# Weakly null sequences in $L_1$*

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**Abstract**

We construct a weakly null normalized sequence $\{f_i\}_{i=1}^{\infty}$ in $L_1$ so that for each $\varepsilon > 0$, the Haar basis is $1 + \varepsilon$-equivalent to a block basis of every subsequence of $\{f_i\}_{i=1}^{\infty}$. In particular, the sequence $\{f_i\}_{i=1}^{\infty}$ has no unconditionally basic subsequence. This answers a question raised by the second author and H. P. Rosenthal in 1977. A similar example is given in an appropriate class of rearrangement invariant function spaces.

## 1 Introduction

In [MR], the second author and H. P. Rosenthal constructed the first examples of weakly null normalized sequences which do not have any unconditionally basic subsequences. Much more recently, the second author and W. T. Gowers [GM] constructed infinite dimensional Banach spaces which do not contain any unconditionally basic sequences. These later examples are of a different character than the examples in [MR]. For example, in [MR], it is shown that if $K$ is a sufficiently complex countable compact metric space, then the space $C(K)$ contains a weakly null normalized sequence which does not have an unconditionally basic subsequence, and it is known [PS] that every infinite dimensional subspace of such a $C(K)$ space contains an unconditionally basic sequence; in fact, a sequence which is equivalent to the unit vector basis of $c_0$. In [MR] it was asked if a similar example exists in $L_1$. The space $L_1$ is similar to $C(K)$ in that every infinite dimensional subspace contains an unconditionally basic sequence. Indeed, if the subspace is not reflexive, then $\ell_1$ embeds into the subspace [KP]. If a subspace $X$ of $L_1$ is reflexive, then it embeds into $L_p$ for some $1 < p \leq 2$ [R]. Since $L_p$ has an unconditional basis, every weakly null normalized sequence in $X$ has an unconditionally basic subsequence.

The main result in this paper, Theorem 1, is that there is a weakly null normalized sequence $\{f_i\}_{i=1}^{\infty}$ in $L_1$ which has no unconditionally basic subsequence. In fact, the

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sequence $\{f_i\}_{i=1}^{\infty}$ has the stronger property that for every $\varepsilon > 0$, the (conditional) Haar basis is $1+\varepsilon$-equivalent to a block basis of every subsequence of $\{f_i\}_{i=1}^{\infty}$. This is analogous to the result in [MR] that if $K$ is a sufficiently complex countable compact metric space, then the space $C(K)$ contains a weakly null normalized sequence $\{x_n\}_{n=1}^{\infty}$ so that every initial segment of the (conditional) summing basis is $1+\varepsilon$-equivalent to a block basis of every subsequence of $\{x_n\}_{n=1}^{\infty}$.

Theorem 1 can also be compared to the result of [MS] that for $1 < p < 2$, there is a 1-symmetric basic sequence $\{g_n\}_{n=1}^{\infty}$ in $L_p$ so that the Haar basis is equivalent to a block basis of every subsequence of $\{g_n\}_{n=1}^{\infty}$. In this result, there is a lower bound $C_p > 1$ to the constant of equivalence to the Haar basis because every block basis of $\{g_n\}_{n=1}^{\infty}$ is monotonely unconditional and the Haar basis for $L_p$ is not.

The approach to proving Theorem 1 also yields information about other rearrangement invariant function spaces. In Theorem 2 we prove that if $X$ is a rearrangement invariant function space on $[0,1]$ such that $\inf_{t>0} \frac{\|\chi_{[0,t]}\|}{t^{1/2}} = 0$ and $\int_0^1 \|\chi_{[0,t]}\| \frac{dt}{t} < \infty$, then there is a weakly null normalized sequence $\{f_i\}_{i=1}^{\infty}$ in $X$ such that for each $\varepsilon > 0$, the Haar basis is $1+\varepsilon$-equivalent to a block basis of every subsequence of $\{f_i\}_{i=1}^{\infty}$. The first condition says that, in some sense, $X$ is not to the right of $L_2$. Some such condition is needed because the theorem is false in $L_p$ for $p > 2$ by the results of [KP]. The second condition in Theorem 2 is technical and probably can be weakened; however, for anything resembling our construction to work, the Rademacher functions in $X$ must be equivalent to an orthonormal sequence in a Hilbert space.

We use standard Banach space theory terminology as can be found in [LT]. By a Haar system we mean here any sequence of functions distributionally equivalent to the ($L_\infty$ normalized, mean zero) traditional Haar functions; i.e., any sequence of functions $\{k_{i,n}\}_{n=0,i=1}^{\infty,\ 2^n}$ with $|k_{i,n}|$ being a characteristic function of a set $A_{i,n}$ of measure $2^{-n}$, $k_{i,n}$ equal to 1 on $A_{2i-1,n+1}$ and to $-1$ on $A_{2i,n+1}$. (In particular $A_{2i,n+1}$ and $A_{2i-1,n+1}$ are disjoint and their union is $A_{i,n}$.)

