Three observations regarding Schatten p classes *

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Abstract

The paper contains three results, the common feature of which is that they deal with the Schatten p class. The first is a presentation of a new complemented subspace of C_p in the reflexive range (and $p \neq 2$). This construction answers a question of Arazy and Lindestrauss from 1975. The second result relates to tight embeddings of finite dimensional subspaces of C_p in C_p^n with small n and shows that ℓ_p^k nicely embeds into C_p^n only if n is at least proportional to k (and then of course the dimension of C_p^n is at least of order k^2). The third result concerns single elements of C_p^n and shows that for $p > 2$ any $n \times n$ matrix of C_p norm one and zero diagonal admits, for every $\varepsilon > 0$, a k-paving of C_p norm at most ε with k depending on ε and p only.

1 Introduction

1.1 Complemented subspaces

Recall that for $1 \leq p < \infty$ C_p denotes the Banach space of all compact operators A on ℓ_2 for which the norm $\|A\|_p = (\mathrm{trace(A^*A)^{p/2}})^{1/\mathrm{p}}$ is finite. Determining the complemented subspaces of these spaces was a subject of investigation for quite a while. In particular Arazy and Lindenstrauss in [AL] list nine isomorphically distinct infinite dimensional complemented subspaces of C_p, $1 < p \neq 2 < \infty$. They are all complemented by quite natural projections which are given by setting certain subsets of the entries of a given

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matrix to zero. They are also all norm one projections. Unlike the situation with L_p there is some hope of isomorphically characterizing all complemented subspaces of C_p. The main result of section 2 of this paper is to introduce a new complemented subspace of C_p, solving some problems from [AL]. To describe the space we need some more notations. For a finite or infinite matrix A we denote by $A(k,l)$ its k,l entry. Z_p will denote the Banach space of matrices whose norm

$$\|A\|_{Z_p} = (\sum_{k=1}^{\infty}(\sum_{l=1}^{\infty}|A(k,l)|^2)^{p/2})^{1/p}$$

is finite. For $p>2$, $\widetilde{Z_p}$ will denote the Banach space of matrices whose norm

$$\|A\|_{\widetilde{Z_p}} = (\|A\|_{Z_p}^p + \|A^*\|_{Z_p}^p)^{1/p}$$

is finite. For $1 \le q < 2$, $\widetilde{Z_q}$ will denote the Banach space of matrices whose norm

$$\|A\|_{\widetilde{Z_q}} = \inf\{(\|B\|_{Z_q}^q + \|C^*\|_{Z_q}^q)^{1/q}\ ;\ A = B + C\}$$

is finite. Note that if $q = p/(p-1)$, $p>2$, then $\widetilde{Z_q}$ is the dual of $\widetilde{Z_p}$ and vice versa.
We also denote by C_p^n, Z_p^n and $\widetilde{Z_p^n}$ the spaces of $n \times n$ matrices with the norms inherited from C_p, Z_p and $\widetilde{Z_p}$ respectively. Define

$$D_p = (\bigoplus_{n=1}^{\infty}\widetilde{Z_p^n})_p.$$

The main result in section 2 is that D_p is complemented in C_p, $1 < p < \infty$, and is not isomorphic to any of the previously known complemented subspaces of C_p.
The proof depends heavily on a result from [L-P] showing that $\widetilde{Z_p}$ is the "unconditional version" of C_p.

1.2 Tight Embeddings

Before describing the results of section 3 we would like to motivate it by describing what is known in the L_p case.
Given a k-dimensional subspace X of $L_p(0,1)$ one can ask what is the minimal

n such that X embeds with constant 2, say, into ℓ_p^n. This was extensively studied and, up to log factors basically solved. Except for a power of $\log k$, n can be taken to be of order k for $1 \le p < 2$ and of order $k^{p/2}$ for $2 < p < \infty$. These orders are best possible, again up to the log factor, as is seen by looking at $X = \ell_2^k$. See [JS] for a survey of these results. In the case that p is an even integer the log factor is not needed ([Sc]).
It is natural to seek similar results in C_p. Given a k-dimensional subspace X of C_p what is the smallest n such that X 2-embeds into C_p^n? If X is ℓ_2^k the answer is known for quite a while: $n \approx \sqrt{k}$ for $1 \le p < 2$ and $n \approx_p k^{\frac{p}{p+2}}$ for $2 < p < \infty$ [FLM]. One could guess that also here the worst case is ℓ_2^k. In section 3 we show that this is wrong and ℓ_p^k is worse than ℓ_2^k in this respect.

