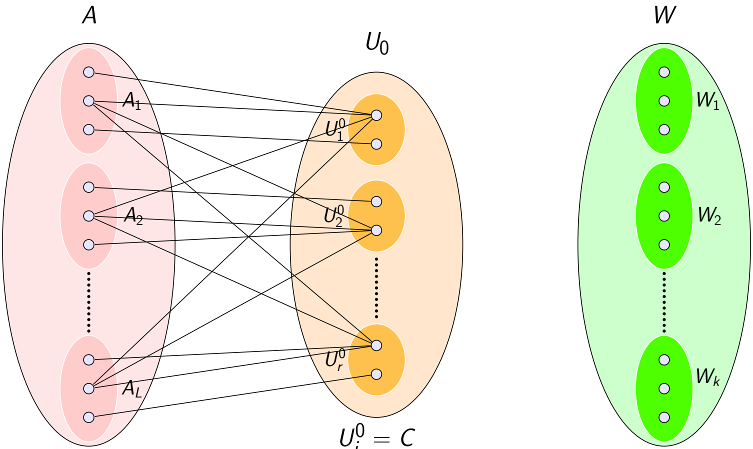
- If $\text{MaxCover}(\Gamma_0) < 1$ then $\text{MaxCover}(\Gamma) \leq 0.75$
- $|\Gamma| = \tilde{O}(2^r \cdot |W| \cdot \log|U_0|)$
- The reduction runs in time $2^{O(r)} \cdot \text{poly}(|\Gamma_0|)$.

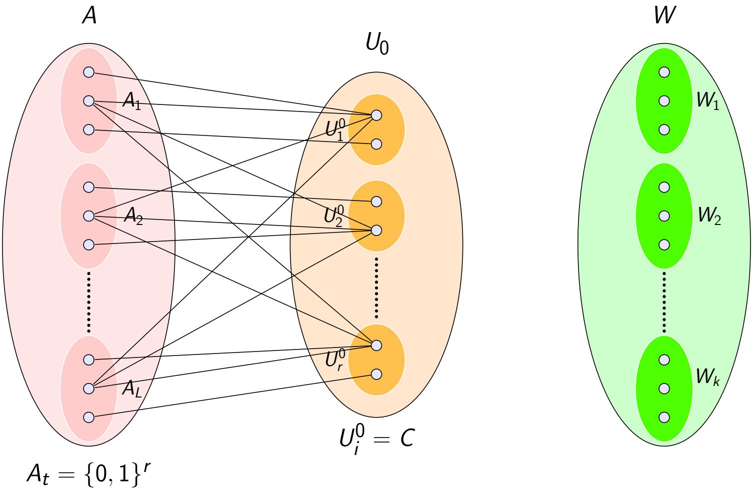
Threshold Graph Construction



$U^0_i = C$

$A_t = \{0,1\}^r$

Threshold Graph Construction



$(u, (q_1, \ldots, q_r)) \in U_i^0 \times A_t$ is an edge $\Leftrightarrow u_t = q_i$

Threshold Graph Properties

Completeness

For every $(u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0$ and every A_t
there exists a unique common neighbor of $(u^1, \ldots, u^r)$ in A_t

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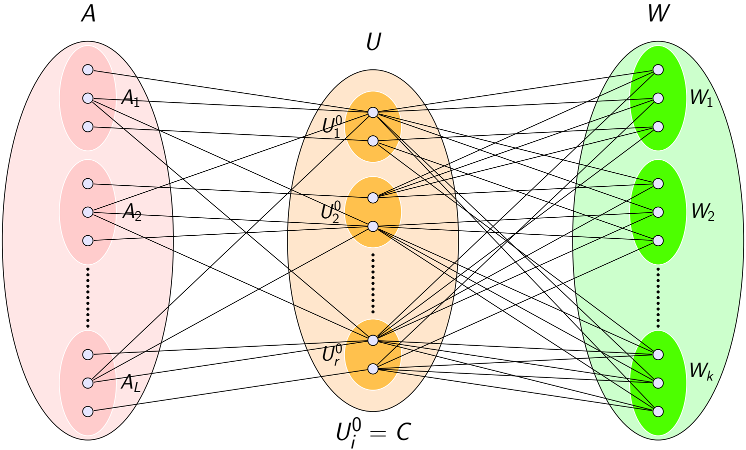
Soundness

For every $u, u' \in U_i^0$, there are at most $L - \Delta(C)$ many supernodes in A
which have a common neighbor of u and u'