## 2 A peculiar weakly null sequence in $L_1$

In proving Theorem 1 we use the well-known fact that $j$ independent sets each having measure less than $1/j$ are essentially disjoint. Rather than hunt for a reference that states this in a form suitable for our use, we formulate what we need as a lemma. Thanks are due S. Kwapień for simplifying the proof.

**Lemma 1** *Let $0 < \theta < 1$ and let $j$ be a positive integer. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $h_1, h_2, \ldots, h_j$ be independent symmetric random variables on $\Omega$ so that for each $i$, $|h_i|$ is the indicator function of a set having probability $\theta$. Then there are disjoint sets $A$ and $B$ in the algebra generated by $h_1, h_2, \ldots, h_j$ so that*

$$\mu(A) = \mu(B) \geq \frac{\theta j}{2}(1 - \theta(j-1)) \quad \textit{and} \tag{1}$$

$$\Big\| \sum_{i=1}^{j} h_i - \chi_{_A} + \chi_{_B} \Big\|_1 \le \theta^2 j(j-1). \tag{2}$$

**Proof:** Since $\int \sum_{i=1}^{j} |h_i| = \theta j$ and

$$\mu\Big[\sum_{i=1}^{j} |h_i| = 1\Big] = \int_{\left[\sum_{i=1}^{j} |h_i| = 1\right]} \sum_{i=1}^{j} |h_i| = \theta j (1-\theta)^{j-1}, \tag{3}$$

$$\begin{aligned} \int_{\left[\sum_{i=1}^{j} |h_i| \ge 2\right]} \textstyle\sum_{i=1}^{j} |h_i| &= \theta j - \int_{\left[\sum_{i=1}^{j} |h_i| = 1\right]} \textstyle\sum_{i=1}^{j} |h_i| \\ &= \theta j - \theta j (1-\theta)^{j-1} \qquad (4) \\ &\le \theta^2 j(j-1). \end{aligned}$$

Set

$$A := \Big[\sum_{i=1}^{j} |h_i| = 1 \ \text{ and } \ \sum_{i=1}^{j} h_i = 1\Big]$$

$$B := \Big[\sum_{i=1}^{j} |h_i| = 1 \ \text{ and } \ \sum_{i=1}^{j} h_i = -1\Big]$$

then, by (3),

$$\mu(A) = \mu(B) \ge \frac{\theta j}{2}(1 - \theta(j-1))$$

and by (4)

$$\Big\| \sum_{i=1}^{j} h_i - \chi_{_A} + \chi_{_B} \Big\|_1 \le \theta^2 j(j-1).$$

■

**Theorem 1** *There is a weakly null normalized sequence $\{f_i\}_{i=1}^{\infty}$ in $L_1$ such that for each $\varepsilon > 0$, the Haar basis is $1+\varepsilon$-equivalent to a block basis of every subsequence of $\{f_i\}_{i=1}^{\infty}$. Consequently, $\{f_i\}_{i=1}^{\infty}$ has no unconditionally basic subsequence.*

**Proof:** Let $\mathcal{A}$ be the algebra generated by the dyadic subintervals of $(0,1)$ and let $\{E_n\}_{n=1}^{\infty}$ be an ordering of the non empty elements of $\mathcal{A}$ so that each element of $\mathcal{A}$ appears infinitely many times in the sequence $\{E_n\}_{n=1}^{\infty}$. We will define by recursion an increasing sequence $\{a_n\}_{n=1}^{\infty}$ of positive number and a rapidly growing sequence $\{k_n\}_{n=1}^{\infty}$ of powers of two to satisfy certain conditions to be specified later. We then define, for each $n$, a sequence $\{h_{i,n}\}_{i=1}^{\infty}$ of functions on $(0,1)$ so that

(i) $|h_{i,n}| = \chi_{_{A_{i,n}}}$ with $A_{i,n} \subset E_n$

(ii) $\int h_{i,n} = 0$

(iii) $\lambda(A_{i,n}) = \lambda(E_n)/k_n$ ($\lambda$ is Lebesgue measure)

(iv) $h_{i,n}$ is $\mathcal{A}$-measurable

(v) $\{h_{i,n}\}_{i=1}^{\infty}$ are independent relative to $E_n$ with normalized Lebesgue measure on $E_n$.

Having done this, we define the desired sequence $\{f_i\}_{i=1}^{\infty}$ by

$$f_i = \sum_{n=1}^{\infty} a_n h_{i,n}. \tag{5}$$

The $a_n$'s are going to be chosen so that $\sum_{n=1}^{\infty} a_n \|h_{i,n}\|_1 = \sum_{n=1}^{\infty} a_n \lambda(E_n)/k_n$ converges and, since each sequence $\{h_{i,n}\}_{i=1}^{\infty}$ is clearly weakly null, also $\{f_i\}$ is. Since the proof that $\{f_i\}_{i=1}^{\infty}$ has the other desired properties is a bit technical, we first describe in words the main part of the construction. The sequence $\{f_i\}_{i=1}^{\infty}$ itself of course cannot be symmetric if it has the properties we claim. However, notice that for each $n$, the sequence $\{a_n h_{i,n}\}_{i=1}^{\infty}$ is a 1-symmetric basic sequence which is equivalent (with constant depending on $n$) to the unit vector basis of $\ell_2$. It turns out that this allows all estimates we make when summing the $f_i$'s to depend only on the number of terms in the sums.