We prove in Theorem 2 that if C_p^n contains a 2-isomorphic copy of ℓ_p^k then $n \gtrsim_p k$. This holds for all $1 \le p \ne 2 < \infty$.
The proof is algebraic in nature, estimating the rank of some operator.

1.3 Paving

Recall that a k-paving of a $n \times n$ matrix A is a matrix of the form $\sum_{i=1}^k P_{\sigma_i} A P_{\sigma_i}$ where $\{\sigma_1, \dots, \sigma_k\}$ is a partition of $\{1, \dots, n\}$ and for a subset $\sigma \subseteq \{1, \dots, n\}$ $P_\sigma A P_\sigma$ is the matrix whose k, l term is $A(k, l)$ if both k and l are in σ and 0 otherwise.

Marcus, Spielman and Srivastava [MSS] solved recently the Kadison–Singer conjecture which is equivalent to the paving conjecture. In our terms they proved:

For each $\varepsilon > 0$ there is a $k = k(\varepsilon)$ such that for all n and all $n \times n$ matrix A with zero diagonal and norm one (as an operator on ℓ_2^n) there is a k-paving of norm at most ε.

Given a reasonable norm on the space of $n \times n$ matrices, one can ask if a similar result holds for that norm. In [BHKW] it was proved that this is the case for self adjoint matrices and the norms C_4 and C_6 (the case of C_2 is easy and was known before). The hope of the authors of [BHKW] was that one will be able to prove a similar result for all zero diagonal matrices for a sequence of C_{p_n} norms with $p_n \to \infty$ and where the function $k(\varepsilon)$ is independent of n. Then the paving conjecture would follow.

In Theorem 3 of section 4 we show that the paving conjecture indeed holds for all the C_p norms, $2 < p < \infty$. However, the function k that we get depends on p.

The proof in [BHKW] (for $p = 4, 6$) is by an averaging argument. Our proof is also quite a simple averaging argument. The difference with the argument in [BHKW] is that instead of estimating the average of $\|P_\sigma AP_\sigma\|$ over all possible σ from above we estimate the average of $\|A - P_\sigma AP_\sigma\|$ from below and then use the uniform convexity of C_p.

2 A new complemented subspace of C_p, $1 < p \neq 2 < \infty$

We start with stating two facts relating the spaces C_p, Z_p and $\widetilde{Z_p}$. The first is relatively simple:

Fact 1 *For $p > 2$, $\|A\|_p \geq \|A\|_{Z_p}$ and consequently also $2^{1/p}\|A\|_p \geq \|A\|_{\widetilde{Z_p}}$.*

Let us reproduce the (known) proof of this fact: Let $E_{k,l}$ denote the matrix with 1 in the k, l place and zeros elsewhere. Also for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k, \dots)$, with $\varepsilon_k = \pm 1$ for all k, denote $\Delta_\varepsilon = \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_k, \dots)$. By the unitary invariance of the C_p norm and the convexity, over positive definite matrices, of $B \to \mathrm{trace(B^{p/2})}$, we get that for all $A = \sum_{k,l} a_{k,l} E_{k,l}$,

$$\|A\|_p^p = \mathrm{Ave}_\varepsilon \|\Delta_\varepsilon A\|_p^p = \mathrm{Ave}_\varepsilon \mathrm{trace}(\Delta_\varepsilon \mathrm{AA}^* \Delta_\varepsilon)^{\mathrm{p}/2} \geq \mathrm{trace}[\mathrm{Ave}_\varepsilon \Delta_\varepsilon \mathrm{AA}^* \Delta_\varepsilon]^{\mathrm{p}/2}.$$

Now, as is easily seen,

$$\begin{aligned} \mathrm{Ave}_\varepsilon \Delta_\varepsilon AA^* \Delta_\varepsilon &= \sum_{k,l} (AA^*)(k,l) \mathrm{Ave}_\varepsilon \varepsilon_k \varepsilon_l E_{k,l} \\ &= \sum_k (AA^*)(k,k) E_{k,k} = \sum_k \sum_l |a_{k,l}|^2 E_{k,k} \end{aligned}$$

and

$$\mathrm{trace}(\sum_\mathrm{k} \sum_\mathrm{l} |\mathrm{a_{k,l}}|^2 \mathrm{E_{k,k}})^{\mathrm{p}/2} = \|\mathrm{A}\|_{\mathrm{Z_p}}^\mathrm{p}.$$

■

As was noted by the referee, one can also prove Fact 1 by interpolating between the simpler cases $p = 2$ and $p = \infty$. A dual statement (extending also to $q = 1$) holds for $1 \leq q < 2$. We'll not use this here.

