A CHARACTERIZATION OF RELATIVELY DECOMPOSABLE BANACH LATTICES

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ABSTRACT

Two Banach lattices of measurable functions X and Y are said to be relatively decomposable if there exists a constant D such that whenever two functions f and g can be expressed as sums of sequences of disjointly supported elements of X and Y respectively, $f = \sum_{n=1}^{\infty} f_n$ and $g = \sum_{n=1}^{\infty} g_n$, such that $\|g_n\|_Y \leq \|f_n\|_X$ for all $n = 1, 2, \ldots$, and it is given that $f \in X$, then it follows that $g \in Y$ and $\|g\|_Y \leq D\|f\|_X$.

Relatively decomposable lattices appear naturally in the theory of interpolation of weighted Banach lattices.

It is shown that X and Y are relatively decomposable if and only if, for some $r \in [1, \infty]$, X satisfies a lower r-estimate and Y satisfies an upper r-estimate. This is also equivalent to the condition that X and ℓ^r are relatively decomposable and also ℓ^r and Y are relatively decomposable.

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1. INTRODUCTION

In a recent paper [CN3] two of us have made a detailed study of the problem of characterizing all the interpolation spaces with respect to couples of weighted Banach lattices of measurable functions. The main results in that paper provide the means to determine for exactly which couples of such Banach lattices it is possible, for all choices of weight functions, to characterize all interpolation spaces in terms of simple monotonicity conditions in terms of K-functionals. Such a characterization is a natural extension of the characterization obtained long ago by Calderón [Ca] and also equivalently by Mityagin [Mi] for describing all interpolation spaces with respect to L^1 and L^∞. One of the key notions required for the formulation of the main results of [CN3] is that of a pair of *relatively decomposable* Banach lattices.

We refer to [CN3] for more details and also to earlier papers [CN2], [CN1] and [Cw] in which the notion of relative decomposability and other closely related notions also play a central role. Some of the concepts in interpolation theory which interact with relative decomposability are also briefly described at the end of this paper. A forthcoming paper [CS] will continue the investigation of relative decomposability and also relative q-decomposability, introduced in [Cw].

Our aim here is to give a characterization of relatively decomposable Banach lattices and show their relation with L^p spaces.

Our result could be deduced from a careful study of parts of the difficult proof of a theorem of Krivine, Maurey and Pisier. See e.g. [MS]. However we choose to give a different, more self–contained and simpler proof.

Throughout this paper we will use the terminology *Banach lattice* to mean a Banach space X of real valued measurable functions on some σ-finite measure space (Ω, Σ, μ) having the property that for all measurable f and g on Ω , if $f \in X$ and $|g| \leq |f|$ a.e. on Ω then also $g \in X$ and $\|g\|_X \leq \|f\|_X$. For the applications which we have in mind we do not need to consider more general abstract Banach lattices as studied e.g. in [LT].

Definition 1.1. (i) Given a Banach lattice X and some positive number D we shall say that X is *decomposable* or that X is *decomposable with constant* D if it has the following property:

Let g be an arbitrary real valued measurable function on the underlying measure space (Ω, Σ, μ) which can be expressed as the sum $g = \sum_{n\in A} g_n$ of a collection of disjointly supported elements of X, for some (finite or infinite) subset $A \subset \mathbb{N}$. Suppose that there exists an element $f \in X$ which can be expressed as the sum $f = \sum_{n\in A} f_n$ of disjointly supported elements such that $\|g_n\|_X \leq \|f_n\|_X$ for all $n \in A$. Then it follows that $g \in X$ and $\|g\|_X \leq D\|f\|_X$.

(ii) Similarly, if X and Y are two Banach lattices, possibly having different underlying measure spaces, then we shall say that X *and* Y *are relatively decomposable with constant* D if, whenever two functions f and g can be expressed as sums of disjointly supported elements of X and Y respectively, $f = \sum_{n\in A} f_n$ and $g = \sum_{n\in A} g_n$, for some $A \subset \mathbb{N}$ such that $\|g_n\|_Y \leq \|f_n\|_X$ for all $n \in A$, and it is given that $f \in X$, then it follows that $g \in Y$ and $\|g\|_Y \leq D\|f\|_X$.

In Proposition 1.4 of [CN1 p. 58] it is observed, using an appropriate form of a representation theorem of Kakutani-Bohnenblust-Tzafriri, that any decomposable lattice X which is also σ-order continuous and has the Fatou property necessarily coincides, for some $p \in [1, \infty)$, with an L^p space of functions supported on some measurable subset Ω' of Ω for some suitable measure defined on Ω'.

The characterization of relatively decomposable Banach lattices which we present here can be considered as an extension of this result. Once again there is a connection, albeit a less rigid one, with the structure of L^p spaces.

Let us preface the formulation of our main result with some elementary remarks and some definitions:

(i) Let X, Y and Z be Banach lattices (on possibly different measure spaces). Suppose that X and Z are relatively decomposable (with constant D_1) and that Z and Y are relatively decomposable (with constant D_2). Suppose also that Z is infinite dimen-

sional. Then its support (cf. Remark 1.3 of [CN3]) contains a nonatomic subset or infinitely many atoms (or both). Thus Z contains an infinite sequence of disjointly supported non-zero elements. With the help of this sequence it is easy to show that X and Y are relatively decomposable (with constant $D_1 D_2$).
(ii) If $X = L^p$ and $Y = L^q$ are defined on measure spaces containing infinitely many disjoint subsets of finite positive measure then X and Y are relatively decomposable if and only if $p \leq q$. (In that case the decomposability constant D can always be taken to be 1.)

