Unipotent characters and ℓ-adic sheaves.

coming from sheaf-f-n corr-nce (with D. Kazhdan, Y. Varshavsky)

Goal: geometric way to write characters of p-adic grps.
(Unipotent

Recall the answer for finite Chevalley groups.
Lusztig's theory of Character sheaves. $\underline{G}$ - reductive $/\mathbb{F}_q$.

1) $G = \underline{G}(\mathbb{F}_q)$

1 a) $\underline{G} = GL_n$.

Then to an ρ - irr. repr-n of G there corresponds an irreducible perverse sheaf on $\underline{G}$, $\mathcal{F}_\rho \in \mathrm{Perv}_{\underline{G}}(\underline{G})$

$$\cap$$

$$D_{\underline{G}}(\underline{G})$$

$$\chi_\rho(g) = \mathrm{Tr}(\mathrm{Fr}, \mathcal{F}_\rho|_g)$$

$$\underline{X_\rho = \mathrm{Tr}(Fr, \mathcal{F}_\rho)}.$$

<u>Example</u> $\rho \subset \mathbb{C}[G/B]$.

Such $\rho \longleftrightarrow$ with irr. reps of $H = \mathbb{C}[B \backslash G / B] \cong \mathbb{C}[W] = S_n$

$$\rho \longleftrightarrow \bar{\rho} \in \mathrm{Irr}(W), \quad W = S_n$$

$$\mathcal{F}_\rho = Spr \otimes_{\mathbb{C}[W]} \rho.$$

$$Spr = \pi_* \, \underline{\overline{\mathbb{Q}}_\ell} \, [d].$$

constant sheaf

$$\pi : \tilde{G} \longrightarrow G, \quad \tilde{G} \subset G/B \times G$$

1 b) $\underline{G}$ - any split (connected center)

There is an assignment $\rho \rightsquigarrow \mathcal{F}_\rho$ - irreducible perverse sheaf

$\rho \in \mathrm{Irr}(G)$ (a character sheaf)

$\chi_\rho \quad \leftarrow - - \rightarrow \quad \mathrm{Tr}(\mathrm{Fr}, \mathcal{F}_\rho)$.

$\mathrm{Irr}(G) = \coprod$ families, for each family $\mathcal{R}$

one defines a finite group $\Gamma = \Gamma_{\mathcal{R}}$.

s.t. $\mathcal{R} \longleftrightarrow \mathrm{Irr}\ \mathrm{Sh}^{\Gamma}(\Gamma) = \{(\gamma, \psi) \mid \gamma \in \Gamma,\ \psi \in \mathrm{Irr}(Z_\Gamma(\gamma))\}$.

<u>Rmk</u> $K(\mathrm{Sh}^{\Gamma}(\Gamma)) \simeq \mathrm{Fun}^{\Gamma}(\mathrm{Comm}(\Gamma)) \circlearrowleft \Phi \quad \big[\ \gamma_1, \gamma_2 \leftrightarrow \gamma_2, \gamma_1$

$$\mathrm{Fun}^{\Gamma}(\mathrm{Comm}(\Gamma)), \qquad \mathrm{Comm}(\Gamma) = \{(\gamma_1, \gamma_2) \in \Gamma^2 \mid \gamma_1\gamma_2 = \gamma_2\gamma_1\}.$$

$[\mathcal{F}] \longmapsto \varphi_{\mathcal{F}}(\gamma_1, \gamma_2) = \mathrm{Tr}(\gamma_2, \mathcal{F}_{\gamma_1})$.

The transformation matrix between χ_ρ, $T\mathcal{U}(F_2, F_\rho)$

$\rho \in \mathcal{R}$

Thm (Lusztig, Shoji ---) is the matrix of Φ.

Ex Γ - abelian. $Sh^\Gamma(\Gamma) = \mathrm{Rep}(\Gamma \times \Gamma^\vee) \supset FT.$

self-dual abelian group.

Rmk 1) $CS \xrightarrow{\sim} Z(\mathcal{H}_f)$, Drinfeld center

$\mathcal{H}_f$ - finite Hecke category on $B \times B$ monodromic sheaves on G.

$$\mathcal{H}_f = D({}_B\backslash G/B)$$

in different contexts this is due to Lusztig. Ben-Zvi - Nadler, RB Finkelberg, Ostrik (D-modules)

Rmk 2. $\mathbb{C}[G]^G \xrightarrow{\sim} \mathbb{C}[G]_G$

$$\begin{array}{ccc} \mathbb{C}[G]^G & \xrightarrow{\sim} & \mathbb{C}[G]_G \\ \shortparallel & & \shortparallel \\ Z(\mathbb{C}[G]) & & \mathbb{C}[G]/[\,,\,] \end{array}$$

$Z(\mathbb{C}[G])$ — center

$\mathbb{C}[G]/[\,,\,]$ — cocenter $= HH_0$.

won't be true for p-adic groups.

---

2) Change notation! $G = \underline{G}(\mathbb{F}_q((t)))$. $\underline{G} = \underline{\underline{G}}(\mathbb{F}_q)$

$\underline{\underline{G}}$ — loop group
group ind-scheme

2a) $\underline{G} = GL_n$.

