No greedy bases for matrix spaces with mixed ℓ_p and ℓ_q norms *

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Abstract

We show that non of the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$, $1 \le p \ne q < \infty$ have a greedy basis. This solves a problem raised by Dilworth, Freeman, Odell and Schlumprect. Similarly, the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{c_0}$, $1 \le p < \infty$, and $(\bigoplus_{n=1}^{\infty} c_o)_{\ell_q}$, $1 \le q < \infty$, do not have greedy bases. It follows from that and known results that a class of Besov spaces on $\mathbb{R}^n$ lack greedy bases as well.

1 Introduction

Given a (say, real) Banach space X with a Schauder basis $\{x_i\}$, an $x \in X$ and an $n \in \mathbb{N}$ it is useful to determine the best n-term approximation to x with respect to the given basis. I.e., to find a set $A \subset \mathbb{N}$ with n elements and coefficients $\{a_i\}_{i \in A}$ such that

$$\|x - \sum_{i \in A} a_i x_i\| = \inf\{\|x - \sum_{i \in B} b_i x_i\|; |B| = n, b_i \in \mathbb{R}\}$$

or, given a $C < \infty$, at least to find such an $A \subset \mathbb{N}$ and coefficients $\{a_i\}_{i \in A}$ with

$$\|x - \sum_{i \in A} a_i x_i\| \le C \inf\{\|x - \sum_{i \in B} b_i x_i\|; |B| = n, b_i \in \mathbb{R}\}.$$

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This problem attracted quite an attention in modern Approximation Theory. Of course one would also like to have a simple algorithm to find such a set $\{a_i\}_{i\in A}$. It would be nice if we could take $\{a_i\}_{i\in A}$ to be just the set of the n largest, in absolute value, coefficients in the expansion of x with respect to the basis $\{x_i\}$. Or, if this set is not unique, any such set. The basis $\{x_i\}$ is called *Greedy* if for some C this procedure works; i.e., for all $x = \sum_{i=1}^{\infty} a_i x_i$, all $n \in \mathbb{N}$ and all $A \subset \mathbb{N}$, $|A| = n$, satisfying $\min\{|a_i|; i \in A\} \geq \max\{|a_i|; i \notin A\}$,

$$\|x - \sum_{i\in A} a_i x_i\| \leq C \inf\{\|x - \sum_{i\in B} b_i x_i\|; |B| = n, b_i \in \mathbb{R}\}.$$

Konyagin and Temlyanov [KT] provided a simple criterion to determine whether a basis is greedy: $\{x_i\}$ is greedy if and only if it is *unconditional* and *democratic.*

Recall that $\{x_i\}$ is said to be unconditional provided, for some $C < \infty$, all eventually zero coefficients $\{a_i\}$ and all sequences of signs $\{\varepsilon_i\}$,

$$\|\sum \varepsilon_i a_i x_i\| \leq C\|\sum a_i x_i\|.$$

$\{x_i\}$ is said to be democratic provided for some $C < \infty$ and all finite $A, B \subset \mathbb{N}$ with $|A| = |B|$,

$$\|\sum_{i\in A} x_i\| \leq C\|\sum_{i\in B} x_i\|.$$

We refer to [DFOS] for a survey of what is known about space that have or do not have greedy bases. In [DFOS] Dilworth, Freeman, Odell and Schlupmrecht determined which of the spaces $X = (\bigoplus_{n=1}^{\infty} \ell_p^n)_{\ell_q}$, $1 \leq p \neq q \leq \infty$ (with c_0 replacing ℓ_∞ in case $q = \infty$) have a greedy basis. It turns out that this happens exactly when X is reflexive. They also raise the question of whether $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$, $1 < p \neq q < \infty$ have greedy bases. Here we show that these spaces (as well as their non-reflexive counterparts) do not have greedy bases. By the Konyagin-Temlyanov characterization it is enough to prove that each unconditional basis of $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$, $1 \leq p \neq q \leq \infty$ (with c_0 replacing ℓ_∞ in case p or q are ∞) has two subsequences, one equivalent to the unit vector basis of ℓ_p (c_0 if $p = \infty$) and one to the unit vector basis of ℓ_q (c_0 if $q = \infty$).

Theorem 1 *Each normalized unconditional basis of the spaces* $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$, $1 \leq p \neq q < \infty$ *has a subsequence equivalent to the unit vector basis of* ℓ_p

and another one equivalent to the unit vector basis of ℓ_q. Similarly, each unconditional basis of the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{c_0}$, $1 \le p < \infty$ (resp. $(\bigoplus_{n=1}^{\infty} c_o)_{\ell_q}$, $1 \le q < \infty$) has a subsequence equivalent to the unit vector basis of ℓ_p (resp. c_0) and another one equivalent to the unit vector basis of c_o (resp. ℓ_q). Consequently, none of these spaces have a greedy basis.

