The serpentine representation of the infinite symmetric group and the basic representation of $\widehat{\mathfrak{sl}_2}$

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Schur–Weyl duality

Classical Schur–Weyl duality: $\boxed{(\mathbb{C}^\ell)^{\otimes N} = \bigoplus_{\lambda \in \mathbb{Y}_N^\ell} \rho_\lambda \otimes \pi_\lambda}$,

- $\mathbb{Y}_N^\ell = \{$Young diagrams with N cells, $\leq \ell$ rows$\}$,
- $\pi_\lambda =$ irreducible representation of $\mathfrak{S}_N$ with Young diagram λ,
- $\rho_\lambda =$ irreducible representation of $GL(\ell, \mathbb{C})$ with signature λ.

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Let us look on this 'dynamically': fix ℓ $(= 2)$ and let $N \to \infty$.

Schur–Weyl duality

Classical Schur–Weyl duality ($\ell = 2,\ N = 2n$):

$$\boxed{(\mathbb{C}^2)^{\otimes N} = \bigoplus_{k=0}^{n} M_{2k+1} \otimes \pi_{(n+k,n-k)}},$$

- $\pi_{(n+k,n-k)} = \mathfrak{S}_N$-irrep with Young diagram $(n+k, n-k)$,
- $M_{2k+1} = (2k+1)$-dimensional $SL(2,\mathbb{C})$-irrep.

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Schur–Weyl embedding: $\boxed{\alpha_N\colon (\mathbb{C}^2)^{\otimes N} \hookrightarrow (\mathbb{C}^2)^{\otimes (N+2)}}$ respecting both the actions of $SL(2,\mathbb{C})$ and $\mathfrak{S}_N$.

In the inductive limit we have an action of $SL(2,\mathbb{C})$ and an action of $\mathfrak{S}_\infty$.

Schur–Weyl representations

Schur–Weyl representation $\Pi^{\{\alpha_N\}}$ of $\mathfrak{S}_\infty$: inductive limit of representations of $\mathfrak{S}_N$ wrt Schur–Weyl embeddings

$$(\mathbb{C}^2)^{\otimes 0} \overset{\alpha_0}{\hookrightarrow} (\mathbb{C}^2)^{\otimes 2} \overset{\alpha_2}{\hookrightarrow} (\mathbb{C}^2)^{\otimes 4} \overset{\alpha_4}{\hookrightarrow} \dots$$

Theorem (Ts.–Vershik 2014)

$$\Pi^{\{\alpha_N\}} = \bigoplus_{k=0}^{\infty} M_{2k+1} \otimes \Pi_k^{\{\alpha_N\}},$$

- *M_{2k+1} is still the $(2k+1)$-dimensional irrep of $\mathfrak{sl}_2$,*
- *$\Pi_k^{\{\alpha_N\}}$ is an irrep of $\mathfrak{S}_\infty$ (an inductive limit of irreps of $\mathfrak{S}_{2k}, \mathfrak{S}_{2k+2}, \dots$ with Young diagrams $(2k), (2k+1,1), (2k+2,2), \dots$).*

Serpentine representation

T_N := the set of Young tableaux with N cells, ≤ 2 rows.

Embedding $i_N : T_N \to T_{N+2}$: put $N+1$ to 1st row and $N+2$ to 2nd row:

1	2	4	5	7	8
3	6				

$\longrightarrow$

1	2	4	5	7	8	9
3	6	10				

Definition

Serpentine representation Π: Schur–Weyl representation of $\mathfrak{S}_\infty$ constructed from the sequence of embeddings i_N.

<u>Structure:</u>

$$\Pi = \sum_{k=0}^{\infty} M_{2k+1} \otimes \Pi_k,$$

- Π_k = irreducible k-serpentine representation,
- M_{2k+1} = $(2k+1)$-dimensional $\mathfrak{sl}_2$-irrep.

k-Serpentine representation

k-vacuum tableau ($k = 0, 1, \ldots$):

$\tau_k =$

1	2	$\cdots$	$2k$	$2k+1$	$2k+3$	$\cdots$
$2k+2$	$2k+4$	$\cdots$				

In particular,

$\tau_0 =$

1	3	5	$\cdots$
2	4	6	$\cdots$

k-serpentine tableau = an infinite tableau tail-equivalent to τ_k.