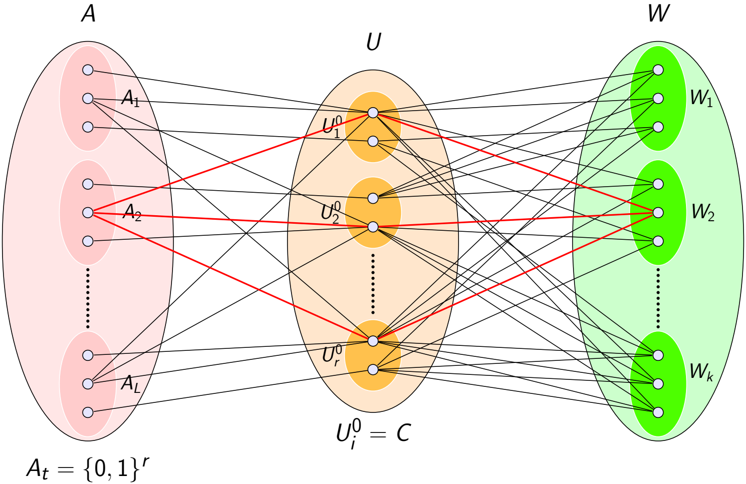
Threshold Graph Composition



$A_t = \{0,1\}^r$

$U_i^0 = C$

Threshold Graph Composition



$(w, (q_1, \ldots, q_r)) \in W_j \times A_t$ is an edge $\Leftrightarrow \exists\, (u^1, \ldots, u^r) \in U_1^0 \times \cdots U_r^0$ such that $\forall i \in [k]$, (w, u^i) and $(u^i, (q_1, \ldots, q_r))$ are both edges

Completeness of Reduction

- Let $(w_1, \ldots, w_k) \in W_1 \times \cdots \times W_k$ be optimal labeling of Γ_0
- Let $(u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0$ be common neighbors of $(w_1, \ldots, w_k)$ in Γ_0

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Completeness of Threshold Graph

For every $(u^1, \ldots, u^r) \in U_1^0 \times \cdots \times U_r^0$ and every A_t
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MaxCover: Gap Creation

Inapproximability of MaxCover using Random Binary Codes

There is a FPT reduction from MaxCover instance $\Gamma_0 = \left(U_0 = \bigcup_{j=1}^{r} U_j^0, W = \bigcup_{j=1}^{k} W_i, E_0\right)$ with projection property to a MaxCover instance $\Gamma = \left(U = \bigcup_{j=1}^{O(\log|U_0|)} U_j, W = \bigcup_{j=1}^{k} W_i, E\right)$ such that

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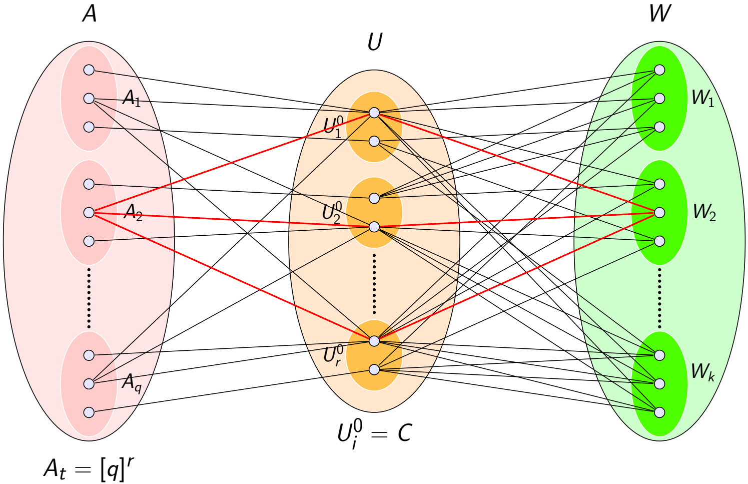
Threshold Graph Composition with Reed Solomon Codes



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Soundness

For every $u, u' \in U_i^0$, there are at most $\log_q |U_0|$ many supernodes in A
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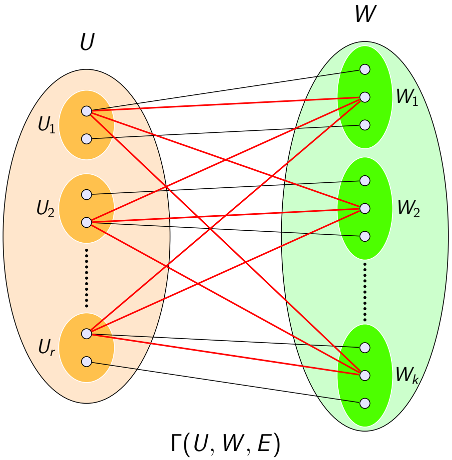
MinLabel

