Hodge theory for non-Archimedean analytic spaces

$\mathbb{K}$ the fraction field of $\mathcal{O}_{\mathbb{C},0}$

$\mathfrak{X}$ proper scheme $/\mathbb{K}$

$\rightsquigarrow \mathfrak{X}^h \xrightarrow{} \mathbb{D}^* \subset \mathbb{C}^*$

topological fibration

$\mathbb{D}^* \ni t \longmapsto H^q(\mathfrak{X}^h_t, \mathbb{Z})$ - local system of mixed Hodge structure (MHS)

$\rightsquigarrow H^q(\overline{\mathfrak{X}^h}, \mathbb{Z})$ limit MHS

$$\begin{tikzcd} \mathcal{X}^h \arrow[r] & D^* \arrow[r, phantom, "\subset"] & \mathbb{C}^* \\ \overline{\mathcal{X}^h_D} \arrow[u] \arrow[r] & \overline{D^*} \arrow[u] \arrow[r] & \mathbb{C} \arrow[u, "\exp"'] \end{tikzcd}$$

$$H^q(\overline{\mathcal{X}^h}, \mathbb{Z}) = \varinjlim_{D \ni 0} H^q(\overline{\mathcal{X}^h_D}, \mathbb{Z})$$

$$W_0 H^q(\overline{\mathcal{X}^h}, \mathbb{Q})$$

$K \longrightarrow \hat{K}$ - non-Archimedean field $\simeq \mathbb{C}((z))$

$\mathcal{X} \rightsquigarrow \mathcal{X}^{an} = (\mathcal{X} \otimes_K \hat{K})^{an}$ - proper $\hat{K}$-analytic space

$\overline{\mathcal{X}^{an}} := (\mathcal{X} \otimes_K \hat{K}^a)^{an}$ — " — $\hat{K}^a$-analytic space

Fact: $H^q(|\overline{\mathcal{X}^{an}}|, \mathbb{Q}) \simeq W_0 H^q(\overline{\mathcal{X}^h}, \mathbb{Q})$

<u>Briefly</u>:

One can define $\forall$ compact $\hat{K}$-analytic space X

$\rightsquigarrow H^q(\overline{X}, \mathbb{Z})$ fin. generated
with a quasi-unipotent action of $\hat{\pi}_1(\mathbb{C}^*)$
functorial in X

if X is proper, it is provided with a MHS functorial in X

s.t. if $X = \mathcal{X}^{an}$ as above, it is $H^q(\overline{\mathcal{X}^h}, \mathbb{Z})$ + limit MHS

Recall: the construction of the limit MHS depends on the choice of a nonzero element of $\mathfrak{m}_0/\mathfrak{m}_0^2$, $\mathfrak{m}_0 \subset \mathcal{O}_{\mathbb{C},0}$ maximal ideal,
W does not depend
F does depend

$\mathbb{K}$ non-Archimedean field with discrete valuation

$$\begin{array}{l} \mathbb{C} \subset \mathbb{K}^0 = \{a \in \mathbb{K} \mid |a| \leq 1\} \\ \qquad\ \ \cup \\ \qquad\ \ \mathbb{K}^{00} = \{a \in \mathbb{K}^0 \mid |a| < 1\} \\ \qquad\ \ \widetilde{\mathbb{K}} := \mathbb{K}^0 / \mathbb{K}^{00} \end{array}$$

$\mathbb{C} \xrightarrow{\sim} \widetilde{\mathbb{K}}$

Example $\hat{\mathbb{K}} = \mathbb{C}((z))$

if ω is a generator of $\mathbb{K}^{00}$

$$\hat{\mathbb{K}} \xrightarrow{\sim} \mathbb{K}$$
$$z \longmapsto \omega$$

$\rightsquigarrow$ $\mathbb{K}_1$ - complex analytic space $\mathbb{K}^{00}/(\mathbb{K}^{00})^2$

$\cup$

$\mathbb{K}_1^* = \mathbb{K}^{00}/(\mathbb{K}^{00})^2 \setminus \{0\}$ - principal homogeneous space for $\mathbb{C}^*$

$\forall \omega \longmapsto$ a universal covering $\Omega^{(\omega)}$ of $\mathbb{K}_1^*$

$\mathbb{K}^{(\omega)}$ an algebraic closure of $\mathbb{K}$

functors on the fundamental groupoid $\Pi(\mathbb{K}_1^*)$

"classifying space" for those

$\Pi(\mathbb{K}_1^\ast)$ the fundamental groupoid of $\mathbb{K}_1^\ast$

ω_1, ω_2 $\quad \Pi(\mathbb{K}_1^\ast, \omega_1, \omega_2) = \{\gamma \in \Gamma \mid \exp(\gamma) = \frac{\omega_1}{\omega_2}\}$