Π_k acts in $\ell^2(\{k\text{-serpentine tableaux}\})$ via Young's orthogonal form.

Major index

Major index of a finite Young tableau:

$$\mathrm{maj}(\tau) = \sum_{i \in \mathrm{des}(\tau)} i,$$

where

$$\mathrm{des}(\tau) = \{ i \le N - 1 : \ i + 1 \text{ in } \tau \text{ is lower than } i \}.$$

Example:

$$\mathrm{maj}\left(\begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 7 & 8 \\ \hline 3 & 6 \\ \cline{1-2} \end{array} \right) = 7.$$

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Observation: $\mathrm{maj}(i_{2n}(\tau)) = \mathrm{maj}(\tau) + (2n+1)$.

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 4 & 5 & 7 & 8 \\ \hline 3 & 6 \\ \cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 4 & 5 & 7 & 8 & 9 \\ \hline 3 & 6 & 10 \\ \cline{1-3}\end{array}$$

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Another observation: $2n+1 = (n+1)^2 - n^2 \implies$

$$\boxed{(n+1)^2 - \mathrm{maj}(i_{2n}(\tau)) = n^2 - \mathrm{maj}(\tau)}$$

Stable major index

Stable major index (well defined for all serpentine tableaux):

$$\boxed{\mathrm{smaj}(\tau) = \lim_{n\to\infty} \left(n^2 - \mathrm{maj}([\tau]_{2n})\right)},$$

where $[\tau]_\ell$ is obtained from τ by removing cells with entries $k > \ell$.

Example:

$$\mathrm{smaj}\left(\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 4 & 6 & 7 & \cdots \\ \hline 3 & 5 & 8 & \cdots \\ \cline{1-4}\end{array}\right) = 3^2 - 6 = 3.$$

For the vacuum tableaux, $\boxed{\mathrm{smaj}(\tau_k) = k^2}$.

Stable major index

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Now, smaj $\rightsquigarrow$ a grading on Π: for $w = u \otimes v \in M_{2k+1} \otimes \Pi_k$,

$$\mathsf{deg}_s(w) := \mathsf{smaj}(v).$$

Thus we have a graded space (Π, deg_s) with an action of $\mathfrak{sl}_2$.

The affine Lie algebra $\widehat{\mathfrak{sl}_2}$

- Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$:
  standard generators $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,
  relations $[e, f] = h$, $[h, e] = 2e$, $[f, e] = -2f$.
- Affine Lie algebra $\widehat{\mathfrak{sl}_2} = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$, where c is a central element and

$$[a \otimes t^n, b \otimes t^m] = [a, b] \otimes t^{n+m} + n\,\mathrm{tr}(ab)\delta_{n+m,0}c.$$

Homogeneous grading on $\widehat{\mathfrak{sl}_2}$: $\deg_H(x \otimes t^n) = n$.

Natural embedding: $\mathfrak{sl}_2 \supset x \mapsto x \otimes 1 \in \widehat{\mathfrak{sl}_2}$.

Main theorem

So, we have two spaces with an action of $\mathfrak{sl}_2$:

- $(\Pi, \deg_s)$ = the serpentine representation of $\mathfrak{S}_\infty$ with stable major index grading,
- $L_{0,1}$ = the basic representation of $\widehat{\mathfrak{sl}_2}$ (level 1 highest weight rep with fundamental weight Λ_0) with homogeneous grading.

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Theorem (Ts.–Vershik 2015)

There is a grading-preserving unitary isomorphism of $\mathfrak{sl}_2$-modules between $(L_{0,1}, \deg_H)$ and $(\Pi, \deg_s)$.