DEFINITION 1.2. (Cf. [LT p. 82].) A Banach lattice X is said to satisfy an *upper*, respectively, *lower* p-estimate for some $p \in [1, \infty]$ if there exists a constant $C < \infty$ such that every finite sequence $\{x_n\}_{n=1}^N$ of disjointly supported elements in X satisfies

$$\left\| \sum_{n=1}^{N} x_n \right\|_X \leq C \left(\sum_{n=1}^{N} \|x_n\|_X^p \right)^{1/p} \tag{1.2a}$$

respectively,

$$\left(\sum_{n=1}^{N} \|x_n\|_X^p \right)^{1/p} \leq C \left\| \sum_{n=1}^{N} x_n \right\|_X \quad . \tag{1.2b}$$

Obviously (1.2b) holds if and only if

$$\left(\sum_{n=1}^{\infty} \|x_n\|_X^p \right)^{1/p} \leq C \left\| \sum_{n=1}^{\infty} x_n \right\|_X$$

for every infinite sequence of disjointly supported elements. This in turn is equivalent to the condition that X and ℓ^p are relatively decomposable with constant C. Similarly, if X has the Fatou property, (1.2a) is equivalent to the condition that

$$\left\| \sum_{n=1}^{\infty} x_n \right\|_X \leq C \left(\sum_{n=1}^{\infty} \|x_n\|_X^p \right)^{1/p}$$

holds for all disjointly supported infinite sequences and also to the condition that ℓ^p and X are relatively decomposable with constant C.

Thus Remarks (i) and (ii) above can be easily generalized:

(iii) Suppose that X satisfies a lower p-estimate (with constant M_1) and Y has the Fatou property and satisfies an upper q-estimate (with constant M_2). Then, if $p \leq q$, it follows that X and Y are relatively decomposable (with constant $M_1 M_2$).

We can now state the main result of this paper, which shows that (iii) is essentially the only possible mechanism which can make a pair of Banach lattices X and Y relatively decomposable:

THEOREM 1.3. *Let X and Y be Banach lattices on σ-finite measure spaces and suppose that Y has the Fatou property. Then the following conditions are equivalent:*

(1.3a) X and Y are relatively decomposable

(1.3b) There exists $r \in [1, \infty]$ such that X and ℓ^r are relatively decomposable and also ℓ^r and Y are relatively decomposable.

(1.3c) There exists $r \in [1, \infty]$ such that X satisfies a lower r-estimate and Y satisfies an upper r-estimate.

We have already seen in the course of the preceding discussion that (1.3b) and (1.3c) are equivalent and imply (1.3a). The proof of the much less trivial implication (1.3a)$\Rightarrow$ (1.3c) will be presented in Section 3, after we have established some preliminary results in Section 2. In Section 4 we shall give an example of how our result can be applied in interpolation theory.

REMARK: The example where $X = \ell^\infty$ and $Y = c_0$ shows that the requirement that Y has the Fatou property cannot be dispensed with. Nevertheless (see [CS]) there is a variant of Theorem 1.3 which holds without this requirement.

2. EQUAL NORM UPPER AND LOWER p–ESTIMATES AND SOME OTHER PRELIMINARY RESULTS.

We will need the following variants of the notions of upper and lower estimates:

DEFINITION. A Banach lattice X is said to satisfy an *equal norm upper p-estimate* respectively, *equal norm lower p-estimate* if there exists a constant $C < \infty$ such that every finite sequence $\{x_n\}_{n=1}^N$ of disjointly supported norm one elements in X satisfies

$$\left\|\sum_{n=1}^{N} x_n\right\|_X \leq CN^{1/p} ,$$

respectively,

$$N^{1/p} \leq C\left\|\sum_{n=1}^{N} x_n\right\|_X .$$

We shall use the abbreviated terminology *e.n.u. p-estimate* and *e.n.l. p-estimate* for these notions. It is interesting to note that there exist Banach lattices X which do not satisfy a lower p-estimate even though they satisfy an equal norm lower p-estimate. For details see Remark 2.15 below.

We shall use the customary notation X' for the associate space or Köthe dual of any Banach lattice X. This is also a Banach lattice of measurable functions defined on a suitable subset of the underlying measure space of X. (Cf. [Z] Section 69 or [CN3].)

For each $p \in [1, \infty]$ we define p' as usual by $p' = p/(p-1)$.

The next three lemmas are all known. For lack of easily accessed references we have provided proofs of Lemmas 2.3 and 2.4 below. We leave the straightforward proof of the next result as an exercise.

LEMMA 2.1. *Let X be a Banach lattice of measurable functions and suppose that $p \in [1, \infty]$.*

(i) If X satisfies an upper p-estimate then X' satisfies a lower p'-estimate.

(ii) If X satisfies an e.n.u. p-estimate then X' satisfies an e.n.l. p'-estimate.

(iii) If X satisfies a lower p-estimate then X' satisfies an upper p'-estimate.

In each case the constant (denoted in the definitions by C) which appears in the estimate satisfied by X' is equal to the constant which appears in the estimate satisfied by X.

REMARK 2.2: If X' is a norming subspace of X^*, the dual of X, then obviously X is a closed isometric subspace of its second Köthe dual X''. In that case it is easy to combine (i) and (iii) of the preceding lemma and to obtain that:

X satisfies an upper, respectively lower, p-estimate if and only if X' satisfies a lower, respectively upper, p'-estimate.