For $1 \le p, q < \infty$ the spaces $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$ are isomorphic to certain Besov spaces on $\mathbb{R}^n$. We refer to [Me] for the definition of the Besov spaces $B_p^{s,q}$ and for the fact that they are isomorphic to $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$. See in particular [Me, Section 6.10, Proposition 7] (and also [Me, Section 2.9, Proposition 4]). We thank P. Wojtaszczyk for this reference.

Corollary 1 *Let $1 \le p \ne q < \infty$ and s any real number then the space $B_p^{s,q}$ does not have a greedy basis.*

Recall that this stand in contrast with the main result in [DFOS] which states that, in the reflexive cases, the corresponding Besov spaces on $[0, 1]$ do have greedy bases.

In the special case of $1 < q < \infty$ and $p = 2$ the theorem above was actually proved in [Sc]. There the isomorphic classification of the span of unconditional basic sequences in $(\bigoplus_{n=1}^{\infty} \ell_2)_{\ell_q}$, $1 < q < \infty$, which span complemented subspaces were characterize. Although it is not stated there, the proof actually established the theorem above in these special cases. Shortly after [Sc] appeared Odell [Sc] strengthened the result and classified *all* the complemented subspace of $(\bigoplus_{n=1}^{\infty} \ell_2)_{\ell_q}$ (thus there is no wonder that [Sc] was forgotten...). We remark in passing that this special case of $p = 2$ was of particular interest since $(\bigoplus_{n=1}^{\infty} \ell_2)_{\ell_q}$ is isomorphic to a complemented subspace of $L_q[0, 1]$.

The first step in the proof in [Sc] is to reduce the case of a general unconditional basic sequence in $(\bigoplus_{n=1}^{\infty} \ell_2)_{\ell_q}$ whose span is complemented to one which is also a block basis of the natural basis of $(\bigoplus_{n=1}^{\infty} \ell_2)_{\ell_q}$. This reduction no longer hold for $p \ne 2$. The complications in the present note stems from this fact. The way we overcome it is by transferring the problem to a larger space (of arrays $\{a_{i,j,k}\}$) of mixed q, p and 2 norms. Unfortunately, this makes the notations quite cumbersome.

2 Preliminaries

$Z_{q,p}$, $1 \le p, q < \infty$ will denote here the space of all matrices $a = \{a(i,j)\}_{i,j=1}^{\infty}$ with norm

$$\|a\| = \|a\|_{q,p} = (\sum_{j=1}^{\infty}(\sum_{i=1}^{\infty} |a(i,j)|^p)^{q/p})^{1/q}.$$

If p or q are ∞ we replace the corresponding ℓ_p or ℓ_q norm by the ℓ_∞ norm and continue to denote by $Z_{q,p}$ the completion of the space of finitely supported matrices under this norm. (Thus, c_0 replacing ∞ would be a more precise notation in this case but, since it would complicated our statements, we prefer the above notation.) The spaces $Z_{q,p}$ are the subject of investigation of this paper. They are more commonly denoted by $\ell_q(\ell_p)$ or $(\bigoplus_{n=1}^{\infty} \ell_p)_{\ell_q}$ (as we have done in the introduction) but since we shall be forced to also consider more complicated spaces with mixed norms we prefer the notation above.

If $\{k_n\}_{n=1}^{\infty}$ is any sequence of positive integers, we shall denote by $Z_{q,p;\{k_n\}}$, the subspace of $Z_{q,p}$ consisting of matrices a satisfying $a(i,j) = 0$ for all $i > k_j$.

We also denote by $Z_{q,p,r}$ (we'll use this only for $r = 2$) the spaces of arrays $a = \{a(u,i,j)\}_{u,i,j=1}^{\infty}$ with norm

$$\|a\| = \|a\|_{q,p,r} = (\sum_{j=1}^{\infty}(\sum_{i=1}^{\infty}(\sum_{u=1}^{\infty} |a(u,i,j)|^r)^{p/r})^{q/p})^{1/q},$$

with the same convention as above when one of p, q (or r) is ∞. Similarly, $Z_{q,p;\{k_n\},r}$ denotes the subspace of $Z_{q,p,r}$ consisting of arrays a satisfying $a(u,i,j) = 0$ for all $i > k_j$.

By P_n we denote the natural projection onto the n-th column in $Z_{q,p}$; i.e, $P_n(\{a(i,j)\}) = \{\bar{a}(i,j)\}$, where $\bar{a}(i,j) = a(i,j)$ if $j = n$ and $\bar{a}(i,j) = 0$ otherwise. Similarly, P_n^k denotes the natural projection onto the first k elements in the n-th column. Q_N denotes $\sum_{n=1}^{N} P_n$.

Given a Banach lattice X, an $1 < r < \infty$ and $x_1, x_2, \dots \in X$ one can define the operation $(\sum |x_n|^r)^{1/r}$ in a manner consistent with what we usually mean by such an operation (when X is a lattice of functions or sequences, for example). See e.g. [LT2, Section 1.d] for this and what follows.

In particular if X has a 1-unconditional basis $\{e_i\}$ (which is the only kind of lattices we'll consider here) then for $x_n = \sum_{i=1}^{\infty} a_i^n e_i$, $n = 1, 2, \dots, N$, $(\sum |x_n|^r)^{1/r} = \sum_{i=1}^{\infty}(\sum_{n=1}^{N} |a_i^n|^r)^{1/r} e_i$.