Suppose that we want to build a Haar function whose support is a set $E$ in $\mathcal{A}$; that is, we want a linear combination of $\{f_i\}_{i=1}^{\infty}$ (for $i$ in a given infinite set of natural numbers) which approximates a mean zero function whose absolute value is approximately $\chi_E$. The definitions of $\{a_n\}_{n=1}^{\infty}$ and $\{k_n\}_{n=1}^{\infty}$ guarantee that for an appropriate $m_n$, if we sum $m_n$ different $f_i$ (say, for $i$ in $B$) and normalize appropriately, the resulting vector $a_n^{-1}\sum_{i\in B} f_i$ is a small perturbation of $\sum_{i\in B} h_{i,n}$, and this latter vector is (by Lemma 1) a small perturbation of an $\mathcal{A}$-measurable function, $h_B$, which takes on only the values $\{-1, 0, 1\}$. Moreover, the measure of the support $S(B)$ of $h_B$ will be a percentage of the measure of the set $E_n$. If that percentage were 100%, we would have achieved our goal since there are arbitrarily large $n$ with $E_n = E$. We cannot make the percentage 100%, but we can repeat the process, replacing $E$ by $E\setminus S(B)$, and get another linear combination of $\{f_i\}_{i=1}^{\infty}$ whose absolute value is approximately $\chi_F$ with $F$ the same percentage of $E \setminus S(B)$. By iterating this process, we obtain a linear combination of $\{f_i\}_{i=1}^{\infty}$ whose absolute value is approximately $\chi_E$.

We turn now to the actual proof of Theorem 1. Suppose we have defined $\{a_1, a_2, \ldots, a_N\}$ and $\{k_1, k_2, \ldots, k_N\}$. Since for each $n$, the sequence $\{a_n h_{i,n}\}_{i=1}^{\infty}$ is equivalent to the unit vector basis of $\ell_2$, there is a constant $M_N$ so that for all finite sets $\sigma$ of natural numbers we have

$$\sum_{n=1}^{N} \Big\| \sum_{i\in\sigma} a_n h_{i,n} \Big\| < |\sigma|^{1/2} M_N. \tag{6}$$

We want to define $k_{N+1}$ and $a_{N+1}$ so that if we add up $\varepsilon k_{N+1}$ (with $\varepsilon > 2^{-N}$, say) of the $f_i$'s, then the terms involving $a_{N+1}h_{i,N+1}$ are dominant. To guarantee that these terms dominate the terms involving $a_n h_{i,n}$ with $n \leq N$, it suffices, as we shall see, to have

$$2^N k_{N+1}^{1/2} M_N \leq a_{N+1} \lambda(E_{N+1}). \tag{7}$$

We shall also see below that in order to guarantee that the terms involving $a_{N+1} h_{i,N+1}$ are negligible with respect to the terms involving $a_n h_{i,n}$ with $n \leq N$ if we add up fewer than $k_N$ of the $f_i$'s, it suffices (in fact, is "over kill") to have

$$\frac{a_{N+1} \lambda\left(E_{N+1}\right)}{k_{N+1}} \leq \frac{\min\{\lambda\left(E_n\right) : 1 \leq n \leq N\}}{2^{N+1} k_N}. \tag{8}$$

There is of course no difficulty in achieving (7) and (8) simultaneously. For example, let $a_{N+1} = k_{N+1}^{3/4}$ with $k_{N+1}$ a sufficiently large power of 2.

This completes the description of how to define the sequence $\{f_i\}_{i=1}^{\infty}$. To see that $\{f_i\}_{i=1}^{\infty}$ satisfies the conclusion of Theorem 1, we need the following:

**Claim 1** *Let $\tau > 0$, let $F_1$ be a non empty set in $\mathcal{A}$, and suppose that $\mathbb{M}$ is an infinite subset of $\mathbb{N}$. Then there is an $f$ in the linear span of $\{f_i\}_{i \in \mathbb{M}}$ and disjoint subsets $A \in \mathcal{A}$, $B \in \mathcal{A}$ of $F_1$ with $\lambda(A) = \lambda(B) > \lambda(F_1)/2 - \tau$ so that $\|f - \chi_A + \chi_B\|_1 < \tau$.*

Once we have the Claim it is of course easy to get the stronger conclusion with $A \cup B = F_1$ keeping the second approximation conclusion as is. Just divide $F_1 \setminus (A \cup B)$ into two disjoint sets of $\mathcal{A}$ of the same measure; add one of them to $A$ and the other to $B$. It is now evident from Claim 1 that for any sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers and any infinite subset $\mathbb{M}$ of $\mathbb{N}$ we can build a Haar system $\{g_n\}_{n=1}^{\infty}$ and a block basis $\{u_n\}_{n=1}^{\infty}$ of $\{f_i\}_{i \in \mathbb{M}}$ so that for each $n$, $\|g_n - u_n\|_1 < \varepsilon_n$. This implies that $\{f_i\}_{i=1}^{\infty}$ satisfies the conclusion of Theorem 1.