This holds, for example, if X is order continuous, since then X' coincides with the dual of X. (See [Z] p. 462). This also holds if X has the Fatou property, since then $X = X''$. (See [Z] p. 470).

We define two "indices" for each given Banach lattice X:

$$p(X) = \sup\{p : X \ \text{ satisfies an upper } p\text{-estimate}\}$$

$$q(X) = \inf\{q : X \ \text{ satisfies a lower } q\text{-estimate}\}.$$

LEMMA 2.3. *Let X be a Banach lattice of measurable functions. Then*

(i) $q(X) = \inf\{q : X \text{ satisfies an equal norm lower } q\text{-estimate }\}$.

If the associate space X' of X is a norming subspace of the dual of X then also

(ii) $p(X) = \sup\{p : X \text{ satisfies an equal norm upper } p\text{-estimate}\}$.

PROOF: Let us denote the quantities on the right hand side of (i) and (ii) by $q_e(X)$ and $p_e(X)$ respectively. Clearly $q(X) \geq q_e(X)$ and $p(X) \leq p_e(X)$. For the proofs of the reverse inequalities we may assume respectively that $q_e(X) < \infty$ and $p_e(X) > 1$. (Otherwise the results are trivial.) First, to show that $q(X) \leq q_e(X)$, it suffices to show that X satisfies a lower s-estimate for every number $s < \infty$ with the property that X satisfies an e.n.l. q-estimate for some $q < s$: Suppose then, for such q and s, that C is the constant appearing in the lower q-estimate inequalities for X. Let $x_1, x_2, \ldots .x_N$ be any finite sequence of disjointly supported functions in X. We may suppose without loss of generality that $\|x_1\| \geq \|x_2\| \geq \ldots . \geq \|x_N\|$. Then, for all $1 \leq k \leq N$,

$$\begin{aligned} \|x_k\| &= \left(\frac{1}{k} \sum_{j=1}^{k} \left\| \frac{\|x_k\|}{\|x_j\|} x_j \right\|^q \right)^{1/q} \leq Ck^{-1/q} \left\| \sum_{j=1}^{k} \frac{\|x_k\|}{\|x_j\|} x_j \right\| \\ &\leq Ck^{-1/q} \left\| \sum_{j=1}^{k} x_j \right\| \leq Ck^{-1/q} \left\| \sum_{j=1}^{N} x_j \right\| . \end{aligned}$$

Consequently

$$\left(\sum_{k=1}^{N}\|x_k\|^s\right)^{1/s} \le C\left(\sum_{k=1}^{N}k^{-s/q}\right)^{1/s}\left\|\sum_{j=1}^{N}x_j\right\|$$

and, since $\sum_{k=1}^{\infty}k^{-s/q}<\infty$, this shows that X satisfies a lower s-estimate and yields (i).

Finally, to establish $p(X)\ge p_e(X)$ and therefore (ii), we must show that, whenever X satisfies an e.n.u. p-estimate for some p and $1<s<p$, then X satisfies an upper s-estimate. By part (ii) of Lemma 2.1, X' satisfies an e.n.l. p'-estimate. So, since $p'<s'<\infty$, we can apply the preceding argument to show that X' satisfies a lower s'-estimate. Then, using Remark 2.2, we see that X satisfies an upper s-estimate as required, and the proof is complete. ∎

We will need to use the following result several times:

LEMMA 2.4. *Let X be a Banach lattice having the property that for some fixed integer $m>1$ and some fixed r in $[1,\infty]$, all choices of m disjointly supported norm 1 functions $g_1,g_2,\ldots,g_m$ satisfy*

$$\left\|\sum_{j=1}^{m}g_j\right\|_X \le m^{1/r} \tag{2.4a}$$

or, respectively,

$$\left\|\sum_{j=1}^{m}g_j\right\|_X \ge m^{1/r}. \tag{2.4b}$$

Then X satisfies an equal norm upper, respectively lower, r-estimate.

PROOF: Let us begin by treating the case where (2.4b) holds. We first claim that

$$\left\|\sum_{j=1}^{n}g_j\right\|_X \ge n^{1/r} \tag{2.5}$$

holds for each choice of n disjointly supported norm one functions $g_1,g_2,\ldots,g_n$ whenever n is an integer of the form $n=m^k$. We already know this for $k=0$ and $k=1$ and we proceed by induction on k. Indeed, suppose that (2.5) holds for $n=m^{k-1}$. Then, given any collection of m^k disjointly supported norm 1 functions, we may write it in the form $\{g_{ij}\}_{i=1}^{m^{k-1}}\,{}_{j=1}^{m}$. For each integer i, $\|\sum_{j=1}^m g_{ij}\|_X \geq m^{1/r}$. Consequently,

$$\left\|\sum_{i=1}^{m^{k-1}}\sum_{j=1}^{m} g_{ij}\right\|_X = m^{1/r}\left\|\sum_{i=1}^{m^{k-1}}\frac{1}{m^{1/r}}\sum_{j=1}^{m} g_{ij}\right\|_X \geq m^{1/r}(m^{k-1})^{1/r} = (m^k)^{1/r}$$

and (2.5) also holds for $n = m^k$.