Fusion product

- Representation ρ of $\mathfrak{sl}_2$, $z \in \mathbb{C} \rightsquigarrow$ evaluation rep. ρ_z of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$: $\rho_z(x \otimes t^i)v = z^i \cdot \rho(v)$;
- irreps $\rho^1, \ldots, \rho^N$ with cyclic vectors $v_1, \ldots, v_N$ and pairwise distinct $z_1, \ldots, z_N \in \mathbb{C} \rightsquigarrow$ representation $V_N := \rho^1_{z_1} \otimes \ldots \otimes \rho^N_{z_N}$ of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$;
- $U^{(m)} \subset \mathfrak{sl}_2 \otimes \mathbb{C}[t]$ is spanned by monomials $e_{i_1} \ldots e_{i_k}$ with $i_1 + \ldots + i_k = m$, where $e_j = e \otimes t^j$;
- $V_N^{(m)} = U^{(m)}(v_1 \otimes \ldots \otimes v_N) \subset V_N$;
- filtration $V_N^{(\leq m)} = \sum_{k \leq m} V_N^{(k)}$.

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Definition (B. Feigin–Loktev 1999)

The **fusion product** of $\rho^1, \ldots, \rho^N$ is the graded representation in

$$V_N^* = V_N^{(\leq 0)} \oplus V_N^{(\leq 1)}/V_N^{(\leq 0)} \oplus V_N^{(\leq 2)}/V_N^{(\leq 1)} \oplus \ldots.$$

Denote the degree wrt this grading by $\widetilde{\deg}$.

Finite-dimensional approximation of $L_{0,1}$

Definition (B. Feigin–Loktev 1999)

The **fusion product** of $\rho^1, \ldots, \rho^N$ is the adjoint graded representation in

$$V_N^* = V_N^{(\leq 0)} \oplus V_N^{(\leq 1)}/V_N^{(\leq 0)} \oplus V_N^{(\leq 2)}/V_N^{(\leq 1)} \oplus \ldots.$$

Theorem (B. Feigin–Loktev 1999)

V_N^* is an $\mathfrak{sl}_2 \otimes (\mathbb{C}[t]/t^N)$-module that does not depend on $z_1, \ldots, z_N$ provided they are pairwise distinct.

Remark: V_N^* is isomorphic to $\rho^1 \otimes \ldots \otimes \rho^N$ as an $\mathfrak{sl}_2$-module.

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<u>Remark:</u> V_N^* is isomorphic to $\rho^1 \otimes \ldots \otimes \rho^N$ as an $\mathfrak{sl}_2$-module.

- Take $\rho^1 = \ldots = \rho^N = \mathbb{C}^2$ with the natural action of $\mathfrak{sl}_2$ $\implies V_N^* \simeq (\mathbb{C}^2)^{\otimes N}$ as an $\mathfrak{sl}_2$-module.

Theorem (B. Feigin–E. Feigin 2003)

An inductive limit of V_N^* is isomorphic to the basic representation $L_{0,1}$ of $\widehat{\mathfrak{sl}_2}$.

Finite-dimensional lemma

Decompose V_N^* into $\mathfrak{sl}_2$-irreducibles: $V_N^* = \bigoplus_{k=0}^{n} M_{2k+1} \otimes \mathcal{M}_k$.

Fusion product grading $\rightsquigarrow \mathcal{M}_k = \bigoplus_{i \geq 0} \mathcal{M}_k[i]$, where $\mathcal{M}_k[i]$ consists of elements of degree i.

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Lemma

There is a grading-preserving unitary isomorphism of $\mathfrak{sl}_2$-modules between $(V_N^*, \widetilde{\deg})$ and $((\mathbb{C}^2)^{\otimes N}, \mathrm{maj})$ such that $\mathcal{M}_k[i]$ is spanned by the standard Young tableaux τ of shape $(n+k, n-k)$ with $\mathrm{maj}(\tau) = i$.

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- Kedem 2004:
$$\sum_{i \geq 0} q^i \dim \mathcal{M}_k[i] = q^{\frac{N(N-1)}{2}} \cdot K_{(n+k,n-k),1^N}(1/q),$$
where $K_{\lambda,\mu}$ is the Kostka–Foulkes polynomial.
- Lascoux–Schützenberger combinatorial description of $K_{\lambda,\mu}$: for two-row λ,
$$K_{\lambda,1^N}(q) = \sum_{\tau \text{ of shape } \lambda} q^{c(\tau)},$$
where $c(\tau)$ is the charge of τ, defined as the sum of $i \leq N-1$ such that in τ the element $i+1$ lies to the right of i.