**Proof of Claim 1:** Let $\varepsilon^{-1}$ be an appropriately large power of 2; say, $\varepsilon^{-1} = 2^m$. We choose an appropriately large $N_1$ with $E_{N_1} = F_1$ and let $\sigma_1$ be a subset of $\mathbb{M}$ which has cardinality $\varepsilon k_{N_1}$. By conditions (i)–(v) and Lemma 1 (applied in the probability space $(F_1, \frac{\lambda}{\lambda(F_1)})$ with $\theta = k_{N_1}^{-1}$ and $j = \varepsilon k_{N_1}$), there are disjoint sets $A_1 \subset F_1$, $B_1 \subset F_1$ in $\mathcal{A}$ so that

$$\lambda(A_1) = \lambda(B_1) \geq \frac{\varepsilon}{2}(1 - \varepsilon)\lambda(F_1) \quad \text{and}$$

$$\Big\| \sum_{i \in \sigma_1} h_{i,k_{N_1}} - \chi_{A_1} + \chi_{B_1} \Big\|_1 < \varepsilon^2.$$

Set $F_2 := F_1 \setminus (A_1 \cup B_1)$ (so that $\lambda(F_2) \leq [1 - \varepsilon(1-\varepsilon)]\lambda(F_1)$) and repeat the construction, replacing $F_1$ by $F_2$. We use the same $\varepsilon$ and choose an appropriately large $N_2 > N_1$ with $E_{N_2} = F_2$ and let $\sigma_2$ be a subset of $\mathbb{M}$ of cardinality $\varepsilon k_{N_2}$ with $\max \sigma_1 < \min \sigma_2$. This time we get disjoint sets $A_2 \subset F_2$, $B_2 \subset F_2$ in $\mathcal{A}$ so that

$$\lambda(A_2) = \lambda(B_2) \geq \frac{\varepsilon}{2}(1 - \varepsilon)\lambda(F_2) \quad \text{and}$$

$$\Big\| \sum_{i\in\sigma_2} h_{i,k_{N_2}} - \chi_{A_2} + \chi_{B_2} \Big\|_1 < \varepsilon^2.$$

Next set $F_3 := F_2 \setminus (A_2 \cup A_3)$ (so that $\lambda(F_3) \leq [1-\varepsilon(1-\varepsilon)]^2\lambda(F_1)$) and repeat. Continue in this way $m\varepsilon^{-1} = m2^m$ steps, thereby obtaining $k_1 < k_2 < \cdots < k_{m2^m}$ (which can grow as fast as we like), disjoint subsets $A_1, A_2, \ldots, A_{m2^m}, B_1, B_2 \ldots, B_{m2^m}$ of $F_1$ which are in $\mathcal{A}$, and subsets $\sigma_1, \sigma_2, \ldots, \sigma_{m2^m}$ of $\mathbb{M}$, with $\max\sigma_{j-1} < \min\sigma_j$ and $|\sigma_j| = \varepsilon k_{N_j}$ so that $\lambda(A_j) = \lambda(B_j)$,

$$\lambda(F_1 \setminus \cup_{j=1}^n (A_j \cup B_j)) \leq [1-\varepsilon(1-\varepsilon)]^n \lambda(F_1) \quad \text{and} \tag{9}$$

$$\Big\| \sum_{i\in\sigma_j} h_{i,k_{N_j}} - \chi_{A_j} + \chi_{B_j} \Big\|_1 < \varepsilon^2. \tag{10}$$

Set $A := \cup_{j=1}^{m2^m} A_j$, $B := \cup_{j=1}^{m2^m} B_j$, and $\sigma := \cup_{j=1}^{m2^m} \sigma_j$. Then from (9) we have that

$$\lambda(A) = \lambda(B) \geq \frac{1-[1-\varepsilon(1-\varepsilon)]^{m/\varepsilon}}{2}\lambda(F_1) \tag{11}$$

while (10) gives

$$\Big\| \sum_{j=1}^{m2^m} \sum_{i\in\sigma_j} h_{i,k_{N_j}} - \chi_A + \chi_B \Big\|_1 < m\varepsilon. \tag{12}$$

We need to verify that $\sum_{j=1}^{m2^m} \sum_{i\in\sigma_j} h_{i,k_{N_j}}$ is close to the linear span of $\{f_i\}_{i\in\mathbb{M}}$. From (6), (7), and (8) we get

$$\begin{aligned} \|\textstyle\sum_{i\in\sigma_j} (a_{N_j}^{-1} f_i - h_{i,k_{N_j}})\|_1 &\leq a_{N_j}^{-1} \Big( (\varepsilon k_{N_j})^{1/2} M_{N_j-1} + \varepsilon k_{N_j} \textstyle\sum_{n=N_j+1}^{\infty} \frac{a_n \lambda(E_n)}{k_n} \Big) \\ &\leq 2^{-N_j+1} \varepsilon^{1/2} \lambda(E_{N_j}) + \varepsilon 2^{-N_j-1} \lambda(E_{N_j}) := (*), \end{aligned}$$

and we can assume that $(*) \leq 2^{-j-1}\varepsilon$ since $\varepsilon$ is specified before $N_1$. From this and (12) we get that

$$\| \sum_{j=1}^{m2^m} a_{N_j}^{-1} \sum_{i\in\sigma_j} f_i - \chi_A + \chi_B \|_1 < (m+1)\varepsilon = \varepsilon(1-\log_2 \varepsilon). \tag{13}$$