Recall that X is said to be r-convex (resp. r-concave) with constant K if for all n and all $x_1, x_2, \ldots, x_n \in X$

$$\|(\sum_{i=1}^n |x_i|^r)^{1/r}\| \le K(\sum_{i=1}^n \|x_i\|^r)^{1/r} \text{ (resp. } (\sum_{i=1}^n \|x_i\|^r)^{1/r} \le K\|(\sum_{i=1}^n |x_i|^r)^{1/r}\|).$$

X is said to be r-convex (resp. r-concave) if it is r-convex (resp. r-concave) with some constant $K < \infty$. $Z_{q,p}$ is easily seen to be $\min\{p,q\}$-convex with constant 1 and $\max\{p,q\}$-concave with constant 1.

It is also known that X is r-convex (resp. r-concave) if and only if its dual X^* is r'-concave (resp. r'-convex) where $r' = r/(r-1)$.

Given a Banach lattice X we denote by $X(\ell_2)$ the (completion of the) space of (finite) sequences $x = (x_1, x_2, \ldots)$ of elements of X for which the norm

$$\|x\| = \|(\sum |x_j|^2)^{1/2}\|$$

is finite. If X has a 1-unconditional basis $\{e_j\}$ then this is just the (completion of the) space of matrices $a = \{a(i,j)\}$ (with only finitely many non-zero entries) with norm

$$\|a\| = \|\sum_{j=1}^\infty (\sum_{i=1}^\infty |a(i,j)|^2)^{1/2} e_i\|.$$

The following two lemmas are well known but maybe hard to find so we reproduce their proofs.

Lemma 1 *Let $\{x_i\}_{i=1}^\infty$ be a normalized unconditional basic sequence in $Z_{q,p}$, $1 \le p < q \le \infty$. If for some $\varepsilon > 0$ and $N \in \mathbb{N}$ $\|Q_N x_i\| > \varepsilon$ for all i then $\{x_i\}_{i=1}^\infty$ has a subsequence equivalent to the unit vector basis of ℓ_p.*

Proof: Assume first $p > 1$. Given a sequence of positive ε_i-s and passing to a subsequence (which without loss of generality we assume is the all sequence) we can assume that there is a sequence of $\{y_i\}$ of vectors disjointly supported with respect to the natural basis of $Z_{q,p}$ such that $\|x_i - y_i\| < \varepsilon_i$ for all i. (Use the fact that $\{x_i\}$ doesn't have a subsequence equivalent to the unit vector basis of ℓ_1 and the argument for Proposition 1.a.12 in [LT1], for example). $\{y_i\}$ is 1-dominated by the unit vector basis of ℓ_p and dominates $\{Q_N y_i\}$

which in turn C-dominates the unit vector basis of ℓ_p for $C = 1/(\varepsilon - \sup \varepsilon_i)$; i.e,

$$(\sum_{i=1}^{\infty} |a_i|^p)^{1/p} \geq \| \sum_{i=1}^{\infty} a_i y_i \| \geq (\varepsilon - \sup \varepsilon_i)(\sum_{i=1}^{\infty} |a_i|^p)^{1/p}$$

for all scalars $\{a_i\}$. If the ε_i-s are small enough a similar inequality holds for the (sub)sequence $\{x_i\}$.

If $p = 1$ then given a sequence of positive ε_i-s and passing to a subsequence (which without loss of generality we assume is the all sequence) we can assume that there is a vector y and sequence of $\{y_i\}$ of vectors all disjointly supported with respect to the natural basis of $Z_{q,p}$ such that $\|x_i - y - y_i\| < \varepsilon_i$ for all i. If $y \neq 0$ and the $\{\varepsilon_i\}$ are small enough then, using the unconditionality $\{x_i\}$ is clearly equivalent to the unit vector basis of ℓ_1. If $y = 0$ the same argument as for $p > 1$ works here too. ■

Lemma 2 *Let $\{x_i\}$ be a K-unconditional basic sequence in a Banach lattice which is r-concave for some $r < \infty$ Let $\bar{x}_i \in X(\ell_2)$ be defined by $(0, \dots, 0, x_i, 0, \dots)$, x_i in the i-th place. Then the sequences $\{x_i\}$ in X and $\{\bar{x}_i\}$ in $X(\ell_2)$ are equivalent.*

If in addition X is also s-convex for some $s > 1$ and $[x_i]$, the closed linear span of $\{x_i\}$, is complemented in X then $[\bar{x}_i]$ is complemented in $X(\ell_2)$.