Now, given any positive integer n, let k be the positive integer which satisfies $m^{k-1} \leq n < m^k$. Then, for any n disjointly supported g_i's of norm 1,

$$\left\|\sum_{i=1}^{n} g_i\right\|_X \geq \left\|\sum_{i=1}^{m^{k-1}} g_i\right\|_X \geq m^{(k-1)/r} = m^{-1/r}m^{k/r} \geq m^{-1/r}n^{1/r}$$

showing that X satisfies an e.n.l. r-estimate with constant $m^{1/r}$.

It remains to deal with the case where (2.4a) holds instead of (2.4b). Here we first use an argument which is almost identical to the preceding induction on k (but of course with most of the inequalities reversed) to obtain that

$$\left\|\sum_{j=1}^{n} g_j\right\|_X \leq n^{1/r}$$

holds for each choice of n disjointly supported norm one functions $g_1, g_2, \ldots, g_n$ whenever n is an integer of the form $n = m^k$. Then, as before, given any positive integer n, let k be the positive integer which satisfies $m^{k-1} \leq n < m^k$. We now need a slightly more elaborate argument than before (since our arbitrary given collection of n disjointly supported g_i's of norm 1 does not necessarily have to be a subcollection of m^k such functions). We express the set $\{1, 2, \ldots, n\}$ as a union of m disjoint sets B_1, B_2, $\ldots$, B_m each containing at most m^{k-1} integers. Then, for each $j = 1, 2, \ldots, m$, we let A_j be a set containing exactly m^{k-1} integers in $\{1, 2, \ldots, n\}$ such that $B_j \subset A_j$. We have $\left\|\sum_{i\in A_j} g_i\right\|_X \leq m^{(k-1)/r}$ for each j. Consequently,

$$\left\|\sum_{i=1}^{n} g_i\right\|_X = \left\|\sum_{j=1}^{m}\sum_{i\in B_j} g_i\right\|_X \leq \sum_{j=1}^{m}\left\|\sum_{i\in B_j} g_i\right\|_X \leq \sum_{j=1}^{m}\left\|\sum_{i\in A_j} g_i\right\|_X \leq m\cdot m^{(k-1)/r} \leq m\cdot n^{1/r}$$

which shows that X satisfies an e.n.u. r-estimate with constant m. ∎

We will also need the following result:

LEMMA 2.6. *Let Z be a Banach lattice which satisfies a lower s-estimate for some $s < \infty$ with constant B. Let m be a positive integer and suppose that the m numbers $\gamma_1, \gamma_2, ..., \gamma_m$ are all in $[1, \infty)$. Suppose that the integer b satisfies*

$$b \geq m\left((B \max_{1\leq j\leq m} \gamma_j)^s + 1\right). \tag{2.7}$$

Let $h_1, h_2, \ldots, h_b$ be disjointly supported norm one elements in Z. Then there exists a family of m disjoint subsets $A_1, A_2, \ldots, A_m$ of $\{1, 2, \ldots, b\}$ and a number $t \geq 1$ such that

$$t\gamma_j \leq \left\|\sum_{k\in A_j} h_k\right\|_Z \leq (t+2)\gamma_j \text{ for } j = 1, 2, \ldots m \tag{2.8}$$

and

$$\bigcup_{j=1}^{m} A_j = \{1, 2, \ldots, b\} \tag{2.9}$$

PROOF: The first step is to show that there exist m disjoint sets A_1, A_2, $\ldots$, A_m and a number t (in fact $t = 1$) satisfying (2.8). We observe that the numbers $\phi_n := \|\sum_{k=1}^{n} h_k\|_Z$ satisfy $\phi_n \leq 1 + \phi_{n-1}$ for each $n = 2, 3, .., b$. Thus, if $\phi_b \geq \gamma_j$ for some j, then there exists an integer $n, 1 \leq n \leq b$, such that $\gamma_j \leq \phi_n \leq 3\gamma_j$. By exactly the same reasoning, if $b \geq mN$ where N is an integer with the property that

$$\left\|\sum_{k=(j-1)N+1}^{jN} h_k\right\|_Z \geq \gamma_j \text{ for } j = 1, 2, \ldots m, \tag{2.10}$$

then it is possible to find disjoint subsets $A_1, A_2, ..., A_m$ of $\{1, 2, \ldots b\}$ such that

$$\gamma_j \leq \left\|\sum_{k\in A_j} h_k\right\|_Z \leq 3\gamma_j \text{ for } j = 1, 2, \ldots m.$$

In view of our hypothesis on Z we can guarantee that (2.10) holds by requiring that $N^{1/s}/B \geq \max_{1\leq j\leq m} \gamma_j$, and of course, if b satisfies (2.7), then $b \geq mN$ for such an N. This completes the first step.

Now let $A_1, A_2, \ldots, A_m$ be any family of disjoint sets satisfying (2.8) for some $t \geq 1$. If (2.9) also holds then we are done. If (2.9) does not hold, then it may happen that $\left\|\sum_{k\in A_j} h_k\right\|_Z < (t+1)\gamma_j$ for some value j^* of j. If so then we can enlarge A_{j^*} by adding an element of $\{1,2,..,b\}\setminus\bigcup_{j=1}^m A_j$ and so obtain a new larger family of m disjoint sets which still satisfies (2.8) for the same value of t. Otherwise $\left\|\sum_{k\in A_j} h_k\right\|_Z \geq (t+1)\gamma_j$ must hold for all $j=1,2,\dots,m$. In this case we can enlarge one of the sets A_j by adding an element of $\{1,2,..,b\}\setminus\bigcup_{j=1}^m A_j$ and again obtain a new larger family of m disjoint sets satisfying (2.8), but with the number t replaced by $t+1$.