By $\alpha \gtrsim_p \beta$ (resp. $\alpha \lesssim_p \beta$) we mean that for some positive constant K, depending only on p, $K\alpha \geq \beta$ (resp. $\alpha \leq K\beta$). $\alpha \approx_p \beta$ means $\alpha \gtrsim_p \beta$ and $\alpha \lesssim_p \beta$. If the subscript p is omitted the constant K is meant to be an absolute constant.

The second Fact is much more profound and follows from a result of Lust-Piquard [L-P] for $1 < p < \infty$ and of Lust-Piquard and Pisier [L-PP] for $p = 1$. See also [PX].

Fact 2 *Let $\{\varepsilon_{k,l}\}_{k,l=1}^{\infty}$ be a matrix of independent random variables each taking the values ± 1 with equal probability. For a matrix $A = \sum_{k,l=1}^{\infty} a_{k,l} E_{k,l}$ in C_p, $1 \leq p < \infty$,*

$$(\mathrm{Ave}_\varepsilon \| \sum_{k,l=1}^{\infty} \varepsilon_{k,l} a_{k,l} E_{k,l} \|_p^p)^{1/p} \approx_p \|A\|_{\widetilde{Z_p}}.$$

Let $\{R_i^n\}_{i=1}^{2^{n^2}}$ be a list of all the $n \times n$ sign matrices; i.e., all matrices each of whose entries is 1 or -1.

Given two matrices of the same dimensions A and B, we denote by $A \circ B$ their Schur product; i.e., the matrix whose i, j component is the product of the i, j component of A with the i, j component of B.

Given an $n \times n$ matrix A let $\overline{A}$ be the $n2^{n^2} \times n2^{n^2}$ matrix given by

$$\overline{A}(k,l) = \begin{cases} (R_i^n \circ A)(k - (i-1)n, l - (i-1)n), & (i-1)n < k, l \leq in, \\ & i = 1, \ldots, 2^{n^2} \\ 0 & \text{otherwise} \end{cases}$$

i.e., $\overline{A}$ is a block diagonal matrix with 2^{n^2} blocks each of size $n \times n$ and the i-th block is $R_i^n \circ A$. Note that by Fact 2 above, for all $1 \leq p < \infty$,

$$\|\overline{A}\|_p \approx_p 2^{n^2/p} \|A\|_{\widetilde{Z_p}}.$$

Set

$$D_p^n = \{\overline{A}\ ;\ A \in C_p^n\}, \text{ equipped with the norm induced by } C_p^{n2^{n^2}}.$$

Then for each $1 \leq p < \infty$, $A \to 2^{-n^2/p}\overline{A} = \overline{2^{-n^2/p}A}$ is an isomorphism, with constant depending only on p, between $\widetilde{Z_p^n}$ and D_p^n.

Proposition 1 *For $1 < p < \infty$ D_p^n is λ_p-complemented in $C_p^{n2^{n^2}}$ where λ_p depends on p only.*

Proof: For $B \in C_p^{n2^{n^2}}$ and $i = 1, \ldots, 2^{n^2}$ let B_i be the $n \times n$ central block of B supported on the coordinates in $((i-1)n, in]$; i.e.,

$$B_i(k,l) = B(k + (i-1)n, l + (i-1)n) \quad , \quad 1 \le k, l \le n.$$

For $B \in C_p^{n2^{n^2}}$ let

$$\Phi(B) = 2^{-n^2} \sum_{i=1}^{2^{n^2}} R_i^n \circ B_i$$

and define

$$Q(B) = \overline{2^{-n^2} \sum_{i=1}^{2^{n^2}} R_i^n \circ B_i}.$$

Hence $Q(B) = \overline{\Phi(B)} \in D_p^n$. Since for every $A \in C_p^n$

$$\Phi(\overline{A}) = 2^{-n^2} \sum_{i=1}^{2^{n^2}} R_i^n \circ R_i^n \circ A = A,$$