Proof: The first assertion, due to Maurey, can be found in [Ma] or [LT2, Theorem1.d.6(i)]. The second is probably harder to find so we reproduce it. Let $P = \sum_{i=1}^{\infty} x_i^* \otimes x_i$, with $x_i^* \in X^*$, be the projection onto $[x_i]$; i.e.,

$$P(x) = \sum_{i=1}^{\infty} x_i^*(x) x_i \quad x \in X.$$

Define $\bar{P} = \sum_{i=1}^{\infty} \bar{x}_i^* \otimes \bar{x}_i$ $(\bar{x}_i^* \in X^*(\ell_2) = X(\ell_2)^*)$; i.e.,

$$\bar{P}(x) = \sum_{i=1}^{\infty} \bar{x}_i^*(x) \bar{x}_i \quad x \in X(\ell_2).$$

Using the facts that $\{\bar{x}_i\}$ is equivalent to $\{x_i\}$, $\{\bar{x}_i^*\}$ is equivalent to $\{x_i^*\}$, and $\{\bar{x}_i^*, \bar{x}_i\}$ is a biorthogonal sequence, it is easy to see that $\bar{P}$ is a bounded projection on $X(\ell_2)$ with range $[\bar{x}_i]$. ■

3 Proof of the main result, the reflexive case

Since the non-reflexive cases (i.e., when p or q are 1 or ∞) of Theorem 1 require a bit different treatment and since the problem raised in [DFOS] was restricted to the reflexive cases only, we prefer to delay the proof of the non-reflexive cases to the next section.

Proposition 1 *Let $\{x_i\}_{i=1}^{\infty}$ be a normalized unconditional basic sequence in $Z_{q,p}$, $1 < p, q < \infty$ such that $[x_i]_{i=1}^{\infty}$ is complemented in $Z_{q,p}$. If no subsequence of $\{x_i\}_{i=1}^{\infty}$ is equivalent to the unit vector basis of ℓ_p then $[x_i]_{i=1}^{\infty}$ isomorphically embeds in $Z_{q,p;\{n\},2}$ as a complemented subspace.*

Proof: We may clearly assume $p \neq q$ and by duality that $q > p$. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers. By Lemma 1 for all n only finitely many of the x_i-s satisfy $\|P_n x_i\| \geq \varepsilon_n$. Consequently, for each $n \in \mathbb{N}$ there is a $k_n \in \mathbb{N}$ such that $\|(P_n - P_n^{k_n})x_i\| < \varepsilon_n$ for all i. We denote $Q = \sum_{n=1}^{\infty} P_n^{k_n}$. In the case $p = 2$ we showed in [Sc] that without loosing generality we can assume that $\{x_i\}$ is a block basis of the natural basis of $Z_{q,p}$ and then $\{Qx_i\}$ and $\{(I - Q)x_i\}$ are also unconditional basic sequences. This is no longer true when $p \neq 2$. We overcome this difficulty by switching to the larger space $Z_{q,p,2}$. Define for each i $\bar{x}_i \in Z_{q,p,2}$ by

$$\bar{x}_i(w, u, v) = \begin{cases} x_i(u, v), & \text{if } w = i; \\ 0, & \text{if } w \neq i. \end{cases}$$

Let the projection P from $Z_{q,p}$ onto $[x_i]$ be given by

$$Px = \sum_{i=1}^{\infty} x_i^*(x) x_i$$

where $\{x_i^*\}$ in $Z_{q',p'} = Z_{q,p}^*$ ($p' = p/(p-1)$, $q' = q/(q-1)$) satisfy $x_i^*(x_j) = \delta_{i,j}$, $i, j = 1, 2, \ldots$. Then by Lemma 2

$$\bar{P}x = \sum_{i=1}^{\infty} \bar{x}_i^*(x) \bar{x}_i$$

is a bounded projection from $Z_{q,p,2}$ onto $[\bar{x}_i]$ and $\{x_i\}_{i=1}^{\infty}$ is equivalent to $\{\bar{x}_i\}_{i=1}^{\infty}$.

We denote by $\bar{P}_n = P_n \otimes I_{\ell_2}$ on $Z_{q,p,2}$; i.e, $\bar{P}_n(x)(w, u, v) = P_n(x(w, \cdot, \cdot))(u, v)$. We also similarly denote $\bar{P}_n^k = P_n^k \otimes I_{\ell_2}$, $\bar{Q}_N = Q_N \otimes I_{\ell_2}$, and $\bar{Q} = Q \otimes I_{\ell_2}$.

Note that now $\{\bar{Q}\bar{x}_i\}$ and $\{(I-\bar{Q})\bar{x}_i\}$ are also unconditional basic sequences. We would like to show that if $\varepsilon_n \to 0$ fast enough, then $\{\bar{Q}\bar{x}_i\}$ is equivalent to $\{\bar{x}_i\}$ and thus to $\{x_i\}$ and that $[\bar{Q}\bar{x}_i]$ is complemented.

