Bernstein conf - actual talk

Monday, May 11, 2020 8:53 AM

Dennis Gaitsgory and I became Bernstein's students in the fall of 1991. During the 1st year he tried to teach us the following things:

1. Derived categories
2. Quantum field theory
3. D-modules
4. Langlands program - special emphasis on class field theory.

Main point today: Combine all these in some new way.

Plan:

1. Review of geometric local class field theory (Laumon '96)
2. Series of new conjectural generalizations. Surprising fact: they "follow from physics" (3d $N=4$ QFT), mathematicians did not come up with them.
3. Discussion of known cases.

1: Local geometric CFT

$K = \mathbb{C}((t)) \supset \mathcal{O} = \mathbb{C}[[t]]$

Theorem:

$$\text{D-mod}(K^*) \simeq \text{QCoh}(LS_1(\mathcal{D}^*))$$

(The LHS of this equivalence should be thought of as a categorical analog of $\mathbb{C}_c(\mathbb{F}_q((t))^*)$)

The RHS is the following. Let $D = \operatorname{Spec} \mathcal{O}$, $\mathcal{D}^* = \operatorname{Spec} K$

let LS_1 = de Rham local on $\mathcal{D}^*$ of rank 1

Explicitly, let $\Omega^1_{\mathcal{D}^*} = \{a(t)dt \mid a(t) \in K\}$

Then $LS_1(\mathcal{D}^*) = \Omega^1_{\mathcal{D}^*} / K^*$ where the action is $f: \omega \mapsto \omega + df/f$.

Want: Generalize this theorem in several directions.

1st direction: Note that $K^* = GL(1, K)$

Replace $GL(1)$ by $GL(n)$

Want: Interpret $\text{D-mod}(GL(n, K))$ in some quasi-coherent terms

Want: Interpret D-mod$(GL(n,K))$ in some quasi-coherent terms.

D-mod$(GL(n,K))$ = ?

D-mod$(Gr_{GL(n)})$ = ?

Whit$(GL(n,K))$ = ?

Here $Gr_{GL(n)} = Gr_n$ = affine Grassmannian

2nd direction – example:

Theorem (Conjectured using 3d mirror symmetry proved recently by S. Raskin):

Let

$$Y = \{(\mathcal{E}, s) \mid \mathcal{E} \in LS_1, \ s\text{- flat section}\}$$

Then $$\text{D-mod}(K) \simeq \text{IndCoh}(Y)$$

Here derived categories are necessary (unlike above)

More generally, suppose we have a short exact seq.

$$1 \to T \to \mathbb{G}_m^n \to T_F \to 1 \quad (T_F = \text{"flavor symmetry"})$$

We can dualize it $\quad 1 \to T_F^\vee \to \mathbb{G}_m^n \to T^\vee \to 1$

Thus both T and $T_F^\vee$ act on $\mathbb{C}^n$.

Theorem

$$\text{D-mod}(K^n / T(K)) \simeq \text{IndCoh}_{\dots}(Y)$$

some singular support condition

where $Y = \{(\mathcal{E}, s) \mid \mathcal{E} \in LS_T(\mathcal{D}^*), \ s\text{- flat } \mathbb{C}^n\text{-valued section}\}$

Now back to the 1st direction

Now back to the 1st direction:

About D-mod(GL(n,K))

Main player on the spectral side

$$W_n = \{ \mathcal{E}_1 \to \mathcal{E}_2 \to \dots \to \mathcal{E}_n \mid \mathcal{E}_i \in LS_i(\mathcal{D}^*) \}$$

$\mathcal{E}_i$ - local system on $\mathcal{D}^*$ of rk i.

if all maps are injective we get a rank n local system with a complete flag. Let

$$\begin{tikzcd} W_n \arrow[d] & W_n^\circ \arrow[l] \arrow[d] \\ LS_n(\mathcal{D}^*) & pt/_{GL(n)} \arrow[l] \end{tikzcd}$$

trivial local system

Conjecture:

① $D\text{-mod}(GL(n,K)) \simeq \mathrm{IndCoh}\left(W_n \underset{LS_n}{\times} W_n\right)$

② $D\text{-mod}(Gr_n) \simeq \mathrm{IndCoh}(W_n^\circ)$

③ $\mathrm{Whit}(GL(n,K)) \simeq \mathrm{QCoh}(W_n)$

Digression on Whit: Let U_n be the standard max. unipotent subgroup.

$\chi_0 : U_n \to \mathbb{G}_a \qquad \chi_0(u) = \sum u_{i,i+1}$

Then we have $\chi : U_n(K) \to \mathbb{G}_a$, $\chi = \mathrm{Res} \circ \chi_0$.