From (11) and (13) it is clear that if $\varepsilon$ is sufficiently small, the claim is satisfied if we set $f := \sum_{j=1}^{m2^m} a_{N_j}^{-1} \sum_{i\in\sigma_j} f_i$. ■

## 3 Rearrangement invariant function spaces

Here we generalize the first statement of Theorem 1 to the case of rearrangement invariant function spaces. It is clear that this statement does not hold for every rearrangement invariant function spaces; even not for nice ones. Indeed, for $2 \le p < \infty$, every weakly null normalized sequence in $L_p$ contains a subsequence equivalent to the unit vector basis in either $\ell_p$ or $\ell_2$. It is natural to conjecture that the theorem may still hold for spaces which are strictly "to the left" of $L_2$, in some sense. Actually a lot more is true:

**Theorem 2** *Let $(X, \|\cdot\|)$ be a rearrangement invariant function space on $[0,1]$ such that*

$$\inf_{t>0} \frac{\|\chi_{[0,t]}\|}{t^{1/2}} = 0 \tag{14}$$

*and*

$$\int_0^1 \|\chi_{[0,t]}\| \frac{dt}{t} < \infty. \tag{15}$$

*Then there is a weakly null normalized sequence $\{f_i\}_{i=1}^\infty$ in $X$ such that for each $\varepsilon > 0$, the Haar basis is $1+\varepsilon$-equivalent to a block basis of every subsequence of $\{f_i\}_{i=1}^\infty$.*

We begin with a lemma which replaces Lemma 1. The main difference is that we give a lattice, rather than norm, estimate for the error.

**Lemma 2** *Let $0 < \theta < 1$ and let $j$ be a positive integer. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $h_1, h_2, \ldots, h_j$ be independent symmetric random variables on $\Omega$ so that for each $i$, $|h_i|$ is the indicator function of a set having probability $\theta$. Denote by $S$ the support of $\sum_{i=1}^j |h_i|$. Then there are disjoint subsets $A$ and $B$ of $S$ in the algebra generated by $h_1, h_2, \ldots, h_j$ and, for each $s = 2, 3, \ldots$, a set $C_s \subset S$ of measure at most*

$$\frac{(\theta j)^{s-1} \exp(\frac{2\theta j}{1-\theta})}{s!} \mu(S)$$

*so that*

$$\mu(A) = \mu(B) \ge \frac{\theta j}{2} \exp\left(\frac{-\theta(j-1)}{1-\theta}\right) \ge \frac{\mu(S)}{2} \exp\left(\frac{-\theta(j-1)}{1-\theta}\right) \tag{16}$$

*and*

$$\Big|\sum_{i=1}^j h_i - \chi_A + \chi_B\Big| \le 2\chi_{C_2} + \sum_{s=3}^\infty \chi_{C_s}. \tag{17}$$

**Remark 1** In particular, if $(\Omega, \mathcal{F}, \mu)$ is a subinterval of $[0,1]$ with normalized Lebesgue measure and $\|\cdot\|$ is a rearrangement invariant norm on $[0,1]$ then, putting $\sigma = \mu(S) \exp\left(\frac{2\theta j}{1-\theta}\right)$,

$$\Big\|\sum_{i=1}^j h_i - \chi_A + \chi_B\Big\| \le 2\|\chi_{[0,\theta j\sigma/2]}\| + \sum_{t=2}^{j-1} \|\chi_{[0,(\theta j)^t\sigma/(t+1)!]}\|. \tag{18}$$

However, this norm estimate of the error does not seem to be enough to deduce Theorem 2 and we will have to use the lattice estimate given in the Lemma.

**Proof:** As in the proof of Lemma 1 set

$$A := \Big[\sum_{i=1}^{j} |h_i| = 1 \ \text{ and } \ \sum_{i=1}^{j} h_i = 1\Big]$$

and

$$B := \Big[\sum_{i=1}^{j} |h_i| = 1 \ \text{ and } \ \sum_{i=1}^{j} h_i = -1\Big].$$

For $s = 2, 3, \ldots$ set

$$C_s := \Big[\sum_{i=1}^{j} |h_i| \geq s\Big].$$

Then clearly

$$\Big|\sum_{i=1}^{j} h_i - \chi_A + \chi_B\Big| \leq \sum_{r=2}^{j} \Big(\sum_{i=1}^{j} |h_i|\Big)\chi_{[\sum_{i=1}^{j}|h_i|=r]} \leq 2\chi_{C_2} + \sum_{s=3}^{\infty} \chi_{C_s}.$$

For $2 \leq s \leq j$ the measure of $C_s$ is

$$\sum_{r=s}^{j} \binom{j}{r} \theta^r (1-\theta)^{j-r} \leq \sum_{r=s}^{j} \frac{(\theta j)^r}{r!} \leq \frac{(\theta j)^s}{s!} e^{\theta j}.$$