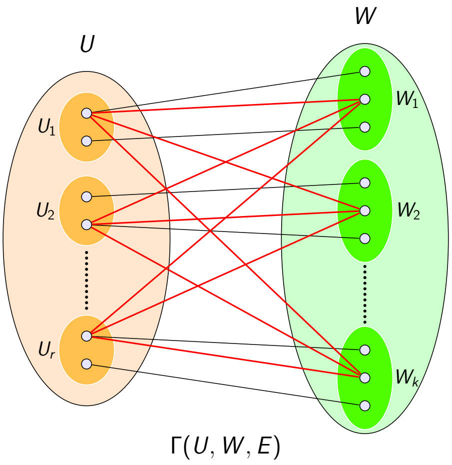


Each W_i is a Right Super Node
Each U_i is a Left Super Node

$S \subseteq W$ is a labeling of W if
$\forall i \in [k], |S \cap W_i| = 1$

S covers U_i if
$\exists u \in U_i,\ \forall v \in S, (u, v) \in E$

MinLabel



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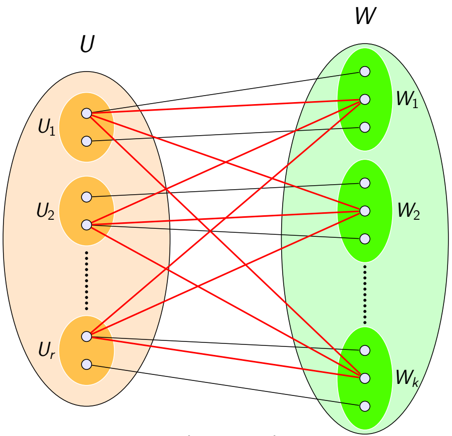
MinLabel(Γ) = smallest $X \subseteq W$:
$\forall i \in [r], \exists$labeling $S \subseteq X$,
S covers U_i

MinLabel



$\Gamma(U, W, E)$

Determine if MinLabel(Γ) $= k$
or MinLabel(Γ) $\geq s \cdot k$

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Maxcover to MinLabel

Reduction from MaxCover to MinLabel

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Completeness is obvious. In Soundness case, if $X \subseteq W$ is a MinLabel solution

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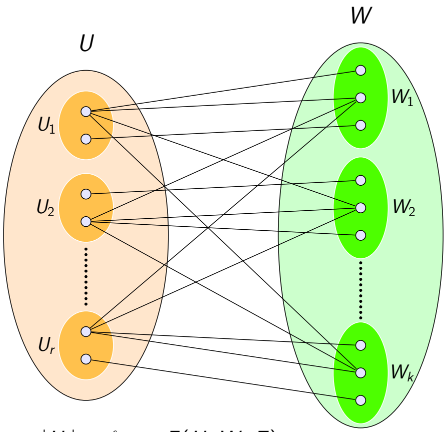
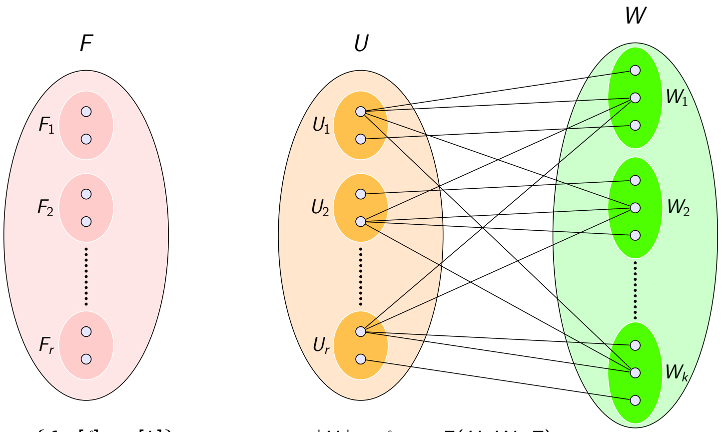
$$\binom{|X|/k}{k} \cdot \varepsilon \geq 1$$