After reiterating and enlarging members of our family of sets sufficiently many times in this way, we must eventually obtain a family which also satisfies (2.9). This completes the proof of the lemma. ∎

We continue presenting further results which will be used in the next section.

Lemma 2.11. *If W and Z are relatively decomposable Banach lattices then Z' and W' are relatively decomposable with the same constant.*

Proof: This is an easy exercise which we leave to the reader. ∎

As usual, for any Banach lattice Z and for any $p\in(1,\infty)$, we shall let $Z^{(p)}$ denote the *p-convexification* of Z, (i.e. the set of all measurable functions f on the underlying measure space for Z such that $|f|^p\in Z$, with norm $\|f\|_{Z^{(p)}} = \||f|^p\|_Z^{1/p}$).

It is obvious that:

(2.12) Any two Banach lattices W and Z are relatively decomposable with constant D if and only if $W^{(p)}$ and $Z^{(p)}$ are relatively decomposable with constant $D^{1/p}$.

(2.13) For each $r\in[1,\infty]$, Z satisfies an upper r-estimate with constant C if and only if $Z^{(p)}$ satisfies an upper pr-estimate with constant $C^{1/p}$.

(2.14) Statements analogous to (2.13) hold for lower estimates and for equal norm upper and lower estimates.

Remark 2.15.: Several proofs in this section, Section 3 and [CS] would be considerably shorter and simpler, were the notions of p-estimate and equal norm p-estimate to coincide. In order to show that we cannot hope for such simplifications, we conclude this section by describing an example of a Banach lattice of measurable functions X

which does not satisfy a lower p-estimate, even though it satisfies an equal norm lower p-estimate:

Let us first consider the case $p = 1$. We will take X to be the *modified Tsirelson's space* defined on p. 48 of [CaSh] and denoted by T_M. Its construction is due to W. Johnson. It was shown by P. Casazza and E. Odell that T_M is in fact isomorphic to the space denoted in [CaSh] by T, which is the dual of the space originally constructed by B. Tsirelson. T_M is a sequence space, i.e. a space of real valued functions on $\mathbb{N}$. In fact it is a Banach lattice on the measure space $\mathbb{N}$ equipped with counting measure. We mention three of its many remarkable properties, which are the ones needed for our purposes.

(2.16) The embeddings $\ell^1 = L^1(\mathbb{N}) \subset T_M \subset \ell^\infty = L^\infty(\mathbb{N})$ hold continuously. (See [CaSh] Proposition I.2.1, p. 9.)

(2.17) T_M is not isomorphic to ℓ^1. (In fact it does not even contain an isomorphic copy of ℓ^1). (See [CaSh] Proposition I.3, p. 9.)

(2.18) For each $k \in \mathbb{N}$, if $f_1, f_2, ..., f_k$ are k disjointly supported elements of T_M such that $f_j(m) = 0$ for all $m = 1, 2, ..., k-1$ and $j = 1, 2, ..., k$, then

$$2 \left\| \sum_{j=1}^{k} f_j \right\|_{T_M} \geq \sum_{j=1}^{k} \|f_j\|_{T_M} . \tag{2.19}$$

When all the functions f_j have finite support, (2.18) follows immediately from the definition of the norm $\|\cdot\|_{T_M}$ for such functions on page 48 of [CaSh]. (Cf. [CaSh] p. 51.) Since T_M is the completion with respect to $\|\cdot\|_{T_M}$ of the space of functions of finite support, it is clear, from (2.16), that each $f \in T_M$ with support in $[k, \infty)$ is the limit of a sequence of finite supported functions, each with support in $[k, \infty)$. Thus the general form of (2.18) follows immediately from the finitely supported case.

Now let $g_1, g_2, ..., g_n$ be an arbitrary finite collection of disjointly supported norm 1 elements of T_M. We claim that

$$5 \left\| \sum_{j=1}^{n} g_j \right\|_{T_M} \geq n = \sum_{j=1}^{n} \|g_j\|_{T_M} . \tag{2.20}$$

This is obviously true for $n \leq 4$. If $n \geq 5$ let k be the integer part of $n/2$, i.e. $n/2 - 1/2 \leq k \leq n/2$. Let E be the set of integers $j \in \{1, 2, .., n\}$ for which $g_j(m) = 0$ for all $m \in \{1, 2, ..., k-1\}$. Since at most $k-1$ of the n functions g_j can have supports including points in $\{1, 2, ..., k-1\}$ it follows that E must contain at least $n - (k-1)$ elements, and therefore at least k elements. If we let A be a subset of E containing exactly k elements, then, by (2.19), we obtain that

$$\left\| \sum_{j=1}^{n} g_j \right\|_{T_M} \geq \left\| \sum_{j \in A} g_j \right\|_{T_M} \geq \frac{1}{2}k \geq \frac{1}{4}(n-1) = \frac{1}{4}\frac{n-1}{n} \cdot n \geq \frac{1}{5}n$$

which establishes (2.20), showing that T_M satisfies an equal norm lower 1-estimate. T_M cannot satisfy a lower 1-estimate because, if it did, since of course it satisfies an upper 1-estimate, it would then be isomorphic to ℓ^1, contradicting (2.17).

Finally, for any $p \in (1, \infty)$, let $X = T_M^{(p)}$, the p-convexification of T_M. In view of (2.14), X does not satisfy a lower p-estimate, but it does satisfy an equal norm lower p-estimate.