The measure of $S$ is

$$1 - (1-\theta)^j \leq \theta j,$$

while $A$ and $B$ both have measure

$$\frac{\theta j (1-\theta)^{j-1}}{2} \geq \frac{\theta j}{2} \exp\Big(\frac{-\theta(j-1)}{1-\theta}\Big).$$

Since $\mu(S) \geq \mu(A) + \mu(B) \geq \theta j \exp\big(\frac{-\theta(j-1)}{1-\theta}\big)$ we also get that

$$\mu(C_s) \leq \frac{(\theta j)^{s-1} \exp(\theta j + \frac{\theta(j-1)}{1-\theta})}{s!} \mu(S) \leq \frac{(\theta j)^{s-1} \exp(\frac{2\theta j}{1-\theta})}{s!} \mu(S).$$

■

The proof of Theorem 2 is very similar to that of Theorem 1 and we shall only sketch it.

**Proof of Theorem 2.** We are going to define inductively an increasing sequence of positive numbers $\{a_n\}$, a sequence $\{k_n\}_{n=1}^{\infty}$ of powers of two and a double sequence $\{h_{i,n}\}$ of functions on $[0, 1]$. Given $k_n$, $\{h_{i,n}\}_{i=1}^{\infty}$ is defined in exactly the same way as in the beginning of the proof of Theorem 1 to satisfy conditions (i)-(v) there. We now describe how, given $a_1, \ldots, a_N$ and $k_1, \ldots, k_N$, to define $a_{N+1}$ and $k_{N+1}$.

We note first that (15) implies in particular that for each $n$, $\{h_{i,n}\}_{i=1}^{\infty}$ is equivalent to an orthonormal sequence. We delay the proof of that to Lemma 3 below. Once we know this we deduce, as in the $L_1$ case, that there is a constant $M_N$ so that for all finite sets $\sigma$ of natural numbers we have

$$\sum_{n=1}^{N}\Big\|\sum_{i\in\sigma}a_n h_{i,n}\Big\| < |\sigma|^{1/2}M_N. \tag{19}$$

We want to define $k_{N+1} > k_N$ (and a power of two) and $a_{N+1} > a_N$ so as to satisfy

$$2^N k_{N+1}^{1/2} M_N \leq a_{N+1}\|\chi_{E_{N+1}}\|. \tag{20}$$

and

$$a_{N+1}\|\chi_{[0,\frac{\lambda(E_{N+1})}{k_{N+1}}]}\| \leq \frac{\min\{\|\chi_{E_n}\| : 1 \leq n \leq N\}}{2^{N+1}k_N}. \tag{21}$$

We can choose $k_{N+1}$ and $a_{N+1}$ to satisfy (20) and (21) simultaneously since, by condition (14) on the space, we clearly can build increasing positive sequences $\{a_n\}$ and $\{k_n\}$ tending to infinity with $a_n/k_n^{1/2} \to \infty$ and $a_n\|\chi_{[0,k_n^{-1}]}\| \to 0$. By passing to a subsequence the convergence to these limits can be made arbitrarily fast.

Condition (21) implies in particular that $\sum_{n=1}^{\infty} a_n\|h_{i,n}\| < \infty$. So we can define $f_i = \sum_{n=1}^{\infty} a_n h_{i,n}$ as before and Lemma 3 guarantees that $\{f_i\}$ is weakly null.

The main part of the proof is contained in the following claim, replacing Claim 1, which is clearly enough to finish the proof of the theorem. Note that since the conditions on the space ensure that it is not $L_\infty$ the measure condition on the sets $A$ and $B$ in the claim imply that $\chi_A - \chi_B$ approximates a Haar function supported on $F_1$ to an arbitrary degree of approximation (see the comment after the statement of Claim 1).

**Claim 2** *Let $\tau > 0$, let $F_1$ be a non empty set in $\mathcal{A}$, and suppose that $\mathbb{M}$ is an infinite subset of $\mathbb{N}$. Then there are $f$ in the linear span of $\{f_i\}_{i\in\mathbb{M}}$ and disjoint subsets $A, B \in \mathcal{A}$ of $F_1$ with $\lambda(A) = \lambda(B) > \lambda(F_1)/2 - \tau$ so that $\|f - \chi_A + \chi_B\| < \tau$.*