Finite-dimensional lemma

Lemma

There is a grading-preserving unitary isomorphism of $\mathfrak{sl}_2$*-modules between* $(V_N^*, \widetilde{\deg})$ *and* $((\mathbb{C}^2)^{\otimes N}, \mathrm{maj})$ *such that* $\mathcal{M}_k[i]$ *is spanned by the standard Young tableaux* τ *of shape* $(n+k, n-k)$ *with* $\mathrm{maj}(\tau) = i$*.*

Feigin–Feigin theorem + approximation argument $\Longrightarrow$

Theorem

There is a grading-preserving unitary isomorphism of $\mathfrak{sl}_2$*-modules between* $(L_{0,1}, \deg_H)$ *and* $(\Pi, \deg_s)$*.*

Fock space: fermions

Fermionic Fock space $\mathcal{F} = \bigwedge^{\frac{\infty}{2}}(V)$: infinite wedge space over V with basis $\{u_k\}_{k\in\mathbb{Z}} \cup \{v_k\}_{k\in\mathbb{Z}}$, spanned by

$$u_{i_1} \wedge \ldots \wedge u_{i_k} \wedge v_{j_1} \wedge \ldots \wedge v_{j_l} \wedge u_N \wedge v_N \wedge u_{N-1} \wedge v_{N-1} \wedge \ldots,$$
$$N \in \mathbb{Z}, \quad i_1 > \ldots > i_k > N, \quad j_1 > \ldots > j_l > N.$$

Fermions ϕ_k (resp. ψ_k) = exterior multiplication by u_k (resp. v_k).

Canonical anticommutation relations:

$$\{\phi_n, \phi_m^*\} = \delta_{nm}, \quad \{\psi_n, \psi_m^*\} = \delta_{nm}, \quad \text{others} = 0.$$

Generating functions:

$$\phi(z) = \sum_{i\in\mathbb{Z}} \phi_i z^{-(i+1)}, \qquad \phi^*(z) = \sum_{i\in\mathbb{Z}} \phi_i^* z^i, \quad \text{the same for } \psi.$$

Vacuum vector in $\mathcal{F}$:

$$\Omega = u_{-1} \wedge v_{-1} \wedge u_{-2} \wedge v_{-2} \wedge \ldots.$$

Fock space: free bosons and the vacuum

Bosons:

$$a_n^{\phi} = \sum_{k \in \mathbb{Z}} \phi_k \phi_{k+n}^* \text{ for } n \neq 0;$$

$$a_0^{\phi} = \sum_{n=1}^{\infty} \phi_n \phi_n^* - \sum_{n=0}^{\infty} \phi_{-n}^* \phi_{-n}.$$

Canonical commutation relations:

$$[a_n^{\phi}, a_m^{\phi}] = m\delta_{m+n,0}, \quad [a_n^{\psi}, a_m^{\psi}] = m\delta_{m+n,0}.$$

Generating functions: $a^{\phi}(z) = \sum_{n \in \mathbb{Z}} a_n^{\phi} z^{-(n+1)}$, the same for a_n^{ψ}.

Fock space realization of $L_{0,1}$

Given $x \in \mathfrak{sl}_2$, denote $X(z) = \sum_{n \in \mathbb{Z}} x_n z^{-(n+1)}$.

Canonical representation of $\widehat{\mathfrak{sl}_2}$ in $\mathcal{F}$:

$$E(z) = \psi(z)\phi^*(z), \qquad F(z) = \phi(z)\psi^*(z),$$
$$h_n = a^{\psi}_{-n} - a^{\phi}_{-n}, \qquad c = 1.$$

Then

$$\boxed{\mathcal{F} = \mathcal{H}_0 \otimes \mathcal{K}_0 + \mathcal{H}_1 \otimes \mathcal{K}_1},$$

where $\mathcal{H}_0 \simeq L_{0,1}$ and $\mathcal{H}_1 \simeq L_{1,1}$.

Recall: $L_{0,1}$ is isomorphic to the serpentine representation of $\mathfrak{S}_\infty$.

Heisenberg algebra

$\frac{1}{\sqrt{2}} h_n$ generate the Heisenberg algebra $\rightsquigarrow$ we have a representation of the Heisenberg algebra in $\mathcal{H}_0$ $\rightsquigarrow$ irreducible decomposition

$$\mathcal{H}_0 = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_0[2k],$$

where $\mathcal{H}_0[2k] = \{v \in \mathcal{H}_0 : h_0 v = 2kv\}$ is the charge $2k$ subspace.