3. COMPLETION OF THE PROOF OF THE MAIN THEOREM

It remains to show that (1.3a)⇒(1.3c). Suppose then that X and Y are relatively decomposable with constant D. Our first step will be to show that

$$q(X) \leq p(Y). \tag{3.1}$$

Suppose on the contrary that $q(X) > p(Y)$ and let r be some number satisfying $q(X) > r > p(Y)$. Then we can apply Lemma 2.3 (i) to obtain that for each constant $K < \infty$ and so, in particular, for $K = D + 2$, there exists an integer $m = m(K)$ and m disjointly supported norm 1 functions in X, $f_1, f_2, ..., f_m$ which satisfy

$$K \left\| \sum_{j=1}^{m} f_j \right\|_X \leq m^{1/r}.$$

Consequently, using the relative decomposability of X and Y, any m disjointly supported norm 1 functions in Y, $g_1, g_2, \ldots, g_m$, must satisfy $\left\| \sum_{j=1}^{m} g_j \right\|_Y \leq \frac{D}{K} m^{1/r} <$

$m^{1/r}$. Since $K > 1$ we have $m > 1$ and so the previous estimate, together with Lemma 2.4, implies that Y satisfies an e.n.u. r-estimate. Since Y has the Fatou property, we can apply Lemma 2.3 (ii) to deduce that $r \le p(Y)$. This contradicts $r > p(Y)$ and so proves that (3.1) holds.

If strict inequality holds in (3.1) then it is easy to check that any index r lying strictly between $q(X)$ and $p(Y)$ satisfies (1.3c) and the proof is complete. Thus, from this point onwards, we may suppose that

$$q(X) = p(Y) = r. \tag{3.2}$$

Our next steps will be to show that

(3.3a) X satisfies an e.n.l. r-estimate,

and then that

(3.3b) Y satisfies an e.n.u. r-estimate.

The first of these steps is very similar to the proof of (3.1). If $r = \infty$ then obviously (3.3a) holds for X (and indeed also for every other Banach lattice). So let us suppose that $1 \le r < \infty$ and that (3.3a) is false. Then, for each constant $K < \infty$, there exists an integer $m = m(K)$ and m disjoint norm 1 functions $f_1, f_2, \ldots, f_m$ in X such that $K \left\| \sum_{j=1}^m f_j \right\|_X \le m^{1/r}$. As before we deduce that any m disjointly supported norm 1 functions in Y, $g_1, g_2, \ldots, g_m$, must satisfy $\left\| \sum_{j=1}^m g_j \right\|_Y \le \frac{D}{K} m^{1/r}$. This time we choose $K > 2D$. Obviously this ensures that $m^{1/r} > 2$ and so there exists a finite number $s > r$ which satisfies $m^{1/r - 1/s} = 2$. Therefore

$$\left\| \sum_{j=1}^m g_j \right\|_Y \le m^{1/s}$$

for all such functions $g_1, \ldots, g_m$ and Lemma 2.4 can be applied to obtain that Y satisfies an e.n.u. s-estimate. This in turn, using Lemma 2.3 (ii) and the fact that Y has the Fatou property, gives that $s \le p(Y)$. But this contradicts the fact that $p(Y) = r < s$ and so proves (3.3a).

If $r = 1$ then (3.3b) is obviously true. The proof of (3.3b) when $1 < r \le \infty$ is a straightforward adaptation of the preceding argument: If (3.3b) is false then, for some constant $K > 2D$ and some integer m, we deduce that $\left\|\sum_{j=1}^{m} g_j\right\|_X \geq \frac{K}{D} m^{1/r}$ for all choices of m disjointly supported norm 1 elements $g_1, g_2, \ldots, g_m$ in X and therefore also $m^{1/r'} > 2$. Since m must satisfy $m^{1/r-1} < 1/2$, there exists a number $s \in (1, r)$ such that $m^{1/r-1/s} = 1/2$, and we can deduce from the second part of Lemma 2.4 that X satisfies an e.n.l. s-estimate. So, by Lemma 2.3 (i), $r = q(X) \leq s$. This contradicts $s < r$ and so proves (3.3b)

The final two steps of the proof will be to strengthen (3.3a) and (3.3b) to

(3.4a) X satisfies a lower r-estimate,

and

(3.4b) Y satisfies an upper r-estimate.

Let us first dispense with the easy case $r = \infty$ for which (3.4a) holds automatically for all choices of X and for which the conditions (3.3b) and (3.4b) are easily seen to be equivalent. From here onwards we can suppose that X is order continuous. If it were not then (see [LT] pp. 6-8) it would contain a sequence of norm 1 disjointly supported functions $\{f_n\}$ such that $\sup_N \left\|\sum_{n=1}^{N} f_n\right\|_X < \infty$. But that is impossible since X satisfies (3.3a) and we can assume that $r < \infty$. In fact, as will be important to know for the last step of the proof, we will not actually need the order continuity of X to establish (3.4b). Instead we will only use another property which follows from order continuity, namely that X' is a norming subspace of X^*.

The fact that X' is a norming subspace of X^* and Remark 2.2 ensure that $p(X') = q(X)' = r'$. Similarly, since Y' is a norming subspace of Y^* (because Y has the Fatou property), we also have $q(Y') = p(Y)' = r'$. Furthermore, by Lemma 2.11, Y' and X' are relatively decomposable with constant D and X' of course has the Fatou property. Thus we can once more apply the arguments above which give (3.3a) and (3.3b), but this time in the setting where X, Y and r are replaced by Y', X' and r' respectively. These arguments give that Y' satisfies an e.n.l. r'-estimate, and also, more importantly for our purposes here,

(3.3b$'$) X' satisfies an e.n.u. r'-estimate.