Q is obviously a projection onto D_p^n. It is also easy to see that it is self adjoint. It is therefore enough to evaluate its norm as an operator on C_p, $p > 2$. As we remarked above,

$$\begin{aligned} \|Q(B)\|_p &\approx_p 2^{n^2/p} \|2^{-n^2} \textstyle\sum_{i=1}^{2^{n^2}} R_i^n \circ B_i\|_{\widetilde{Z_p}} \\ &= 2^{n^2/p} (\|2^{-n^2} \textstyle\sum_{i=1}^{2^{n^2}} R_i^n \circ B_i\|_{Z_p}^p + \|2^{-n^2} (\textstyle\sum_{i=1}^{2^{n^2}} R_i^n \circ B_i)^*\|_{Z_p}^p)^{1/p}. \end{aligned} \tag{1}$$

We'll show that

$$2^{n^2/p} \|2^{-n^2} \sum_{i=1}^{2^{n^2}} R_i^n \circ B_i\|_{Z_p} \le \|B\|_p. \tag{2}$$

Since the treatment of the other term on the right hand side of (1) is identical this will prove the Proposition. (2) is really just the statement that Q is a norm 1 projection on Z_p but let us repeat the proof.

$$
\begin{aligned}
\| \sum_{i=1}^{2^{n^2}} R_i^n \circ B_i \|_{Z_p}^p &= \sum_{k=1}^n \| \sum_{i=1}^{2^{n^2}} (R_i^n \circ B_i)(k, \cdot) \|_{\ell_2}^p \\
&\leq \sum_{k=1}^n (\sum_{i=1}^{2^{n^2}} \|(R_i^n \circ B_i)(k, \cdot)\|_{\ell_2})^p \\
&\qquad \text{by the triangle inequality,} \\
&= \sum_{k=1}^n (\sum_{i=1}^{2^{n^2}} \|B_i(k, \cdot)\|_{\ell_2})^p \\
&\leq \sum_{k=1}^n 2^{n^2(p-1)} \sum_{i=1}^{2^{n^2}} \|B_i(k, \cdot)\|_{\ell_2}^p \\
&\qquad \text{by Holder's inequality,} \\
&\leq 2^{n^2(p-1)} \sum_{k'=1}^{n2^{n^2}} \|B(k', \cdot)\|_{\ell_2}^p \\
&\qquad \text{by denoting } k' = in + k, \\
&= 2^{n^2(p-1)} \|B\|_{Z_p}^p \leq 2^{n^2(p-1)} \|B\|_p^p \\
&\qquad \text{by Fact 1.}
\end{aligned}
$$

This proves (2) and thus concludes the proof. ■

Recall the notation from [AL]: $S_p = (\bigoplus_{n=1}^\infty C_p^n)_p$; i.e., S_p is the subspace of C_p spanned by disjoint diagonal blocks, the n-th one being of size $n \times n$. We also denote: $D_p = (\bigoplus_{n=1}^\infty \widetilde{Z_p^n})_p$. As a corollary we immediately get the main result of this section.

Theorem 1 *For $1 < p \neq 2 < \infty$ D_p is isomorphic to a complemented subspace of S_p and thus of C_p. It is not isomorphic to any of the nine previously known infinite dimensional complemented subspaces of C_p listed in Theorem 5 of [AL].*

Proof: The first assertion follows immediately from Proposition 1. Since D_p has an unconditional basis and cotype $p \vee 2$ and since C_p^n does not uniformly embed into a lattice with non trivial cotype with constant independent of n (see [Pi, Theorem 2.1]), it remains to show that D_p is not isomorphic to any of the spaces listed in Theorem 5 of [AL] having an unconditional basis. These are the four spaces ℓ_2, ℓ_p, $\ell_2 \oplus \ell_p$ or Z_p. These four spaces are isomorphic to complemented subspaces of $L_p(0,1)$, so it is enough to show that D_p is not. Indeed, for $1 < p < 2$ D_p is not even isomorphic to a subspace of $L_p(0,1)$. This follows for example from [HRS]. Indeed, if the $\widetilde{Z_p^n}$-s, $1 \leq p < 2$, uniformly embed in $L_p(0,1)$ then by a simple limiting argument so would $\widetilde{Z_p}$. However, Corollary 1.3 in [HRS] states in particular that RadC_p, which is isomorphic to $\widetilde{Z_p}$, does not isomorphically embed in