$\mathrm{Whit}(GL(n,K)) = (U_n(K), \chi)$-equivariant $\mathcal{D}$-modules

Relation of the conjecture to some classical

Relation of the conjecture to some classical things:

F - local non-archimedian field

$Whit(GL(n,F))$ = direct integral of all non-degenerate irreducible rep's of $GL(n,F)$

$Whit(GL(n,K))$ — should be "direct integrall of non-degenerate irreducible categories with $GL(n,K)$-action.

Example $n=2$

$$W_2 = \{(\mathcal{E}_1 \to \mathcal{E}_2)\} \xrightarrow{\pi_2} LS_2(\mathcal{D}^*)$$

Fix $\mathcal{E}_2$ and assume it is irreducible. Then any $\mathcal{E}_1 \to \mathcal{E}_2$ is 0. In other words, we have

$\pi_2^{-1}(\mathcal{E}_2) = LS_1$. Hence the conjecture implies:

$Whit(GL(2,K))$ is a sheaf of categories over $LS_2(\mathcal{D}^*)$

fiber at $\mathcal{E}_2 \simeq QCoh(LS_1) \simeq Dmod(K^*)$

This is related to a well known classical statement:

Let V be an irreducible cuspidal rep. of $GL(2,F)$

Then $V \simeq \mathbb{C}_c(F^*)$ (as an F^*-module)

The map from LHS to RHS is easy (come from Whittaker model). Now we would like to categorify.

Whittaker model). Now we would like to categorify. The above conjecture says that if $\mathcal{C}$ is an irreducible cuspidal $GL(2,K)$-category then $\mathcal{C} \simeq \mathcal{D}\text{-mod}(K^*)$. The above operator is easy to categorify BUT it gives a functor to $\mathcal{D}\text{-mod}(K \setminus \{0\})$ which is very different from $\mathcal{D}\text{-mod}(K^*)$. We do have a map $K^* \to K \setminus \{0\}$ and the conjecture is that the above functor factorizes through the direct image w.r. to this map. This is non-trivial - I checked it in one example of a particular cuspidal irreducible $\mathcal{C}$.

What is known? Nothing if $n>2$.
For $n=2$ Parts 1 and 3 of the conjecture are not known either. Let us look at part 2.
We want:

$$\mathcal{D}\text{-mod}(GR_2) \simeq \mathrm{IndCoh}(W_2^{\circ})$$

LHS has an action of $\mathcal{D}\text{-mod}(K^*)$ (using the embedding $K^* \to GL(2,K)$ $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$)
RHS lives over $LS_1(\mathcal{D}^*)$. According to Laumon these are the same structures.
B-Finkelberg: Proved the above equivalence

B – Finkelberg: Proved the above equivalence fiberwise over LS_1 (i.e. equivalence of fibers) – this is published.
Recently S. Raskin (unpublished) proved the full statement.

Relation to other works:

1) Bezrukavnikov: G – any reductive group

$$D({}_I\backslash G(K)/I) \simeq \operatorname{IndCoh}(St/G^\vee)$$

St – (derived) Steinberg variety of $G^\vee$
Similar statement for $D(I\backslash G(K)/G(\mathcal{O}))$ (get the $G^\vee$-equivariant IndCoh of the derived fiber over 0 of the map $\tilde{\mathfrak{g}}^\vee \to \mathfrak{g}^\vee$).
How is this related to our conjecture?
I DON'T KNOW!

Note that if S denotes the above derived Springer fiber then $S/G^\vee$ is an open subset of W_n°

(corresponds to all maps $\Sigma_1 \to \Sigma_2 \to \dots$ being injective)

But I still don't know what to say next.

② Local geometric Langlands:

Roughly speaking it says for $GL(n)$ that categories with $GL(n, K)$-action should be the same as sheaves of categories over $LS_n(\mathcal{D}^*)$.

The former notion is the same categories with an action of $\mathcal{D}$-mod$(GL(n,K))$ and the latter notion is the same as categories with an action of $QCoh(LS_n(\mathcal{D}^*))$. So, the naive formulation says that the monoidal categories $\mathcal{D}$-mod$(GL(n,K))$ and categories $\mathcal{D}$-mod$(GL(n,K))$ and $(QCoh(LS_n(\mathcal{D}^\times))$ are Morita equivalent in some sense. Unfortunately, this is not quite right. Recently Arinkin gave a precise formulation ($\forall G$) – it is too long to reproduce here.

On the other hand, we conjecture an equivalence

$$D\text{-mod}(GL(n,K)) \simeq IndCoh\left(W_n \underset{LS_n}{\times} W_n\right)$$

It should be possible to see that this equivalence implies Arinkin's conjecture (but I haven't done it).