Our strategy now will be to first obtain (3.4b) while temporarily making the additional assumption:

(3.5) X' satisfies a lower s-estimate for some $s < \infty$ with constant B .

Let us suppose that (3.4b) does not hold. Then, by Remark 2.2, Y' does not satisfy a lower r' estimate. So, given any finite constant K, there exists an integer $m = m(K)$ and m disjointly supported elements $f_1, f_2, \ldots, f_m$ in Y' such that

$$K \left\| \sum_{j=1}^{m} f_j \right\|_{Y'} \leq \left(\sum_{j=1}^{m} \|f_j\|_{Y'}^{r'} \right)^{1/r'} . \tag{3.6}$$

In particular this holds for

$$K = 6DC \tag{3.7}$$

where D is the decomposability constant of X and Y (and so also of Y' and X') and C is the constant with respect to which X' satisfies an e.n.u. r'-estimate.

We may suppose without loss of generality that $\min \|f_j\|_{Y'} = 1$.

Now, in view of (3.5), we can apply Lemma 2.6 in the case where $Z{=}X'$, m is the integer $m = m(K) = m(6DC)$ just introduced, and the numbers γ_j are given by $\gamma_j = \|f_j\|_{Y'}$ for $j = 1, 2, ..., m$. Let b be an integer satisfying (2.7) and let $h_1, h_2, \ldots, h_b$ be any b disjointly supported norm one elements in X'. Then, using the disjoint sets A_j and the number $t \geq 1$ supplied by Lemma 2.6, and also (3.6) and (3.7), we obtain that

$$\begin{aligned}\left\|\sum_{k=1}^{b} h_k\right\|_{X'} &= \left\|\sum_{j=1}^{m}\sum_{k\in A_j} h_k\right\|_{X'} \le (t+2)D\left\|\sum_{j=1}^{m} f_j\right\|_{Y'} \\ &\le \frac{(t+2)D}{K}\left(\sum_{j=1}^{m}\|f_j\|_{Y'}^{r'}\right)^{1/r'} \\ &\le \frac{(t+2)}{6tC}\left(\sum_{j=1}^{m}\left\|\sum_{k\in A_j} h_k\right\|_{X'}^{r'}\right)^{1/r'} \\ &\le \frac{(t+2)}{6t}\left(\sum_{j=1}^{m}\sum_{k\in A_j}\|h_k\|_{X'}^{r'}\right)^{1/r'}.\end{aligned}$$

Since $(t+2)/t \le 3$, this shows that

$$\left\|\sum_{k=1}^{b} h_k\right\|_{X'} \le \frac{1}{2}\left(\sum_{j=1}^{m}\sum_{k\in A_j}\|h_k\|_{X'}^{r'}\right)^{1/r'} = \frac{1}{2}b^{1/r'}.$$

If $r'=\infty$ we adopt the usual convention of replacing sums by maxima in the preceding estimates. Thus they give that $\left\|\sum_{k=1}^{b} h_k\right\|_{X'} \le \frac{1}{2}$. But this is a contradiction since $\left\|\sum_{k=1}^{b} h_k\right\|_{X'} \ge \|h_1\|_{X'} \ge 1$. Otherwise, if $r'<\infty$, there exists a number w satisfying $w>r'$ and $b^{1/r'-1/w} \le 2$, and so we can apply Lemma 2.4 to obtain that X' satisfies an e.n.u. w-estimate. But this too is a contradiction, since $r'=p(X')$. Thus, for all $r\in[1,\infty)$, we have established (3.4a) subject to (3.5).

We shall next use p-convexifications to obtain (3.4b) without the need for (3.5). No longer assuming that X' satisfies (3.5), we choose some $s\in(1,\infty)$ and deduce from (2.12) that $X^{(s')}$ and $Y^{(s')}$ are relatively decomposable. We also have from (2.13) and (2.14) that $q(X^{(s')}) = s'r = p(Y^{(s')})$.

We now wish to apply all steps of the preceding arguments once more, from (3.2) onwards, but now in the setting where X, Y and r are replaced by $X^{(s')}$, $Y^{(s')}$ and $s'r$ respectively. Note that since Y has the Fatou property, so does $Y^{(s')}$ and so its Köthe dual is a norming subspace of its dual. We have previously established that we only have to consider the case where X is order continuous, or at least X' is a norming subspace of X^*, and $r < \infty$. These two conditions now ensure respectively that the analogues of these conditions for the current setting also hold, i.e. $X^{(s')}$ is order continuous and $s'r < \infty$. Since of course X satisfies an upper 1-estimate, we deduce, using (2.13) and then Remark 2.2, that $(X^{(s')})'$ satisfies a lower s-estimate. This corresponds exactly to condition (3.5). So we can indeed apply all of the proof from (3.2) onwards, to deduce the analogue of (3.4b), namely that $Y^{(s')}$ satisfies an upper $s'r$-estimate. Another application of (2.13) gives (3.4b) itself.

