Regular Densities and Differential Operators for Singular Varieties.

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Let X be an irreducible complex algebraic variety. We denote by $Omod(X)$ the category of quasi-coherent sheaves of O-modules on X. If $X = Spec(A)$ is affine, this is just the category of A-modules.

If X is smooth, then we have two important coherent sheaves of O-modules on X – sheaf O_X and the sheaf ω_X of densities (= volume forms). Usually ω_X is realized as the sheaf of top differential forms.

I claim that for arbitrary irreducible variety X there is a natural (canonical) local construction that associates to X a coherent O-module ω_X that I call the sheaf of **Regular Densities** on X.

On the open dense subset X_{sm} of smooth points it coincides with the sheaf of densities. So my claim is that this familiar sheaf of densities on the subset X_{sm} has a **canonical** extension to the whole variety X.

This construction is non-trivial and is based on the theory of D-modules.

It has important applications when we switch from algebra to analysis, where we can use it to define the space of smooth densities on a singular variety.

It is also useful in discussing the following informal general

Question. What is a "correct" notion of a differential operator on a singular variety X.

First let me remind some standard facts about D-modules.

Let X be an affine algebraic variety, $A = O(X)$ the algebra of regular functions on X.

Grothendieck defined the algebra of differential operators $D(X)$ as the subalgebra of the algebra $End(A)$ of all $\mathbf{C}$-linear endomorphisms of A.

Namely, the algebra $End(A)$ is naturally a bimodule over the algebra A (with respect to left and right multiplications). Thus $End(A)$ is a module over the algebra $B = A \otimes_{\mathbf{C}} A = O(X \times X)$. Denote by $J \subset B$ the ideal of functions on $X \times X$ that vanish on the diagonal $\Delta X \subset X \times X$.

Grothendieck's Definition. A differential operator on X of order $\leq k$ is an operator $D \in End(A)$ that is killed by the ideal J^{k+1}.

We denote by $D^k(X)$ the space of operators of order $\leq k$ and define the algebra $D(X)$ of differential operators on X as $D(X) = \bigcup D^k(X)$.

In fact, this definition is easy to generalize.
Let M, N be two A-modules. As before we can define the space of differential homomorphisms $DiffHom(M, N)$.
In particular, we define the algebra $DiffEnd(M)$ of differential operators on the module M by $DiffEnd(M) := DiffHom(M, M)$.

This construction is compatible with localization if the module M is finitely generated. As a result, we can sheafify this construction. Namely, for a general variety X given a coherent O-module M and a quasi coherent O-module N on X we will define the sheaf $DiffHom(M, N)$ of differential morphisms.

Grothendieck's definition of the sheaf D_X of differential operators on X works perfectly well for smooth varieties. It turns out that for a singular variety X it can give an answer that is not satisfactory. We will discuss this later.

D-modules.

Let X be a smooth variety. By definition, a D-module on X is a sheaf F of left D_X-modules that is quasi-coherent as a sheaf of O-modules. The abelian category of such modules we denote by $Dmod(X)$.

We can also consider the category $Dmod^r$ of right D-modules. For example, the O-module ω_X has a natural structure of a right D-module.
In fact, there is a functor
$\Omega : Dmod(X) \to Dmod^r(X), F \mapsto F \otimes \omega_X$ that is an equivalence of categories.

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A convenient way to think is that there is one category of D-modules $Dmod(X)$ that has two realizations as a category of sheaves related by the functor Ω. For us it will be more convenient to work with right D-modules.

Kashiwara's lemma.

Let W be a smooth variety and $X \subset W$ a closed smooth subvariety. Then the category $Dmod^r(X)$ can be naturally identified with the subcategory $Dmod^r(W)_X \subset Dmod^r(W)$ that consists of sheaves supported on X.

Namely, let $i : X \to W$ be the embedding, $i^* : O_W \to O_X$ the corresponding morphism of sheaves. Denote by J the kernel of the morphism i^* and set $D_{X \to W} := D_W / J \cdot D_W$.

This sheaf is a (D_X, D_W)- bimodule and it defines a functor $i_* : Dmod^r(X) \to Dmod^r(W)$.

Geometrically we can describe sections of the sheaf $i_*(F)$ as sections of F and their transversal derivatives of all orders.

Kashiwara's Lemma. The functor
$i_* : Dmod^r(X) \to Dmod^r{}_X(W)$ is an equivalence of categories. The inverse functor T is given by $T(H) = Hom(D_{X\to W}, H) =$ Annihilator of the Ideal J in H.