Now,

$$(I-\bar{Q})\bar{P}\bar{Q}\bar{x}_n = \sum_{i=1}^{\infty} \bar{x}_i^*(\bar{Q}\bar{x}_n)(I-\bar{Q})\bar{x}_i, \quad n = 1, 2, \ldots.$$

The operator $(I-\bar{Q})\bar{P}$ sends the span of the unconditional basic sequence $\{\bar{Q}\bar{x}_n\}$ to the span of the unconditional basic sequence $\{(I-\bar{Q})\bar{x}_n\}$ thus the diagonal operator D defined by

$$D\bar{Q}\bar{x}_n = \bar{x}_n^*(\bar{Q}\bar{x}_n)(I-\bar{Q})\bar{x}_n, \quad n = 1, 2, \ldots.$$

is bounded (see e.g. [To] or [LT1, Proposition 1.c.8]). If we show that $\bar{x}_n^*(\bar{Q}\bar{x}_n)$ are uniformly bounded away from zero this will show that $\{\bar{Q}\bar{x}_n\}$ dominates $\{(I-\bar{Q})\bar{x}_n\}$ and thus also $\{\bar{x}_n\} = \{(I-\bar{Q})\bar{x}_n + \bar{Q}\bar{x}_n\}$. That $\{\bar{Q}\bar{x}_n\}$ is dominated by $\{\bar{x}_n\}$ is clear from the boundedness of $\bar{Q}$. This will show that $\{\bar{Q}\bar{x}_n\}$ is equivalent to $\{x_n\}$. To show that $\bar{x}_n^*(\bar{Q}\bar{x}_n)$ are uniformly bounded away from zero note that

$$\bar{x}_n^*(\bar{Q}\bar{x}_n) = 1 - \bar{x}_n^*((I-\bar{Q})\bar{x}_n)$$

and that

$$|\bar{x}_n^*((I-\bar{Q})\bar{x}_n)| \le \|\bar{P}(I-\bar{Q})\bar{x}_n\| \le \|\bar{P}\| \sum_{i=1}^{\infty} \varepsilon_i.$$

So, if $\|\bar{P}\| \sum_{i=1}^{\infty} \varepsilon_i < 1/2$, then $\bar{x}_n^*(\bar{Q}\bar{x}_n) \ge 1/2$ for all n.

We still need to show that $[\bar{Q}\bar{x}_n]$ is complemented. Note that $\{\frac{\bar{x}_n^*}{\bar{x}_n^*(\bar{Q}\bar{x}_n)}, \bar{Q}\bar{x}_n\}$ is a biorthogonal sequence such that $\{\bar{Q}\bar{x}_n\}$ is equivalent to $\{\bar{x}_n\}$ and $\{\frac{\bar{x}_n^*}{\bar{x}_n^*(\bar{Q}\bar{x}_n)}\}$ is dominated by $\{x_n^*\}$. It follows that

$$x \to \sum_{n=1}^{\infty} \frac{\bar{x}_n^*(x)}{\bar{x}_n^*(\bar{Q}\bar{x}_n)} \bar{Q}\bar{x}_n$$

defines a bounded projection with range $[\bar{Q}\bar{x}_n]$.

We have shown that $[x_i]$ embeds complementably into $Z_{q,p;\{k_n\},2}$ for some sequence of positive integers $\{k_n\}$. This last space is clearly isometric to a norm one complemented subspace of $Z_{q,p;\{n\},2}$. ∎

In the case $p = 2$ the argument above simplifies and actually shows that under the assumptions of Proposition 1 we can strengthen the conclusion to: $[x_i]$ embeds complementably in $Z_{q,2;\{n\}}$ (which is isomorphic to ℓ_q). We will not dwell on it farther as this is contained in [Sc]. The next proposition combained with the previous one will show in particular that any unconditional basis of $Z_{q,p}$ contains a subsequence equivalent to the unit vector basis of ℓ_p. We'll need to use this also in the next section so we include the non-reflexive cases here as well.

Proposition 2 *Let $1 \leq p, q \leq \infty$. If $p \neq 2, q$, then ℓ_p (c_0 in case $p = \infty$) does not embed into $Z_{q,p;\{n\},2}$.*

Proof: Assume ℓ_p or c_0 embeds into $Z_{q,p;\{n\},2}$. Passing to a subsequence of the image of the unit vector basis of ℓ_p or c_0, taking successive differences (this is needed only in the case $p = 1$) and using a simple perturbation argument, we may assume that some normalized block basis $\{x_i\}$ of the natural basis of $Z_{q,p;\{n\},2}$ is equivalent to the unit vector basis of ℓ_p (c_0 if $p = \infty$). Let $P_{n,m}$, $m = 1, 2 \dots$, $1 \leq n \leq m$, denote the canonical projection onto the n, m copy of ℓ_2 in $Z_{q,p;\{n\},2}$:

$$P_{n,m}(\{a(w, u, v)\}) = \{\bar{a}(w, u, v)\}$$

where

$$\bar{a}(w, u, v) = \begin{cases} a(w, u, v), & \text{if } u = n,\ v = m; \\ 0, & \text{otherwise.} \end{cases}$$