$\gamma \rightsquigarrow [0,1] \to \mathbb{K}_1^\ast : t \mapsto \exp(-t\gamma)\omega_1$

The functor $\Pi(\mathbb{K}_1^\ast) \to G(K) : \omega \mapsto K^{(\omega)}$, algebraic closure of K

$\rightsquigarrow$ {étale sheaves on $\mathrm{Spec}(K)$} $\to \mathcal{L}con(\mathbb{K}_1^\ast)$ locally constant sheaves

$\Lambda \longmapsto (\omega \mapsto \Lambda(K^{(\omega)}))_\omega$

X - K-analytic space

$X^{(\omega)} := X \hat{\otimes}_K \widehat{K}^{(\omega)}$. $\widehat{K}^{(\omega)}$-analytic space

$\forall$ étale abelian sheaf Λ on $\mathrm{Spec}(K)$, $\forall q \geq 0$

$\omega \longmapsto H^q_{ét}(X^{(\omega)}, \Lambda(K^{(\omega)}))$

$\rightsquigarrow \mathcal{H}^q_{ét}(\overline{X}, \Lambda)$ local system on $\mathbb{K}_1^\ast$

<u>Fact</u>: One can construct $\forall$ compact (strictly) K-analytic space X, $\forall q \geq 0$

$$\mathcal{L}con(\mathbb{K}_1^*) \longrightarrow \mathcal{L}con(\mathbb{K}_1^*) \quad \text{(abelian)}$$

$$\Lambda \longmapsto \mathcal{H}^q(\overline{X}, \Lambda)$$

such that (1) if Λ is finite étale on $\mathrm{Spec}(K)$

$$\mathcal{H}^q(\overline{X}, \Lambda) \xrightarrow{\sim} \mathcal{H}^q_{ét}(\overline{X}, \Lambda)$$

(2) $\mathcal{H}^q(\overline{X}, \mathbb{Z})$ is a quasi-unipotent local system of fin. generated ab. groups

(3) if $X = \mathcal{X}^{an}$ as above, $\mathcal{X}$ proper$/\mathcal{K} \ni z$

$$\mathcal{H}^q(\overline{X}, \mathbb{Z})_z = H^q(\overline{X^h}, \mathbb{Z})$$

$\Rightarrow \forall$ prime l, $\mathcal{H}^q(\overline{X}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_l \xrightarrow{\sim} \mathcal{H}^q_{ét}(\overline{X}, \mathbb{Z}_l)$

de Rham cohomology (another ingredient)

if X is rig-smooth, $H^q_{dR}(X/K) := R^q\Gamma(X, \Omega^{\bullet}_{X/K})$

Gauss-Manin connection $+ \nabla : H^q_{dR}(X/K) \to H^q_{dR}(X/K) \otimes_K \Omega^1_K$

Facts: (1) If X is compact $\Rightarrow \dim_K(H^q_{dR}(X/K)) < \infty$

& $\exists\, H^q_{dR}(X/K^0) \subset H^q_{dR}(X/K)$ a lattice + Gauss-Manin

$\nabla : H^q_{dR}(X/K^0) \to H^q_{dR}(X/K^0) \otimes_{K^0} \omega^1_{K^0}$, where

$\omega^1_{K^0}$ is the K^0-module of log differentials

$= K^0\, d\log(\varpi)$ $\forall$ generator ϖ of K^{00}

(2) One can extend the above to a functor on compact K-analytic spaces

$$X \longmapsto (H^q_{dR}(X/K^0), \nabla)$$

$$H^q_{dR}(X/K^\circ_1) := H^q_{dR}(X/K^\circ) \big/ (K^{\circ\circ})$$ - fin. gen. $\mathbb{C}$-vector space

$+ \ \delta : \circlearrowright H^q_{dR}(X/K^\circ_1)$ the residue of the connection ∇

Defn A distinguished de Rham structure on $\mathbb{K}^\ast_1$ is a triple $\mathcal{H} = (\mathcal{H}_{\mathbb{Z}}, E, \alpha)$ where

(1) $\mathcal{H}_{\mathbb{Z}}$ is a local system of fin. gen. ab. groups on $\mathbb{K}^\ast_1$