$L_p(0,1)$. Since $D_p^* = D_{p/(p-1)}$ it follows that, for $p > 2$, D_p is not isomorphic to a complemented subspace of $L_p(0,1)$. ■

1. Theorem 1 solves the problem in Remark (ii) on page 107 of [AL]. In particular, D_p is complemented in S_p but is not isomorphic to S_p or ℓ_p. One can of course ask whether these three spaces exhaust all isomorphic types of infinite dimensional complemented subspaces of S_p.
2. Combining the space D_p with the previously known complemented subspaces, we can get two more isomorphically different complemented subspaces of C_p. These are $\ell_2 \oplus_p D_p$ and $Z_p \oplus_p D_p$. It is not hard to show that they are not isomorphic to any of the other spaces and not isomorphic to each other. So all together we get twelve isomorphically different infinite dimensional complemented subspaces of C_p, $1 < p \neq 2 < \infty$. This leaves open the main problem of whether there are infinitely many isomorphic classes of infinite dimensional complemented subspaces of C_p.
We also remark that for any subset $\{m_n\}_{n=1}^{\infty}$ of the natural numbers with $\sup m_n = \infty$, the space $(\bigoplus_{n=1}^{\infty} \widetilde{Z_p^{m_n}})_p$ is isomorphic to D_p.
3. Is $\widetilde{Z_p}$ isomorphic to a complemented subspace of C_p? We believe not. Probably, for $1 \le p < 2$, $\widetilde{Z_p}$ does not even isomorphically embed into C_p. Note that for $p > 2$ the situation is different: $\widetilde{Z_p}$ isometrically embeds even in Z_p.
4. It is easy to see (using [L-PP]) that the norm of Q in the proof of Proposition 1 is of order $\sqrt{p}$ for $p > 2$.

3 Tight embeddings in C_p^n

The result of this section was obtained when I visited the University of Alberta where I greatly benefitted from interaction on the subject with Sasha Litvak and Nicole Tomczak–Jaegermann. The main result of this section is

Theorem 2 *If $T_1, \dots, T_k$ are $n \times n$ matrices with $\|T_i\|_{C_p^n} \ge 1$ for all i and*

$$\mathrm{Ave}_{\pm} \| \sum_{i=1}^{k} \pm T_i \|_{C_p^n}^p \le Kk, \quad \text{if} \ \ p > 2,$$

or $\|T_i\| \le 1$ for all i and

$$\mathrm{Ave}_{\pm} \| \sum_{i=1}^{k} \pm T_i \|_{C_p^n}^p \ge K^{-1}k, \quad \text{if} \ \ 1 \le p < 2,$$

then $n \geq K^{\frac{-2}{|p-2|}} k$. *In particular if* $T : \ell_p^k \to X \subseteq C_p^n$ *is a linear isomorphism then* $n \geq (\|T\| \|T^{-1}\|)^{\frac{-2p}{|p-2|}} k$.

We start with two claims.

Claim 1 *Let* $p > 2$ *and let* A_i, $i = 1, 2, \ldots, k$, *be* $n \times n$ *positive definite matrices with* $\mathrm{trace} A_i^{p/2} \geq 1$ *for all* i. *Assume* $\mathrm{trace}(\sum_{i=1}^k A_i)^{p/2} \leq Kk$. *Then* $n \geq \mathrm{rank}(\sum_{i=1}^k A_i) \geq K^{\frac{-2}{p-2}} k$.

Proof: Let $d = \mathrm{rank}(\sum_{i=1}^k A_i)$ and let $\lambda_1, \ldots, \lambda_d$ be the non zero eigenvalues of $\sum_{i=1}^k A_i$. Then

$$Kk \geq \mathrm{trace}(\sum_{i=1}^k A_i)^{p/2} = \sum_{i=1}^d \lambda_i^{p/2} \geq d^{\frac{2-p}{2}} (\sum_{i=1}^d \lambda_i)^{p/2} \quad \text{by Holder's inequality}$$
$$= d^{\frac{2-p}{2}} (\sum_{i=1}^k \mathrm{trace} A_i)^{p/2} \geq d^{\frac{2-p}{2}} (\sum_{i=1}^k (\mathrm{trace} A_i^{p/2})^{2/p})^{p/2} \geq d^{\frac{2-p}{2}} k^{p/2}.$$