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Corollary

The multiplicity-free serpentine representation $\Pi[0] := \bigoplus_{k=0}^{\infty} \Pi_k$ (spanned by all serpentine tableaux) has a structure of the zero charge representation of the Heisenberg algebra.

Thus we have an action of the Heisenberg algebra on serpentine tableaux.

Virasoro algebra

Virasoro algebra Vir: generated by L_n, $n \in \mathbb{Z}$, and a central element c,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c.$$

Sugawara construction: given free bosons a_n $(= \frac{1}{\sqrt{2}}h_n)$,

$$L_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-j}a_{j+n}, \quad n \neq 0; \quad L_0 = \sum_{j=1}^{\infty} a_{-j}a_j.$$

Thus we have a representation of Vir in $\mathcal{H}_0$.

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Thus we have a representation of Vir in $\mathcal{H}_0$.

So, in the Fock space:

- $\widehat{\mathfrak{sl}_2}$: $\mathcal{F} = \mathcal{H}_0 \otimes \mathcal{K}_0 + \mathcal{H}_1 \otimes \mathcal{K}_1$, where $\mathcal{H}_0 \simeq L_{0,1} \simeq \Pi$.
- Heisenberg: $\mathcal{H}_0 = \bigoplus_{k\in\mathbb{Z}} \mathcal{H}_0[2k]$, where $\mathcal{H}_0[2k]$ = charge $2k$ subspace.
- Virasoro: $\mathcal{H}_0 = \ldots$.

Virasoro algebra and k-serpentine representation

Vir and $\mathfrak{sl}_2 \subset \widehat{\mathfrak{sl}_2}$ are mutual commutants in $\mathcal{H}_0$, and

$$\boxed{\mathcal{H}_0 = \bigoplus_{k=0}^{\infty} M_{2k+1} \otimes L(1, k^2)},$$

where $L(1, k^2)$ = irreducible lowest weight rep of Vir with central charge (eigenvalue of c) 1 and conformal dimension (eigenvalue of L_0) k^2 (and M_{2k+1} are still irreducible representations of $\mathfrak{sl}_2$).

Virasoro algebra and k-serpentine representation

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Cf. for the serpentine representation:

$$\boxed{\Pi = \sum_{k=0}^{\infty} M_{2k+1} \otimes \Pi_k}.$$

Virasoro algebra and k-serpentine representation

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Corollary

The k-serpentine representation of the infinite symmetric group has a natural structure of the Virasoro module $L(1, k^2)$.

An action of Vir in Π_k, an action of $\mathfrak{S}_\infty$ in $L(1, k^2)$.

Virasoro algebra and k-serpentine representation

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Corollary

In this realization of $L(1, k^2)$, the Young basis of Π_k (consisting of the Young tableaux tail-equivalent to τ_k) is the eigenbasis of L_0, the eigenvalues being given by the stable major index:

$$L_0 \tau = \mathrm{smaj}(\tau) \tau.$$

A representation-theoretic meaning of maj!

Symmetric functions realization

Λ := algebra of symmetric functions.

Boson-fermion correspondence: $\Lambda \leftrightarrow \mathcal{H}_0[0] \simeq \Pi[0]$.

Representation of the Heisenberg algebra in Λ:

$$h_n \leftrightarrow 2n\frac{\partial}{\partial p_n}, \quad h_{-n} = p_n, \qquad n > 0,$$

where p_j are Newton's power sums.

Inner product: $\langle p_\lambda, p_\mu \rangle_2 := \delta_{\lambda\mu} \cdot z_\lambda \cdot 2^{\ell(\lambda)}$, where $z_\lambda = \prod_i i^{m_i} m_i!$ for $\lambda = (i^{m_i})$ and $\ell(\lambda)$ is the number of nonzero rows in λ.

The isomorphism in more detail

$\Phi :=$ isomorphism between $\Pi[0] = \bigoplus_{k=0}^{\infty} \Pi_k$ and Λ:

a serpentine tableau $\tau \mapsto$ a symmetric function with $\deg = \mathrm{smaj}(\tau)$.