Using Kashiwara's lemma we will define the category $Dmod^r(X)$ for any algebraic variety X. Namely, if X is affine we embedd it into a smooth affine variety W and set
$Dmod^r(X) := Dmod^r(W)_X$.

Using Kashiwara's lemma it is easy to check that this construction does not depend on a choice of the embedding (up to canonical equivalence). Then we can define the category $Dmod^r(X)$ for any X by glueing.

For any variety X we have a canonical functor
$T : Dmod^r(X) \to Omod(X)$.

Namely, Let us fix a closed embedding $i : X \to W$ as before and denote by $J \subset O_W$ the ideal defining X (J equals to the kernel of the morphism $i^* : O_W \to O_X$). The functor T is defined by
$T(H) :=$ Kernel of J in H.

Using Kashiwara's lemma it is easy to see that this functor is well defined. In case when X is smooth and we consider $Dmod^r(X)$ as the category of D_X-modules, this functor is just a restriction functor from right D-modules to O-modules.

Construction of the O-module ω_X.

Let X be an irreducible algebraic variety. Choose an open dense subset $U \subset X$ consisting of smooth points. Consider ω_U as a right D-module on U.

Now, in the theory of D-modules there exists a procedure of "perverse" extension of this D-module to X. Shortly, the module ω_U is holonomic and hence its direct image $F = j_*(\omega_U) \in Dmod^r(X)$ under open embedding $j : U \to X$ is also holonomic. This implies that the D-module F has finite length.

We define the $Dmod^r$ on $j_{*!}(\omega_U)$ to be the minimal submodule of this direct image F that restriction to U is ω_U.

Now we define $\omega_X := T(j_{*!}(\omega_U)) \in Omod(X)$.

Applications.

Smooth regular densities.

Let us assume that the variety X is defined over $\mathbf{R}$. Consider the space $Y = X(\mathbf{R})$ of its real points. This is a singular space, but we assume that the space $Y_{sm} = Y \bigcap X_{sm}$ is dense in Y.

By imbedding X into a smooth variety we define the sheaf of smooth algebras S_Y. It has a natural structure of O-module on X (not quasi-coherent). For any O-module H on X we define a sheaf of its smooth sections on Y to be $S(H)H \otimes_O S$.

In particular, We call the sheaf $S(\omega_X)$ the **Sheaf of smooth regular densities.**

The global sections of this sheaf on the space Y are just smooth volume forms on the open set Y_{sm} that have very definite asymptotic behavior in neighborhoods of singular points.
The analysis of these spaces usually gives a lot of information about the structure of Y and its singularities.

Spaces of smooth regular sections for other sheaves – Schwartz spaces.

Smooth regular densities over p-adic fields..

Let F be a p-adic field of characteristic 0 and X an irreducible variety defined over F.

As before, we set $Y = X(F)$ and consider an open subset $Y_{sm} \subset Y$.

This is a p-adic space. It turns out that using the O-module ω_X we can construct the sheaf $S(\omega_X$ of complex vector spaces on the topological space Y that on the open subset Y_{sm} coincides with the sheaf of locally constant complex valued measures.

Again, the study of spaces of global sections of these sheaves gives a lot of information about the original variety X.

In fact, we also can consider a variety X over a global field K (say $K = \mathbf{Q}$) and study these global spaces for all completions of K simultaneously (adelic picture).
Integrating sections of interesting sheaves on adelic spaces is a standard way to construct automorphic L-functions.

"Correct" differential operators. I am also interested in one particular purely algebraic problem related to O-module ω_X.

Namely, Grothendieck definition of differential operators work well for smooth varieties. If X is not smooth, the definition formally works, but produces the algebra $D(X)$ that is not good.

Example. Let C be a non-degenerate cubic cone in $\mathbf{C}^3$. Then the algebra $D(X)$ is bad – it is not Noetherian and not finitely generated.

My suggestion is that in fact there exists a "correct" algebra $D^{corr}(X)$ of differential operators, just in this case Grothendieck's definition gives a wrong answer.

In fact, I even have a tentative conjectural answer for the correct definition.

Conjecture.. Let X be a normal irreducible affine complex variety.

Consider the O-module ω_X and define the algebra $D^{corr}(X) := DiffEnd(\omega_X)$.
Then this algebra is Noetherian.

II can prove this result for some cases (e.g. the cubic cone). For me the proof of this conjecture will be an indication that this is the "correct" definition of the algebra of differential operators.