So $d \geq K^{\frac{-2}{p-2}} k$. ∎

Claim 2 *Let* $0 < p < 2$ *and let* A_i, $i = 1, 2, \ldots, k$, *be* $n \times n$ *positive definite matrices with* $\mathrm{trace} A_i^{p/2} \leq 1$ *for all* i. *Assume* $\mathrm{trace}(\sum_{i=1}^k A_i)^{p/2} \geq ck$. *Then* $n \geq \mathrm{rank}(\sum_{i=1}^k A_i) \geq c^{\frac{2}{2-p}} k$.

Proof: Let $d = \mathrm{rank}(\sum_{i=1}^k A_i)$ and let $\lambda_1, \ldots, \lambda_d$ be the non zero eigenvalue of $\sum_{i=1}^k A_i$.

$$ck \leq \mathrm{trace}(\sum_{i=1}^k A_i)^{p/2} = \sum_{i=1}^d \lambda_i^{p/2} \leq d^{\frac{2-p}{2}} (\sum_{i=1}^d \lambda_i)^{p/2}$$
$$= d^{\frac{2-p}{2}} (\sum_{i=1}^k \mathrm{trace} A_i)^{p/2} \leq d^{\frac{2-p}{2}} (\sum_{i=1}^k (\mathrm{trace} A_i^{p/2})^{2/p})^{p/2} \leq d^{\frac{2-p}{2}} k^{p/2}.$$

So $d \geq c^{\frac{2}{2-p}} k$. ∎

Proof of Theorem 2 By the easy part of the inequality in [L-P] (or see [PX]) which was actually known before and whose proof is quite easy and similar to that of Fact 1,

$$\mathrm{Ave}_{\pm}\|\sum_{i=1}^{k}\pm T_i\|_{C_p^n}^p \geq \mathrm{tr}(\sum_{i=1}^{k}T_i^*T_i)^{p/2}, \quad \text{if } \ p>2,$$

and

$$\mathrm{Ave}_{\pm}\|\sum_{i=1}^{k}\pm T_i\|_{C_p^n}^p \leq \mathrm{tr}(\sum_{i=1}^{k}T_i^*T_i)^{p/2}, \quad \text{if } \ 0<p<2.$$

Now apply Claim 1 or 2 with $A_i = T_i^*T_i$.

To prove the last claim in the statement of the theorem, assume $p > 2$ and assume (as we may by multiplying T by a constant) that $\|T^{-1}\| = 1$. Letting T_i be the image by T of the i-th unit basis vector in ℓ_p^k we see that $\|T_i\|_{C_p^n} \geq 1$ and that $\mathrm{Ave}_{\pm}\|\sum_{i=1}^{k}\pm T_i\|_{C_p^n}^p \leq \|T\|^p k$. Now apply the first part of the theorem with $K = \|T\|^p$. The case $1 \leq p < 2$ is treated similarly starting with $\|T\| = 1$. ■

Is ℓ_p^k the worst (maybe up to log factors) k-dimensional subspace for tight embedding in C_p^n? i.e., does any k dimensional subspaces of C_p 2-embed into C_p^n with n proportional to k, except maybe for a multiplicative factor of a power of $\log k$?

4 Paving in C_p, $p > 2$

The main result here is the following Theorem which clearly gives, by iteration, the result claimed in the third subsection of the Introduction.

Theorem 3 *Let A be a $2m \times 2m$ matrix with zero diagonal. Then there are mutually orthogonal diagonal (i.e., with range a span of a subset of the natural basis) projections P, Q of rank m such that, for all $2 \leq p < \infty$,*

$$\|PAP + QAQ\|_p \leq \left(1 - \frac{1}{2^p}\right)^{1/p} \|A\|_p.$$

Given a $2m \times 2m$ matrix $A = \{a_{i,j}\}_{i,j=1}^{2m}$ and a subset $\sigma \subset \{1, 2, \ldots, 2m\}$ of cardinality m, let A_σ denote the matrix whose i, j element is 0 if i, j both belong to σ or both belong to the complement σ^c of σ, and $a_{i,j}$ otherwise.