Corollary

- *The vacuum tableaux correspond to Schur functions with square Young diagrams:* $\boxed{\Phi(\tau_k) = \mathrm{const} \cdot s_{(k^k)}}$.
- $\Pi^{(N)} :=$ *the subspace in* $\Pi[0]$ *spanned by the infinite tableaux coinciding with some vacuum tableau from the* N*th level. Then* $\boxed{\Phi(\Pi^{(2k)}) = \Lambda_{k\times k}}$, *where* $\Lambda_{k\times k}$ *is the subspace in* Λ *spanned by the Schur functions indexed by Young diagrams in the* $k \times k$ *square.*

The isomorphism in more detail

Let $F_{2n} = \mathbb{C}[e_0, \ldots, e_{-(2n-1)}]\Omega_{-2n}$ and $F_{2n}[0] = F_{2n} \cap \mathcal{H}_0[0] \implies \Pi^{(2n)} \simeq F_{2n}[0] \simeq \Lambda_{n\times n}$.

Lemma

A basis in $F_{2n}[0]$ is $\{\prod e_0^{i_0} e_{-1}^{i_1} \ldots e_{-n}^{i_n} : i_0 + i_1 + \ldots + i_n = n\}\Omega_{-2n}$.

Theorem

The correspondence between the Schur function basis in $\Lambda_{n\times n}$ and the above basis in $\Pi^{(2n)} \simeq F_{2n}[0]$ is given by the following formula: for $\nu \subset (n^n)$,

$$s_\nu = \sum_{\mu\subset(n^n)} \frac{K_{\nu\mu}}{\prod_{j=0}^n r_j!} \cdot e_{-(n-\mu_1)} \ldots e_{-(n-\mu_n)}\Omega_{-2n},$$

where $\mu = (0^{r_0}1^{r_1}2^{r_2}\ldots)$ and $K_{\lambda\mu}$ are Kostka numbers.

Examples

smaj(τ)	τ	$\Phi(\tau)$
0	τ_0	$1 = s_\emptyset$
1	$\tau_1 = \begin{smallmatrix}1 & 2\end{smallmatrix}$	$p_1 = s_{(1)}$
2	$\begin{smallmatrix}1 & 2 & 4 \\ 3\end{smallmatrix}$	p_2
2	$\begin{smallmatrix}1 & 2 \\ 3 & 4\end{smallmatrix}$	p_1^2
3	$\begin{smallmatrix}1 & 2 & 3 \\ 4\end{smallmatrix}$	$p_1^3 - p_3$
3	$\begin{smallmatrix}1 & 2 & 4 & 6 \\ 3 & 5\end{smallmatrix}$	$p_1^3 + 8p_3$
3	$\begin{smallmatrix}1 & 2 & 4 \\ 3 & 5 & 6\end{smallmatrix}$	$p_1 p_2$
4	$\tau_2 = \begin{smallmatrix}1 & 2 & 3 & 4\end{smallmatrix}$	$p_1^4 + 3p_2^2 - 4p_1p_3 = s_{(2^2)}$
4	$\begin{smallmatrix}1 & 3 & 4 \\ 2 & 5 & 6\end{smallmatrix}$	$p_1^4 - 3p_2^2 + 2p_1p_3$
4	$\begin{smallmatrix}1 & 2 & 4 & 6 \\ 3 & 5 & 7 & 8\end{smallmatrix}$	$p_1^4 + 12p_2^2 + 32p_1p_3$
4	$\begin{smallmatrix}1 & 3 & 4 & 6 \\ 2 & 5\end{smallmatrix}$	$p_1^2p_2 - p_4$
4	$\begin{smallmatrix}1 & 2 & 4 & 6 & 8 \\ 3 & 5 & 7\end{smallmatrix}$	$p_1^2p_2 + 4p_4$

Open questions

- Explicit general formulas for $\Phi(\tau) \rightsquigarrow$ a family of symmetric functions orthonormal with respect to $\langle \cdot, \cdot \rangle_2$.
- Explicit formulas for the Virasoro (Heisenberg, $\widehat{\mathfrak{sl}_2}$) action on infinite serpentine Young tableaux (or, conversely, for the $\mathfrak{S}_\infty$ action in the Fock space).

Thank you for your midsummer day attention!