MinLabel to Dominating Set



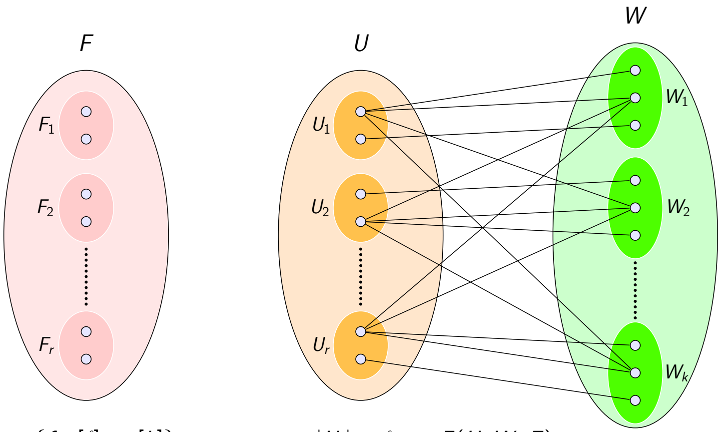
$|U_i| = \ell$ $\Gamma(U, W, E)$

MinLabel to Dominating Set



$F_i = \{f : [\ell] \to [k]\}$ $|U_i| = \ell$ $\Gamma(U, W, E)$

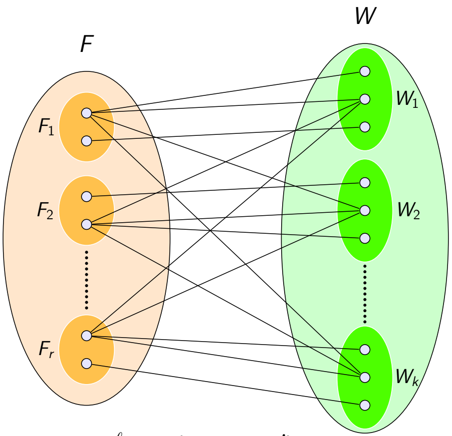
MinLabel to Dominating Set



$F_i = \{f : [\ell] \to [k]\}$ $\quad |U_i| = \ell \quad \Gamma(U, W, E)$

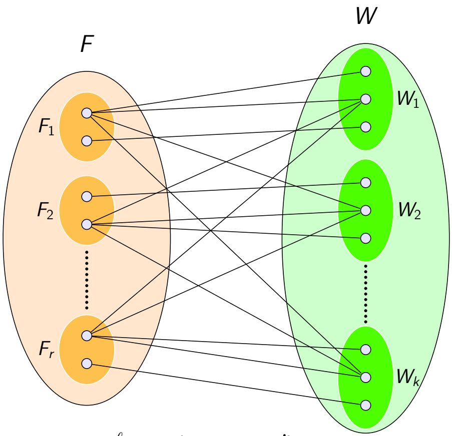
Edge between $f \in F_i$ and $w \in W_j \Leftrightarrow$
$\exists u \in U_i$ such that $(u, w) \in \Gamma$ and $f(u) = j$

MinLabel to Dominating Set



$|F_i| = k^{\ell}$ $\quad H(F \cup W, E')$

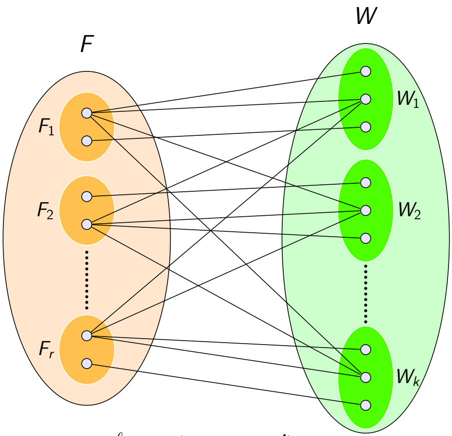
MinLabel to Dominating Set



$|F_i| = k^\ell$ $\quad H(F \cup W, E')$

$$F = \{(i, f) \mid i \in [r], f : [\ell] \to [k]\}$$

MinLabel to Dominating Set



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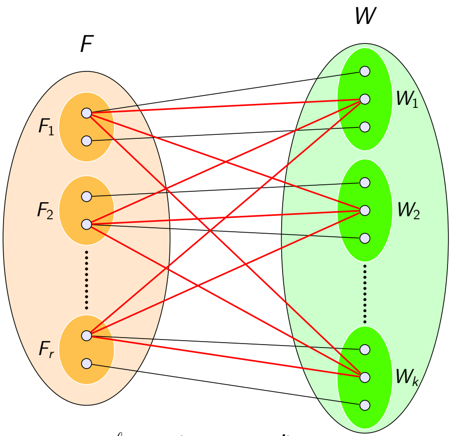
$$F = \{(i, f) \mid i \in [r], f : [\ell] \to [k]\}$$

$$((i, f), w) \in H \Leftrightarrow \exists u \in U_i$$
$$(u, w) \in \Gamma \text{ and } f(u) = j$$