**Proof of Claim 2:** Let $\varepsilon^{-1}$ be an appropriately large power of 2; say, $\varepsilon^{-1} = 2^m$. We choose an appropriately large $N_1$ with $E_{N_1} = F_1$ and let $\sigma_1$ be a subset of $\mathbb{M}$ which has cardinality $\varepsilon k_{N_1}$. By conditions (i)–(v) and Lemma 2 (applied in the probability space $(F_1, \frac{\lambda}{\lambda(F_1)})$ with $\theta = k_{N_1}^{-1}$ and $j = \varepsilon k_{N_1}$), there are disjoint sets $A_1, B_1$ in $\mathcal{A}$ and another set $S_1$ in $\mathcal{A}$ such that $A_1, B_1 \subset S_1 \subset F_1$ and, for each $s = 2, 3, \ldots$, a set $C_{1,s} \subset S_1$ of measure at most

$$\frac{\varepsilon^{s-1}e^{3\varepsilon}}{s!}\lambda(S_1)$$

so that

$$\lambda(A_1) = \lambda(B_1) \geq \frac{\varepsilon e^{-2\varepsilon}}{2}\lambda(F_1) \geq \frac{e^{-2\varepsilon}}{2}\lambda(S_1) \quad \text{and}$$

$$\Big|\sum_{i\in\sigma_1} h_{i,k_{N_1}} - \chi_{A_1} + \chi_{B_1}\Big| \le 2\chi_{C_{1,2}} + \sum_{s=3}^{\infty} \chi_{C_{1,s}}.$$

Set $F_2 := F_1 \setminus S_1$ and repeat the construction, replacing $F_1$ by $F_2$. We use the same $\varepsilon$ and choose an appropriately large $N_2 > N_1$ with $E_{N_2} = F_2$ and let $\sigma_2$ be a subset of $\mathbb{M}$ of cardinality $\varepsilon k_{N_2}$ with $\max\sigma_1 < \min\sigma_2$. This time we get sets $A_2, B_2 \subset S_2 \subset F_2$ in $\mathcal{A}$ with $A_2, B_2$ disjoint and, for each $s = 2, 3, \ldots$, a set $C_{2,s} \subset S_2$ of measure at most

$$\frac{\varepsilon^{s-1}e^{3\varepsilon}}{s!}\lambda(S_2)$$

so that

$$\lambda(A_2) = \lambda(B_2) \ge \frac{\varepsilon e^{-2\varepsilon}}{2}\lambda(F_2) \ge \frac{e^{-2\varepsilon}}{2}\lambda(S_2) \quad \text{and}$$

$$\Big|\sum_{i\in\sigma_2} h_{i,k_{N_2}} - \chi_{A_2} + \chi_{B_2}\Big| \le 2\chi_{C_{2,2}} + \sum_{s=3}^{\infty} \chi_{C_{2,s}}.$$

We continue in an obvious way, setting $F_3 = F_1 \setminus (S_1 \cup S_2)$ ..., getting subsets $\sigma_1, \ldots, \sigma_l$ of $\mathbb{M}$ with $\max\sigma_i < \min\sigma_{i+1}$, disjoint subsets $S_1, S_2, \ldots, S_l$ of of $F_1$, a disjoint couple of sets $A_j, B_j \subset S_j$, $j = 1, 2, \ldots, l$, all in $\mathcal{A}$, and a double sequence of sets $C_{j,s}$, $j = 1, 2, \ldots, l$, $s = 1, 2, \ldots$, satisfying for all $j = 1, 2, \ldots, l$,

$$\lambda(C_{j,s}) \le \frac{\varepsilon^{s-1}e^{3\varepsilon}}{s!}\lambda(S_j),$$

$$\lambda(A_j) = \lambda(B_j) \ge \frac{\varepsilon e^{-2\varepsilon}}{2}\lambda(F_1 \setminus \cup_{r=1}^{j-1} S_r) \ge \frac{e^{-2\varepsilon}}{2}\lambda(S_j) \quad \text{and}$$

$$\Big|\sum_{i\in\sigma_j} h_{i,k_{N_j}} - \chi_{A_j} + \chi_{B_j}\Big| \le 2\chi_{C_{j,2}} + \sum_{s=3}^{\infty} \chi_{C_{2j,s}}.$$

We choose $l$ so that $\lambda(F_1 \setminus \cup_{r=1}^{l} S_r) < \varepsilon\lambda(F_1)$. This clearly can be done since $S_j$ "eats" at least a $\varepsilon e^{-2\varepsilon}$ portion of $F_1 \setminus \cup_{r=1}^{j-1} S_r$. Set

$$A = \cup_{j=1}^{l} A_j, \;\; B = \cup_{j=1}^{l} B_j$$

and for all $s = 2, 3, \ldots$

$$C_s = \cup_{j=1}^{l} C_{j,s}.$$

Then

$$\lambda(C_s) \le \frac{\varepsilon^{s-1}e^{3\varepsilon}}{s!}\lambda(F_1),$$

$$\lambda(A) = \lambda(B) \ge \frac{e^{-2\varepsilon}}{2}\lambda(\cup_{r=1}^{l} S_r) \ge \frac{e^{-2\varepsilon} - \varepsilon}{2}\lambda(F_1), \tag{22}$$

and

$$\Big|\sum_{j=1}^{l}\sum_{i\in\sigma_j} h_{i,k_{N_j}} - \chi_A + \chi_B\Big| \le 2\chi_{C_2} + \sum_{s=3}^{\infty}\chi_{C_s}.$$

Consequently,

$$\Big\|\sum_{j=1}^{l}\sum_{i\in\sigma_j} h_{i,k_{N_j}} - \chi_A + \chi_B\Big\| \le 2\|\chi_{[0,\varepsilon e^{3\varepsilon}/2]}\| + \sum_{t=2}^{\infty}\|\chi_{[0,\varepsilon^t e^{3\varepsilon}/(t+1)!]}\|. \tag{23}$$