To complete the proof of Theorem 1.3 it now remains only to establish (3.4a). We will do this by applying all preceding steps, from (3.2) until the proof of (3.4b) which we have just completed, but now in yet another setting. This time the previous roles of X, Y and r are played by $(Y^{(s)})'$, $(X^{(s)})'$ and $(rs)'$ respectively, where s is some number in $(1,\infty)$. We first need to check that $(Y^{(s)})'$, $(X^{(s)})'$ and $(rs)'$ satisfy the analogues of all conditions which we needed to impose earlier on X, Y and r for the proof of (3.4b). We use (2.12) followed by Lemma 2.11 to obtain that $(Y^{(s)})'$ and $(X^{(s)})'$ are relatively decomposable. For reasons already given above, $(Y^{(s)})'$ and $(X^{(s)})'$ are norming subspaces of $(Y^{(s)})^*$ and $(X^{(s)})^*$ respectively. So we can apply properties (2.12) and (2.13) and also Remark 2.2 to show that $q\left((Y^{(s)})'\right) = \left(p(Y^{(s)})\right)' = (rs)'$ and $p\left((X^{(s)})'\right) = \left(q(X^{(s)})\right)' = (rs)'$. This is the counterpart of (3.2). $(Y^{(s)})'$ and $(X^{(s)})'$, like all other Köthe duals, have the Fatou property, so their own respective Köthe duals are norming subspaces of their own respective duals. Since $r \in [1,\infty)$ we have that $(rs)' \in (1,\infty)$.

We have checked that all required conditions on $(Y^{(s)})'$, $(X^{(s)})'$ and $(rs)'$ are indeed fulfilled and so we can deduce the analogue of (3.4b), namely that $(X^{(s)})'$ satisfies an upper $(rs)'$-estimate. In view of Remark 2.2, $X^{(s)}$ satisfies a lower rs-estimate, and then property (2.14) gives us (3.4a) and so completes the proof of Theorem 1.3. ∎

4. APPLICATION TO THE PROBLEM OF CHARACTERIZING INTERPOLATION SPACES

Let us briefly describe one form of the corollary of Theorem 1.3 for the theory of interpolation spaces. We refer to [BL], [CN1] and [CN3] for any unexplained notions or notation.

Perhaps the simplest way to generate interpolation spaces and relative interpolation spaces is by means of the Peetre K-functional. For example, given any two Banach couples $\mathbf{X} = (X_0, X_1)$ and $\mathbf{Y} = (Y_0, Y_1)$, we can use the K-functional to define the spaces $\mathbf{X}_{\theta,p}$ and $\mathbf{Y}_{\theta,p}$, which are relative exact interpolation spaces with respect to $\mathbf{X}$ and $\mathbf{Y}$ ([BL] p. 40). More generally, if X and Y are normed intermediate spaces with respect to $\mathbf{X}$ and $\mathbf{Y}$ respectively then we say that they are *relative* K spaces if whenever $x \in X$ and $y \in Y_0 + Y_1$ satisfy

$$K(t, y; \mathbf{Y}) \leq K(t, x; \mathbf{X}) \quad \text{for all } t > 0$$

then it follows that $y \in Y$. It is very easy to see that if X and Y are relative K spaces then they are also relative interpolation spaces with respect to $\mathbf{X}$ and $\mathbf{Y}$. ([BL] p. 41) But, on the other hand, there are many known examples of relative interpolation spaces which are not relative K spaces.

For a reasonably large class of suitable couples $\mathbf{X}$ and $\mathbf{Y}$, which we refer to as *relative Calderón-Mityagin couples* or *relative* $\mathcal{CM}$ *couples*, the relative interpolation spaces are precisely the relative K spaces. The characterization of all such couples is a long standing and apparently very difficult problem in the theory of interpolation spaces.

Theorem 1.3 of this paper combined with Theorem 2.2 of [CN3] yields the following formulation of the solution of a variant of that problem:

THEOREM 4.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two couples of saturated* σ*-order continuous Banach lattices with the Fatou property on the* σ*-finite measure spaces* (Ω, Σ, μ) *and* (Ξ, Ψ, ν) *respectively. Then the following conditions are equivalent:*

(i) The couples of weighted lattices $\mathbf{X}_\mathbf{w} = (X_{0,w_0}, X_{1,w_1})$ *and* $\mathbf{Y}_\mathbf{v} = (Y_{0,v_0}, Y_{1,v_1})$ *are*

relative $\mathcal{CM}$ couples for all choices of weight functions w_0, w_1 on Ω and v_0 and v_1 on Ξ.

(ii) There exist p_0 and p_1 in $[1,\infty)$ such that X_j satisfies a lower p_j-estimate and Y_j satisfies an upper p_j-estimate for $j = 0,1$.

(iii) There exist p_0 and p_1 in $[1,\infty)$ such that $\mathbf{X_w}$ and $(L^{p_0}_{u_0}, L^{p_1}_{u_1})$ are relative $\mathcal{CM}$ couples and also $(L^{p_0}_{u_0}, L^{p_1}_{u_1})$ and $\mathbf{Y_v}$ are relative $\mathcal{CM}$ couples for all choices of weight functions w_0, w_1 on Ω and v_0 and v_1 on Ξ and u_0 and u_1 on the underlying measure space of the L^p spaces.

Note that in Theorem 4.1 it is specified that the exponents p_0 and p_1 do not take the value ∞. This is simply because the fact that Y_0 and Y_1 are σ-order continuous precludes these spaces from satisfying an upper ∞-estimate (except in the trivial case where they are finite dimensional).

We do not know at this stage whether Theorem 2.2 of [CN3] is true without the assumption of σ-order continuity. We suspect that at least a partial version can be obtained without this assumption and that, consequently, there is also a variant of Theorem 4.1 in which p_0 and/or p_1 may assume the value ∞.

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