MinLabel to Dominating Set



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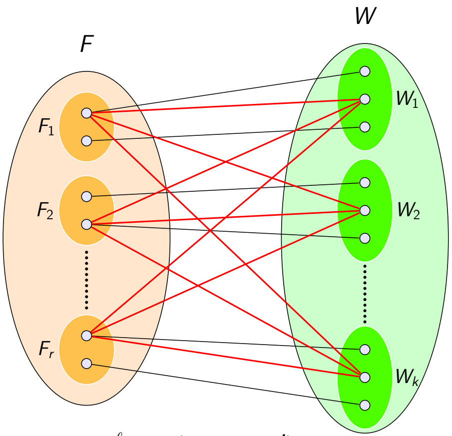
$(w_1, \ldots, w_k)$ is labeling
that covers every U_i $\Rightarrow$
$(w_1, \ldots, w_k)$ dominates H

MinLabel to Dominating Set



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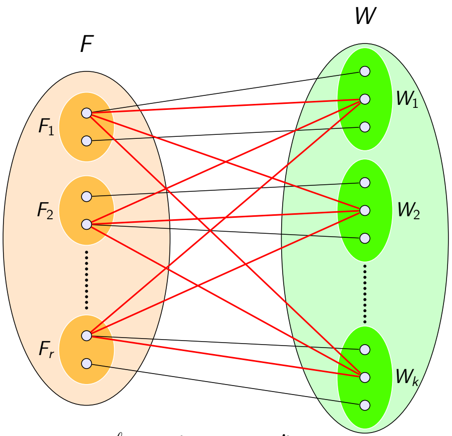
that covers every U_i $\Rightarrow$
$(w_1, \ldots, w_k)$ dominates H

$\forall (i, f) \in F, \ \exists u \in U_i,$
$(u, w_j) \in \Gamma \ (\forall j \in [k])$

MinLabel to Dominating Set



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$(w_1, \ldots, w_k)$ dominates H

$$\forall (i,f) \in F,\ \exists u \in U_i,$$
$$(u, w_j) \in \Gamma\ (\forall j \in [k])$$

Determine if $\text{DomSet}(H) = k$
or $\text{DomSet}(H) \geq s \cdot k$ is hard!

MinLabel to Dominating Set: Soundness Analysis

- $F = \{(i, f) \mid i \in [r], f : [\ell] \to [k]\}$
- $((i, f), w) \in H \Leftrightarrow \exists u \in U_i : (u, w) \in \Gamma$ and $f(u) = j$

MinLabel to Dominating Set: Soundness Analysis

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- Suppose X is a Dominating Set of size $sk - 1$

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- Suppose X is a Dominating Set of size $sk - 1$
- $\exists U_i$ not covered by any labeling in X

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- Suppose X is a Dominating Set of size $sk - 1$
- $\exists U_i$ not covered by any labeling in X
- For every $u \in U_i$ there is some $j \in [k]$ such that $W_j \cap X \cap N(u)$ is empty

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- Construct f using above u

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- For every $u \in U_i$ there is some $j \in [k]$ such that $W_j \cap X \cap N(u)$ is empty
- Construct f using above u
- (i, f) is not dominated by X

Parameterized Inapproximability of Dominating Set

Inapproximability of Dominating Set

There is a FPT reduction from k-clique instance $G([n], E)$ to a Dominating Set instance H such that

- If G has a k-clique then $\binom{k}{2}$ vertices in H form a dominating set
- If G has no k-clique then $(\log n)^{1/k^2}$ vertices in H are needed to form a dominating set
- $|H| \leq \text{poly}(n)$
- The reduction runs in time $2^{\text{poly}(k)} \cdot \text{poly}(n)$.

Outline

Part 1: Handwaving Introduction ✓

Part 2: Dominating Set ✓

Part 3: Hardness of Approximation ✓

- Hardness of Approximation in NP ✓
- Hardness of Approximation in Parameterized Complexity ✓

Part 4: Coding Theory ✓

- Definition and Geometric Intuition ✓
- Random Codes ✓
- Algebraic Codes ✓

Part 5: Hardness of Approximating MaxCover ✓

- MaxCover with Projection Property ✓
- Gap Creation ✓

Part 6: Hardness of Approximating Dominating Set ✓

- MinLabel ✓
- Gap Translation to Dominating Set ✓