Assume first $p > 2$. For each n, m $P_{n,m}$ acts as a compact operator from $[x_i]$ to ℓ_2 as every bounded operator from ℓ_p, $p > 2$ or c_0 to ℓ_2 do. Consequently, given a sequence of positive numbers $\{\varepsilon_{n,m}\}$, we can find $k_{n,m} \in \mathbb{N}$ such that $\|(P_{n,m} - P_{n,m}^{k_{n,m}})_{|[x_i]}\| < \varepsilon_{n,m}$ for all n, m. Then, if $\sum_{n,m} \varepsilon_{n,m}$ is small enough

$$(\sum_{n,m} P_{n,m}^{k_{n,m}})_{|[x_i]}$$

is an isomorphism and we get that $[x_i]$ embeds into $Z_{q,p;\{n\},2;\{k_{n,m}\}}$. Now for each finite m and k the ℓ_p^m sum of ℓ_2^k-s 2-embeds into ℓ_p^N for some N depending only on p, m and k. It thus follows that $[x_i]$ embeds into $Z_{q,p;\{k_n\}}$ for some sequence of positive integers $\{k_n\}$. Passing to a farther subsequence of $\{x_i\}$, we get that the unit vector basis of ℓ_p (or c_0 in the case $p = \infty$) is equivalent to that of ℓ_q which is a contradiction.

The case $1 \le p < 2$ is just a bit more complicated. Here $P_{n,m}$ doesn't act as a compact operator from $[x_i]$ to ℓ_2 but it is still strictly singular. Consequently, for each n, m and l we can find a normalised block basis of $\{x_i\}_{i=l}^\infty$ such that $\|(P_{n,m})_{|[x_i]_{i=l}^\infty}\| < \varepsilon_{n,m}$ and consequently there is a block basis of $\{x_i\}$ *whose first* $l-1$ *terms are just* $x_1, \ldots, x_{l-1}$, and $k_{n,m,l}$ such that

$$\|(P_{n,m} - P_{n,m}^{k_{n,m,l}})_{|[x_i]}\| < \varepsilon_{n,m}.$$

A simple diagonal argument will now produce a normalised block basis $\{z_i\}$ of $\{x_i\}$ and natural numbers $k_{n,m}$-s such that

$$(\sum_{n,m} P_{n,m}^{k_{n,m}})_{|[z_i]}$$

an isomorphism. Since $\{z_i\}$ is equivalent to the unit vector basis of ℓ_p we get that ℓ_p embeds into $Z_{q,p;\{n\},2;\{k_{n,m}\}}$. The rest of the proof in this case is the same as in the case $p > 2$. ■

We are now aiming at proving that every normalized unconditional basis of $Z_{q,p}$ contains a subsequence equivalent to the unit vector basis of ℓ_q.

Proposition 3 *Let* $\{x_i\}_{i=1}^\infty$ *be a normalized unconditional basic sequence in* $Z_{q,p}$, $1 < p, q < \infty$ *such that* $[x_i]_{i=1}^\infty$ *is complemented in* $Z_{q,p}$. *If no subsequence of* $\{x_i\}_{i=1}^\infty$ *is equivalent to the unit vector basis of* ℓ_q *then* $[x_i]_{i=1}^\infty$ *isomorphically embeds in* $Z_{p,2}$ *as a complemented subspace.*

Proof: We may assume $q < p$. We first claim that for each $\varepsilon > 0$ there is an N such that $\|(I - Q_N)x_i\| < \varepsilon$ for each $i = 1, 2, \ldots$. Indeed if this is not the case then there is an $\varepsilon > 0$, a sequence $0 = N_1 < N_2 < \cdots$ in $\mathbb{N}$ and a subsequence $\{y_i\}$ of $\{x_i\}$ such that $\|(Q_{i+1} - Q_i)y_i\| \ge \varepsilon$ for all i. Passing to a further subsequence and a small perturbation we may assume that $\{y_i\}$ is a block basis of the natural basis of $Z_{q,p}$. Then, since $q < p$, for all scalars $\{a_i\}$,

$$(\sum_{i=1}^\infty |a_i|^q)^{1/q} \ge \|\sum_{i=1}^\infty a_i y_i\| \ge \|\sum_{i=1}^\infty a_i (Q_{i+1} - Q_i) y_i\| \ge \varepsilon(\sum_{i=1}^\infty |a_i|^q)^{1/q}$$

in contradiction to the fact that no subsequence of the $\{x_i\}$ is equivalent to the unit vector basis of ℓ_q.

The rest of the proof is now similar to that of Proposition 1, only a bit simpler. Fix an $\varepsilon > 0$ and let N be as in the beginning of this proof. Let $P = \sum_{i=1}^{\infty} x_i^* \otimes x_i$ be the projection onto $[x_i]$ and let $\{\bar{x}_i\}$ (in $Z_{q,p,2}$), $\bar{P}$ and $\bar{Q}_N$ be as in the proof of Proposition 1. Consider the operator $(I - \bar{Q}_N)\bar{P}$ as acting from the span of the unconditional basic sequence $\{\bar{Q}_N \bar{x}_i\}$ to the span of the unconditional sequence $\{(I - \bar{Q}_N)\bar{x}_i\}$:

$$(I - \bar{Q}_N)\bar{P}\bar{Q}_N\bar{x}_n = \sum_{i=1}^{\infty} \bar{x}_i^*(Q_N\bar{x}_n)(I - \bar{Q}_N)\bar{x}_i, \ n = 1, 2, \ldots .$$