(2) E - fin. dim. $\mathbb{C}$-vector space with an operator $\delta : E \to E$ whose eigenvalues lie in $\mathbb{Q} \cap [0,1)$

(3) α an isomorphism of vector bundles with ∇

$$\alpha : \mathcal{O}_{\mathbb{K}^\ast_1} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{O}_{\mathbb{K}^\ast_1} \otimes_{\mathbb{C}} E, \quad x \in E, \ \nabla(x) = \delta(x)\, d\log(z)$$

$$\rightsquigarrow \forall w \in \mathbb{K}^\ast_1 : \ \alpha^{(w)} : \mathcal{H}_{\mathbb{C},w} = \mathcal{H}_{\mathbb{Z},w} \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} E$$

(isomorphism of fibers)

<u>Fact</u> $\forall \gamma \in \Pi(\mathbb{K}_1^{A}, \omega_1, \omega_2)$: $\exp(\gamma) = \frac{\omega_1}{\omega_2}$

$\rightsquigarrow$ commutative diagram

$$\begin{tikzcd} \mathcal{H}_{\mathbb{Z},\omega_1} \arrow[r, "T_\gamma"] \arrow[d, "\alpha^{(\omega_1)}"] & \mathcal{H}_{\mathbb{Z},\omega_2} \arrow[d, "\alpha^{(\omega_2)}"] \\ E \arrow[r, "\exp(-\gamma\delta)"] & E \end{tikzcd}$$

$\Rightarrow \mathcal{H}_{\mathbb{Z}}$ is a quasi-unipotent local system

<u>Example</u> $\gamma = 2\pi i : \omega \to \omega \Rightarrow \sigma = \exp(-2\pi i \delta)$

<u>Thm</u> $\forall$ compact K-analytic space X, $\forall q \geq 0$

$\rightsquigarrow$ distinguished de Rham structure

$$(\mathcal{H}^q(\bar{X}, \mathbb{Z}), H^q_{dR}(X/K_1^0), \alpha)$$

functorial in $\bar{X}$

<u>Defn</u> A distinguished mixed Hodge structure on $\mathbb{K}_1^*$ is a triple $\mathcal{H} = (\mathcal{H}_{\mathbb{Z}}, (\mathcal{H}_{\mathbb{Q}}, W), (E, W, F), \alpha)$, where

(1) $(\mathcal{H}_{\mathbb{Z}}, E, \alpha)$ is a distinguished de Rham structure with a finite increasing filtration W on $(\mathcal{H}_{\mathbb{Q}}, E, \alpha)$

(2) F a finite decreasing filtration of E by $\mathbb{C}$-vector subspaces

s.t. if $\delta = \delta_{ss} + \delta_{ne}$ is the additive Jordan decomposition

then

(a) $\delta_{ss}(W_k) \subset W_k$ & $\delta_{ne}(W_k) \subset W_{k-2}$

(b) $\delta_{ss}(F^k) \subset F^k$ & $\delta_{ne}(F^k) \subset F^{k-1}$ $\Big\}\ \forall k$

(c) $\forall \omega \in \mathbb{K}_1^*$, the triple $\mathcal{H}_\omega = (\mathcal{H}_{\mathbb{Z},\omega}, W_\omega, F_\omega)$ is MHS

$\left(\alpha^{(\omega)} : \mathcal{H}_{\mathbb{C},\omega} \xrightarrow{\sim} E \right.$
$\left. F_\omega = (\alpha^{(\omega)})^{-1}(F) \right)$

Thm $\forall$ <u>proper</u> K-analytic space X

$$\rightsquigarrow \mathcal{H}dg^q(X/K_1^0) := \left(H^q(\bar{X}, \mathbb{Z}), (H^q(\bar{X}, \mathbb{Q}), W), (H^q_{dR}(X/K_1^0), W, F), \alpha\right)$$

a distinguished mixed Hodge structure

s.t. (1) functorial in X

(2) $\forall$ finite extension K'/K, $\exists$ an isomorphism

$$\underbrace{\mathcal{H}dg^q(X/K_1'^0)\Big|_{K_1^0}}_{\text{(non-evident defn)}} \xrightarrow{\sim} \mathcal{H}dg^q(X/K_1^0)$$

(3) if $X = \mathfrak{X}^{an}$ for $\mathfrak{X}$ proper $/\mathbb{K}$

then $\mathcal{H}dg^q(X/\hat{\mathbb{K}}_1^0)_{\mathbb{Z}} = \left(H^q(\overline{\mathfrak{X}^h, \mathbb{Z}})\right)$ with limit mixed Hodge structure