Proposition 2 *For all $2 \le p < \infty$ and every $2m \times 2m$ matrix A with zero diagonal,*

$$\mathrm{Ave}_\sigma \mathrm{Tr}(\mathrm{A}_\sigma^* \mathrm{A}_\sigma)^{\mathrm{p}/2} \ge \frac{1}{2^{\mathrm{p}}} \mathrm{Tr}(\mathrm{A}^*\mathrm{A})^{\mathrm{p}/2},$$

where the average is taken over all subsets of $\{1, 2, \ldots, 2m\}$ of cardinality m.

Proof: Applying Proposition 2 in [Pe] we get that

$$\mathrm{Ave}_\sigma \mathrm{Tr}(\mathrm{A}_\sigma^* \mathrm{A}_\sigma)^{\mathrm{p}/2} \ge \mathrm{Tr}(\mathrm{Ave}_\sigma \mathrm{A}_\sigma^* \mathrm{A}_\sigma)^{\mathrm{p}/2}. \tag{3}$$

Now, the i, j element of $A_\sigma^* A_\sigma$ is

$$A_\sigma^* A_\sigma(i,j) = \begin{cases} \sum_{k \notin \sigma} \bar{a}_{k,i} a_{k,j}, & i, j \in \sigma \\ \sum_{k \in \sigma} \bar{a}_{k,i} a_{k,j}, & i, j \notin \sigma \\ 0 & \text{otherwise.} \end{cases}$$

It follows that for all i and j,

$$\binom{2m}{m} \mathrm{Ave}_\sigma \mathrm{A}_\sigma^* \mathrm{A}_\sigma(\mathrm{i},\mathrm{j}) \\ = 2 \sum_{\{\sigma;\ i,j \in \sigma\}} \sum_{k \notin \sigma} \bar{a}_{k,i} a_{k,j} = 2 \sum_{k \ne i,j} \#\{\sigma;\ i, j \in \sigma,\ k \notin \sigma\} \bar{a}_{k,i} a_{k,j}.$$

If $i = j$ and $k \ne i$, then $\#\{\sigma;\ i, j \in \sigma,\ k \notin \sigma\} = \binom{2m-2}{m-1}$. If $i \ne j$ and $k \ne i, j$, then $\#\{\sigma;\ i, j \in \sigma,\ k \notin \sigma\} = \binom{2m-3}{m-2}$. This translates to

$$\mathrm{Ave}_\sigma \mathrm{A}_\sigma^* \mathrm{A}_\sigma(\mathrm{i},\mathrm{j}) = \begin{cases} 2\binom{2m-2}{m-1}\binom{2m}{m}^{-1} \sum_{k \ne i} |a_{k,i}|^2, & i = j \\ 2\binom{2m-3}{m-2}\binom{2m}{m}^{-1} \sum_{k \ne i,j} \bar{a}_{k,i} a_{k,j}, & i \ne j. \end{cases} \tag{4}$$

Since

$$\binom{2m-2}{m-1}\binom{2m}{m}^{-1} = \frac{m}{4m-2} \quad \text{and} \quad \binom{2m-3}{m-2}\binom{2m}{m}^{-1} = \frac{m}{8m-4},$$

we get from (3), (4) and the fact that A has zero diagonal that

$$\mathrm{Ave}_\sigma \mathrm{Tr}(\mathrm{A}_\sigma^* \mathrm{A}_\sigma)^{\mathrm{p}/2} \ge \mathrm{Tr}\left(\left(\frac{\mathrm{m}}{4\mathrm{m}-2}\right)\mathrm{A}^*\mathrm{A} + \left(\frac{\mathrm{m}}{4\mathrm{m}-2}\right)\mathrm{B}\right)^{\mathrm{p}/2},$$

where B is the diagonal matrix whose i-th element is $\sum_k |a_{k,i}|^2$. Proposition 1 in [Pe] now implies that

$$\mathrm{Ave}_\sigma \mathrm{Tr}(\mathrm{A}_\sigma^* \mathrm{A}_\sigma)^{\mathrm{p}/2} \geq \left(\frac{\mathrm{m}}{4\mathrm{m}-2}\right)^{\mathrm{p}/2} \mathrm{Tr}(\mathrm{A}^*\mathrm{A})^{\mathrm{p}/2} \geq \frac{1}{2^{\mathrm{p}}} \mathrm{Tr}(\mathrm{A}^*\mathrm{A})^{\mathrm{p}/2}. \qquad \blacksquare$$