Its diagonal defined by

$$D\bar{Q}_N\bar{x}_n = \bar{x}_n^*(Q_N\bar{x}_n)(I - \bar{Q}_N)\bar{x}_n, \ n = 1, 2, \ldots$$

is bounded ([To] or [LT1]). So if we show that $\bar{x}_n^*(\bar{Q}_N\bar{x}_n)$ are bounded away from zero then the sequence $\{\bar{Q}_N\bar{x}_i\}$ will dominate the sequence $\{(I - \bar{Q}_N)\bar{x}_i\}$ and thus also $\{\bar{x}_i\}$ and $\{x_i\}$. This will also show that

$$x \to \sum_{n=1}^{\infty} \frac{\bar{x}_n^*(x)}{\bar{x}_n^*(\bar{Q}_N\bar{x}_n)} \bar{Q}_N\bar{x}_n$$

defines a bounded projection from $\bar{Q}_N Z_{q,p,2}$ (which is isomorphic to $Z_{p,2}$) onto $[\bar{Q}_N\bar{x}_i]$ (which is isomorphic to $[x_i]$).

To show that $\bar{x}_n^*(\bar{Q}_N\bar{x}_n)$ are bounded away from zero note that

$$\bar{x}_n^*(\bar{Q}_N\bar{x}_n) = 1 - \bar{x}_n^*((I - \bar{Q}_N)\bar{x}_n)$$

and that

$$|\bar{x}_n^*((I - \bar{Q}_N)\bar{x}_n)| \le \|\bar{P}(I - \bar{Q}_N)\bar{x}_n\| \le \|\bar{P}\|\varepsilon.$$

So, if $\|\bar{P}\|\varepsilon < 1/2$, then $\bar{x}_n^*(\bar{Q}\bar{x}_n) \ge 1/2$ for all n. ∎

Remark 1 *With a bit more effort one can strengthn the conclusion of Proposition 3 to: $[x_i]_{i=1}^{\infty}$ is isomorphic to ℓ_p. This is done by first showing that one can embed $[x_i]_{i=1}^{\infty}$ as a complemented subspace in $Z_{p,2;\{n\}}$ which is isomorphic to ℓ_p and using the fact that any infinite dimensional complemented subspace of ℓ_p is isomorphic to ℓ_p.*

Proof of Theorem 1 in the reflexive case: Propositions 1 and 2 show that any normalized unconditional basis of $Z_{q,p}$, $1 < p, q < \infty$, has a subsequence equivalent to the unit vector basis of ℓ_p. To show that any such basis also has a subsequence equivalent to the unit vector basis of ℓ_q we need, in view of Proposition 3, only prove that $Z_{q,p}$ doesn't embed complementably into $Z_{p,2}$ for $1 < q \neq p < \infty$. This can probably be done directly (especially in the case $q \neq 2$ in which case it is also true that ℓ_q does not embed into $Z_{p,2}$) but it also follows from the main theorems of [Sc] and [Od] in which the complemented subspaces of $Z_{p,2}$ (in [Sc] only those with unconditional basis) where characterized. ∎

4 Proof of the main result, the non-reflexive case

Recall that the subscript ∞ in $Z_{\infty,p}$ refers, by our convention, to the c_0 (rather than ℓ_∞) sum. Similatly, the subscript ∞ in $Z_{q,\infty}$ refers to the q sum of c_0. We are going to show that any unconditional basis of each of the spaces $Z_{q,p}$, $p \neq q$, when at least one of p or q is 1 or ∞ contains a subsequence equivalent to the unit vector basis of ℓ_p (c_0 if $p = \infty$) and another subsequence equivalent to the unit vector basis of ℓ_q (c_0 if $q = \infty$).

The spaces $Z_{1,\infty}$ and $Z_{\infty,1}$ (as well as $Z_{1,2}$ and $Z_{\infty,2}$) have unique, up to permutation, unconditional bases [BCLT]. These bases clearly contain a subsequence equivalent to the unit vector basis of c_0 and another one equivalent to the unit vector basis of ℓ_1, so we only need to deal with the spaces $Z_{\infty,p}$, $1 < p < \infty$, and their duals $Z_{1,p'}$ and with $Z_{q,\infty}$, $1 < q < \infty$, and their duals $Z_{q',1}$.

We shall need some classical results concerning unconditional bases and duality. These can be found conveniently in sections 1.b. and 1.c. of [LT1]. ℓ_1 does not isomorphically embed into $Z_{\infty,p}$, $1 < p < \infty$, (resp. into $Z_{q,\infty}$, $1 < q < \infty$) (this follows for example from the fact that these spaces are p (resp. q) convex). It thus follows from a theorem of James [Ja] or see [LT1, Theorem 1.c.9] that any unconditional basis of these spaces is shrinking. See [LT1, Proposition 1.b.1] for the the definition of a shrinking basis as well as for the fact that then the biorthogonal basis is an unconditional basis of the dual space $Z_{1,p'}$, $1 < p < \infty$, (resp. $Z_{q',1}$, $1 < q < \infty$). Thus, in order to prove Theorem 1 in the non-reflexive cases, if would be enough to show that any normalized unconditional basis of $Z_{1,p}$, $1 < p < \infty$, (resp. $Z_{q,1}$, $1 < q < \infty$) contains a subsequence equivalent to the unit vector basis of ℓ_1 and another subsequence equivalent to the unit vector basis of ℓ_p (resp. ℓ_q).