ρ — a (n irreducible) representation of G $\supset \underline{G}(\mathcal{O}) \supset I$ — Iwahori

$$\begin{array}{ccc} \underline{G}(\mathcal{O}) & \supset & I \\ \downarrow & & \downarrow \\ G & \supset & B \end{array}$$

$\rho^I \neq 0$, ρ is generated by ρ^I.

$G \supset G_c$ — union of compact subgroups.

Thm $\chi_\rho|_{G_c^{reg}} = \mathrm{Tr}(\mathrm{Fr}, \, Spr \overset{L}{\otimes}_{\mathbb{C}[W_{aff}]} \bar{\rho})$.

$\underline{G} = GL_n$.

$\rho \leftrightarrow$ rep of $H_{aff} = \mathbb{C}_c({}_I\backslash G/I) \underset{\text{degen-n}}{\rightsquigarrow} \mathbb{C}[W_{aff}]$

$\bar{\rho}$ - degeneration of ρ.

$Spr = \pi_* \, \underline{\bar{\mathbb{Q}}_e} \circlearrowleft W_{aff}$ affine Springer action. (Lusztig, Yun ...)

$$\pi : \underline{\underline{\widetilde{G}}} \longrightarrow \underline{\underline{G}}$$
$$\cap$$
$$\mathcal{H} \times \underline{\underline{G}}$$
$$\shortparallel$$
affine flag variety G/I

Idea of proof

It's enough for every parahoric P to prove equality after restricting to P and then also after taking push-forward to $\bar{P} = P/$prounip. radical. - reduces to the f.dim-statement.

2 b) G - general (split).

If H is a (reductive) grp $/\mathbb{C}$, $\mathrm{Comm}(H)$ - variety of comm-ng pairs

$$\mathcal{O}(\mathrm{Comm}(H))^H \supset \mathcal{O}_e(\mathrm{Comm}(H))^H = \{f \mid \forall y,\ f|_{Z(y)\times\{y\}} \text{ is loc. constant}\}$$

$$\mathcal{O}_2 = \{f \mid \forall x,\ f|_{\{x\}\times Z(x)}\}$$

$$\mathcal{O}_{e2} = \mathcal{O}_e \cap \mathcal{O}_2.$$

Recall G - p-adic grp. $S = C_c^\infty(G)$

$C(S) = S/[S,S]$, $\mathrm{Dist}(G) = C(S)^*$.

$\mathrm{Rep}(G) = \mathrm{Rep}(G)_u \oplus \mathrm{Rep}(G)_{nu}$

unipotent, nonunipotent

$\mathrm{Rep}(G)_u \supset \{\rho \mid \rho^I \neq 0\}$ & their L-packets.

$S = S_u \oplus S_{nu}$, $\Rightarrow$ $C(S) = C(S)_u \oplus C(S)_{nu}$

$$G = G_c \amalg G_{nc}$$

$$S(G) = S(G_c) \oplus S(G_{nc}).$$

$$C(S) = S_G = C(S)_c \oplus C(S)_{nc}$$

<u>Lemma</u> $C(S)_u = C(S)_{u,c} \oplus C(S)_{u,nc}$ (Thm under construction)

<u>Conj</u> There are canonical isomorphisms.

a) $$C_{u} = \bigoplus_{u \in \mathcal{U}/\check{G}} \mathcal{O}_Z\left(\mathrm{Comm}(Z_{red}(u)).\right)^{Z(u)_{red}}$$

b) $$C_{u,nc} = \bigoplus \mathcal{O}_{ez}.$$

$\check{G} \supset \mathcal{U}$ unipotent set

Langlands group

This agrees with characters & almost characters.

irr. unipotent reps $\longleftrightarrow$ (u, s, ψ)

$us = su,\ s, u \in \check{G}$

$\psi \in \mathrm{Irr\,Rep}(\pi_0(Z(s,u)))$.

s - s. simple, u - unipotent.

$\rho_{u,s,\psi}$ - <u>standard</u> repr-n.

$$\chi_{\rho_{u,s,\psi}} : C_{\mathrm{un}} \longrightarrow \mathbb{C}.$$

$$\begin{array}{c} \rotatebox{90}{$\in$} \\ f \longrightarrow \langle f_u |_{Z(s),s},\ \psi \rangle. \end{array}$$

$C_{\underline{u}.\underline{c}}$ has an obvious involution.

This sends character to almost character - f-n coming from a character sheaf.

<u>Rmk.</u> Partly inspired by <u>Lusztig's</u> paper on unipotent CS on loop groups.

---

cocenter of $\mathcal{H}$ - affine Hecke category.

$$\mathcal{H} \simeq D\,Coh^{G^\vee}(St)$$

$$\mathcal{Z}(\mathcal{H}) \simeq D\,Coh\,(\,Comm\,(\check{G}))$$

BZ - N - P

$$\Big\downarrow F_{s,u,\psi}$$

$$Vect$$

Conj b) $\Longrightarrow$ $C_{u,c}$ carries an involution Φ.

idea: Φ : characters of standard mod/G_c

to almost characters =

f-n coming from CS.

Rmk Bouthier - Kazhdan - Varshavsky

define $\mathrm{Perv}^{\underline{G}}(\underline{G})$

we expect these CS to lie in $\mathrm{Perv}^{\underline{G}}(\underline{G})$

with Varshavsky have a similar result for cuspidal depth 0 L-packets.

for cuspidal depth 0 L-packets.