Proof of Theorem 3 We shall use the well known (and easy to prove by interpolation) inequality

$$\left(\frac{1}{2}(\|x+y\|_p^p + \|x-y\|_p^p)\right)^{1/p} \leq (\|x\|_p^{p/(p-1)} + \|y\|_p^{p/(p-1)})^{(p-1)/p} \tag{5}$$

for all $2 \leq p < \infty$ and all matrices x, y (see e.g. [PX, Th. 5.1]). By Proposition 2, we can find a $\sigma \subset \{1, 2, \ldots, 2m\}$ of cardinality m such that, putting P to be the natural projection onto the span of $\{e_i\}_{i\in\sigma}$, $Q = I - P$, and $v = PAQ + QAP$,

$$\|v\|_p^p \geq \frac{1}{2^p}\|A\|_p^p.$$

Put also $u = PAP + QAQ$ and notice that $A = u + v$ and that $\|u - v\|_p = \|u + v\|_p = \|A\|_p$. Indeed, $P - Q$ is a unitary transformation, so $\|u - v\|_p = \|(P - Q)A(P - Q)\|_p = \|A\|_p$. Apply now (5) to $x = u + v, y = u - v$, to get

$$\left(\frac{1}{2}(\|2u\|_p^p + \|2v\|_p^p)\right)^{1/p} \leq (\|u+v\|_p^{p/(p-1)} + \|u-v\|_p^{p/(p-1)})^{(p-1)/p}$$

or

$$(\|u\|_p^p + \|v\|_p^p)^{1/p} \leq \|A\|_p.$$

Thus,

$$\|u\|_p \leq \left(1 - \frac{1}{2^p}\right)^{1/p} \|A\|_p.$$

$\blacksquare$

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References

[AL] J. Arazy, J. Lindenstrauss, Some linear topological properties of the spaces C_p of operators on Hilbert space. Compositio Math. 30 (1975), 81–111.

[BHKW] K. Berman, H. Halpern, V. Kaftal, G. Weiss, Some C_4 and C_6 norm inequalities related to the paving problem. Operator theory: operator algebras and applications, Part 2 (Durham, NH, 1988), 29–41, Proc. Sympos. Pure Math., 51, Part 2, Amer. Math. Soc., Providence, RI (1990) .

[FLM] T. Figiel, J. Lindenstrauss, V. D. Milman, The dimension of almost spherical sections of convex bodies. Acta Math. 139 (1977), no. 1–2, 53–94.

[HRS] U. Haagerup, H. P. Rosenthal, F. A. Sukochev, Banach embedding properties of non-commutative Lp-spaces. Mem. Amer. Math. Soc. 163 (2003), no. 776, vi+68 pp.

[JS] W. B. Johnson, G. Schechtman, Finite dimensional subspaces of L_p. Handbook of the geometry of Banach spaces, Vol. I, 837–870, North-Holland, Amsterdam, 2001.

[L-P] F. Lust-Piquard, Inégalités de Khintchine dans C_p $(1 < p < \infty)$. (French) [Khinchin inequalities in C_p $(1 < p < \infty)$] C. R. Acad. Sci. Paris Sér. I Math. 303 (1986), no. 7, 289–292.

[L-PP] F. Lust-Piquard, G. Pisier, Noncommutative Khintchine and Paley inequalities. Ark. Mat. 29 (1991), no. 2, 241–260.

[MSS] A. Marcus, D. A. Spielman, N. Srivastava, Interlacing Families II: Mixed Characteristic Polynomials and the Kadison-Singer Problem, http://arxiv.org/abs/1306.3969

[Pe] D. Petz, A survey of certain trace inequalities. Functional analysis and operator theory (Warsaw, 1992), 287–298, Banach Center Publ., 30, Polish Acad. Sci., Warsaw (1994).

[Pi] G. Pisier, Some results on Banach spaces without local unconditional structure. Compositio Math. 37 (1978), no. 1, 3–19.

[PX] G. Pisier, Q. Xu, Non-commutative L_p-spaces. Handbook of the geometry of Banach spaces, Vol. 2, 1459–1517, North-Holland, Amsterdam (2003).

[Sc] G. Schechtman, Tight embedding of subspaces of L_p in ℓ_p^n for even p. Proc. Amer. Math. Soc. 139 (2011), no. 12, 4419–4421.

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