Let $\{x_n\}$ be a normalized unconditional basis of $X^* = Z_{1,p}$, $1 < p < \infty$, (resp. $X^* = Z_{q,1}$, $1 < q < \infty$) such a basis is boundedly complete and its biorthogonal basis spans a space isomorphic to $X = Z_{\infty,p'}$ (resp. $X = Z_{q',\infty}$).

We begin with a proposition which replaces Propositions 1 and 2 for the current cases.

Proposition 4 *Let $\{x_n\}$ be a normalized unconditional basis of $Z_{1,p}$, $1 < p < \infty$, (resp. $Z_{q,1}$, $1 < q < \infty$). Then $\{x_n\}$ contains a subsequence equivalent to the unit vector basis of ℓ_p (resp. ℓ_1).*

Proof: By proposition 2, ℓ_p does not embed into $Z_{1,p:\{n\},2}$ for $1 < p < \infty$ and ℓ_1 does not embed into $Z_{q,1:\{n\},2}$ for $1 < q < \infty$. It is thus enough to show that if $\{x_n\}$ contains no subsequence equivalent to the unit vector basis of ℓ_p (resp. ℓ_1) then $[x_n]$ embeds in $Z_{1,p:\{n\},2}$ (resp. $Z_{q,1:\{n\},2}$).

The case of $Z_{q,1}$, $1 < q < \infty$: We proceed as in the proof of Proposition 1. Since $q > 1$ the beginning of the proof works for $p = 1$ as well. The problem arise when we need to show that $\bar{P}$ is bounded as this no longer follow from Lemma 2. But here we can use instead [LT2, Theorem 1.d.6(ii)] to prove that $\bar{P}$ is bounded in a very similar way to the proof of Lemma 2. The rest of the proof of Proposition 1 carries over.

The case of $Z_{1,p}$, $1 < p < \infty$: Assume $\{x_n\}$ be a basis of $Z_{1,p}$, $1 < p < \infty$. Let $\{x_n^*\}$ be the biorthogonal basis (of $Z_{\infty,p'}$). By the assumption that $\{x_n\}$ doesn't contain a subsequence equivalent to the unit vector basis of ℓ_p, $[x_n^*]$ doesn't contain a subsequence equivalent to the unit vector basis of $\ell_{p'}$. The proof of Proposition 1 works for $Z_{\infty,p'}$, $1 < p' < \infty$, as well, with the same modification for the proof that $\bar{P}$ is bounded as in the previous paragraph, to show that in this case $[x_n^*]$ embeds (even complementably) into $Z_{\infty,p':\{n\},2}$.
■

The next proposition replaces Proposition 3 in the non-reflexive case.

Proposition 5 *(i) Let $\{x_n\}$ be a normalized unconditional basis of $Z_{1,p}$, $1 < p < \infty$. Then the unit vector basis of ℓ_1 is equivalent to a subsequence of $\{x_n\}$.*

(ii) Let $\{x_n\}$ be a normalized unconditional basis of $Z_{q,1}$, $1 < q < \infty$. Then the unit vector basis of ℓ_q is equivalent to a subsequence of $\{x_n\}$.

Proof: The proof of Proposition 3 works for $Z_{q,p}$ also in the case $q = 1 < p < \infty$ and we get that under the assumption of (i), if no subsequence of $\{x_n\}$ is equivalent to the unit vector basis of ℓ_1 then $[x_n]$ embeds into $Z_{p,2}$ but this space has type $\min\{p, 2\}$ so ℓ_1 and thus also $Z_{1,p}$, $1 < p < \infty$, do not embed into it. This proves (i).

(ii) It is enough to show that the unit vector basis of $\ell_{q'}$ is equivalent to a subsequence of $\{x_n^*\}$ (the biorthogonal basis to $\{x_n\}$) which is an unconditional basis of $Z_{q',\infty}$. The proof of Proposition 3 gives that if this is not the case then $Z_{q',\infty}$ isomorphically embeds as a complemented subspace in $Z_{\infty,2}$. Now if $Z_{q',\infty}$ isomorphically embeds as a complemented subspace in $Z_{\infty,2}$ then an easy application of Pełczynski's decomposition method gives that $Z_{q',\infty} \oplus Z_{\infty,2}$ is isomorphic to $Z_{\infty,2}$ but this immediately presents an unconditional basis for $Z_{\infty,2}$ which is not equivalent to a permutation of the cannonical basis of $Z_{\infty,2}$. This stands in contradiction to a result from [BCLT] and thus proves (ii). ■

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