For $\varepsilon$ small enough the last expression is smaller than

$$10\int_0^{\varepsilon}\|\chi_{[0,t]}\|\frac{dt}{t}$$

and condition (15) implies that the last quantity is smaller than $\tau/2$ if $\varepsilon$ is small enough. Also, (22) implies that, if $\varepsilon$ is small enough, then

$$\lambda(A), \lambda(B) > \lambda(F_1)/2 - \tau.$$

The rest of the proof is very similar to that of the $L_1$ case. We need to verify that $\sum_{j=1}^{l}\sum_{i\in\sigma_j} h_{i,k_{N_j}}$ is close to the linear span of $\{f_i\}_{i\in\mathbb{M}}$. From (19), (20), and (21) we get

$$\begin{aligned}\|\textstyle\sum_{i\in\sigma_j}(a_{N_j}^{-1}f_i - h_{i,k_{N_j}})\| &\le a_{N_j}^{-1}\Big((\varepsilon k_{N_j})^{1/2}M_{N_j-1} + \varepsilon k_{N_j}\textstyle\sum_{n=N_j+1}^{\infty} a_n\|\chi_{[0,\frac{\lambda(E_n)}{k_n}]}\|\Big)\\ &\le 2^{-N_j+1}\varepsilon^{1/2}\|\chi_{E_{N_j}}\| + \varepsilon 2^{-N_j}\|\chi_{E_{N_j}}\| := (*),\end{aligned}$$

and we can assume that $(*) \le 2^{-j-1}\varepsilon$ since $\varepsilon$ is specified before $N_1$. From this and (23) we get that

$$\|\sum_{j=1}^{l} a_{N_j}^{-1}\sum_{i\in\sigma_j} f_i - \chi_A + \chi_B\|_1 < \varepsilon + \tau/2 < \tau \tag{24}$$

for small enough $\varepsilon$. Thus the claim is satisfied for $f := \sum_{j=1}^{l} a_{N_j}^{-1}\sum_{i\in\sigma_j} f_i$. ■

**Lemma 3** *Assume $(X, \|\cdot\|)$ is a rearrangement invariant function space on $[0,1]$ and $\int_0^1\|\chi_{[0,t]}\|\frac{dt}{t} < \infty$. Let $\{r_i\}$ be a sequence of three valued symmetric random variables whose absolute values are characteristic functions of sets $\{A_i\}$ of equal measure, all contained in one set $A$, and assume the $r_i$'s are independent as random variables in the probability space $A$ with the normalized Lebesgue measure. Then, $\{r_i\}$ is equivalent to an orthonormal sequence.*

**Proof:** By a simple and classical computation in the space $L_4(A, \frac{dt}{\lambda(A)})$ we get

$$\|\sum a_j r_j\|_4^4 = m\sum a_j^4 + 6m^2 \sum_{j<k} a_j^2 a_k^2 \leq 3m\Big(\sum a_j^2\Big)^2$$

where $m = \lambda(A_j)/\lambda(A)$, hence the $L_2$ and $L_4$ norms are equivalent, and this yields that the $L_1$ and $L_2$ norms are also equivalent. By the fact that the norm in $X$ dominates the $L_1$ norm we get that $\{r_i\}$, as a sequence of elements of $X$, dominates the same sequence considered as a sequence in $L_1(A, \frac{dt}{\lambda(A)})$. The later is equivalent to an orthogonal sequence by the preceding argument.

For the other inequality we first assume as me may that $A = [0,1]$. For each sequence of coefficients $\{a_i\}$ with $\sum a_i^2 = 1$ and each $t > 0$ we have the well known inequality

$$\lambda(|\sum a_i r_i| > t) \leq 2e^{-t^2/2}.$$

Since

$$|\sum a_i r_i| \leq \sum_{j=1}^{\infty} j\chi_{(j-1<|\sum a_i r_i|\leq j)} = \sum_{t=0}^{\infty} \chi_{(|\sum a_i r_i|>t)},$$

we get that $\|\sum a_i r_i\|$ is dominated by

$$\sum_{t=0}^{\infty} \|\chi_{(|\sum a_i r_i|>t)}\| \leq 2\sum_{t=0}^{\infty} \|\chi_{[0,e^{-t^2/2}]}\|.$$

The last sum is comparable to

$$\int_0^{\infty} \|\chi_{[0,e^{-t^2/2}]}\| dt = \int_0^1 \|\chi_{[0,s]}\| \frac{ds}{s\sqrt{\log \frac{1}{2s}}},$$

which, by (15), is finite. ■

**Remark 2** The proofs of Theorems 1 and Theorem 2 actually show that the weakly null sequence obtained has the stronger property that every subsequence has a block basis which is an arbitrarily small perturbation of a Haar system. Under some mild extra condition on the rearrangement invariant space (separability, for example, is sufficient), the closed linear span of a Haar system is norm one complemented (via a conditional expectation), which implies that the corresponding block basis of the weakly null sequence spans a complemented subspace.

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