What happens when you push measures using polynomial maps – a dictionary between algebraic geometry and analysis.

based on joint works with N. Avni, and S. Carmeli

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  $$(\phi_*(\mu))(A) := \mu(\varphi^{-1}(A))$$

We study the image $\varphi_*(\mathcal{M}_c^\infty(X(F)))$

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A form ω on X gives a measure by:

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$$\frac{\varphi_*(f \cdot |\omega_X|)}{|\omega_Y|}(y) = \int f|\omega_{\varphi^{-1}(y)}|$$

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TFAE:

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Proof.

Radon-Nikodym. □

Dictionary between algebraic geometry and analysis

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L^1_{loc}	Generically smooth
C^0 or L^∞_{loc}	?

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When are the integrals of the G-L form converging?

Integrability and rational singularities

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If Z is CM and $char(F) = 0$ we can also add

- *The singularities of Z are rational.*

FRS maps

Theorem (Miracle flatness)

φ is flat iff its fibers have constant dimension.

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Definition

φ is called FRS if it is flat and its fibers (are reduced and) have rational singularities.

Continuty criterion

Theorem (A. - Avni 2013, Reiser - 2018)

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Proof.

- Reduce for the case $\dim(Y) = 1$

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Proof.

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- Resolve the singularities of a fiber.

□

More facts for dominant maps

Theorem (Cluckers-Loeser-Gordon-Halupczok 2010-2014)

For p-adic fields if φ is (locally) dominant then,

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where $Const(Y(F))$ is the algebra generated by $|f|, \ln(|f|)$ for definable $f : X \to F$.

Theorem (Glazer-Hendel 2021-2026, Cluckers-Miller 2013)

if φ is generically smooth, then there is $\varepsilon > 0$ s.t.

$$\frac{\varphi_*(\mathcal{M}_c^\infty(X(F)))}{\nu_0} \subset L^{1+\varepsilon}_{loc}(Y(F))$$

Theorem (Glazer-Hendel-Sodin 2024)

Explicit bounds for ε in terms of the degree/complexity of φ.

Finiteness questions

Question

When are $\varphi_*(\mathcal{M}_c^\infty(X(F)))_x := \varphi_*(\mathcal{M}_c^\infty(X(F)))/C_c^\infty(X(F) \smallsetminus \{x\})\varphi_*(\mathcal{M}_c^\infty(X(F)))$ *finite dimensional?*

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Consider the characteristic polynomial map

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- We will discuss a generalization of this phenomenon to the infinitesimal setting.

The Scheme $\mathfrak{B}_\varphi$ and Quasitransitivity

Definition (The scheme $\mathfrak{B}_\varphi$)

Let $\varphi : X \to Y$ be as above. Let

$$D\varphi : \mathcal{T}_X \to \varphi^*(\mathcal{T}_Y)$$

be the differential, and let $\mathrm{Im}(D\varphi)$ be its image sheaf. Define

$$\mathfrak{B}_\varphi := \mathrm{Spec}_X \left(\mathrm{Sym}(\mathrm{Im}(D\varphi)) \right).$$

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Theorem (A., Avni, Carmeli - 2026)

Let $\varphi : X \to Y$ be a quasi-transitive morphism of smooth algebraic varieties over a p-adic field F. Then $\varphi_*(\mathcal{M}_c^\infty(X(F)))$ has finite stalks

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2. exponentiate the fields to get local automorphisms

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Theorem (A., Avni, Carmeli - 2026)

Let $\varphi : X \to Y$ be a quasi-transitive morphism of smooth algebraic varieties over a p-adic field F. Then $\varphi_*(\mathcal{M}_c^\infty(X(F)))$ has finite stalks

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$\square$

Step 3 does not work. Non-archimedean exponentiation is too local.